1. **Introduction:** Just like with differentiability it’s helpful to establish rules for when functions are and are not integrable.

2. **Lemma:** Suppose \( f \) is integrable and \( \{P_n\} \) is an Archimedean sequence of partitions. Then if \( P^*_n \) is a refinement of \( P_n \) for each \( n \) then \( \{P^*_n\} \) is also an Archimedean sequence of partitions.

   **Proof:** We know that:
   \[
   \{U(f, P_n) - L(f, P_n)\} \to 0
   \]

   and by the Refinement Theorem:
   \[
   U(f, P^*_n) - L(f, P^*_n) \leq U(f, P_n) - L(f, P_n)
   \]

   Then by the Comparison Lemma we have
   \[
   \{U(f, P^*_n) - L(f, P^*_n)\} \to 0
   \]

3. **Theorem (Additivity Over Intervals):** Suppose \( f \) is integrable on \([a, b]\) and let \( c \in (a, b) \). Then \( f \) is integrable on \([a, c]\) and on \([c, b]\) and

   \[
   \int_a^b f = \int_a^c f + \int_c^b f
   \]

   **Proof:** Let \( \{P_n\} \) be an Archimedean sequence of partitions for \( f \). By the above lemma adding \( x = c \) to each \( P_n \) still results in an Archimedean sequence of partitions so we just assume that \( x = c \) is in each \( P_n \). Write \( P_n = P^*_n \cup P''_n \) where \( P^*_n \) and \( P''_n \) are the partitions induced by \( P_n \) on just \([a, c]\) and \([c, b]\) respectively. By the definition of upper and lower sums we have:

   \[
   U(f, P_n) - L(f, P_n) = [U(f, P^*_n) - L(f, P^*_n)] + [U(f, P''_n) - L(f, P''_n)]
   \]

   Since the second bracket on the right is nonnegative we have

   \[
   U(f, P_n) - L(f, P_n) \geq U(f, P^*_n) - L(f, P^*_n)
   \]

   So that by the Comparison Lemma \( \{P^*_n\} \) is an Archimedean sequence of partitions for \( f \) on \([a, c]\) so \( f \) is integrable on \([a, c]\) and \( U(f, P^*_n) \to \int_c^c f \). A similar argument shows that \( \{P''_n\} \) is an Archimedean sequence of partitions for \( f \) on \([c, b]\) so \( f \) is integrable on \([c, b]\) and \( U(f, P''_n) \to \int_c^b f \). Therefore since

   \[
   \{U(f, P_n)\} \to \int_a^b f
   \]

   and

   \[
   \{U(f, P_n)\} = \{U(f, P^*_n) + U(f, P''_n)\} \to \int_a^c f + \int_c^b f
   \]

   we have the result.
4. **Theorem (Monotonicity):** Suppose $f, g : [a, b] \to \mathbb{R}$ are integrable and for all $x \in [a, b]$ we have $f(x) \leq g(x)$. Then

$$\int_a^b f \leq \int_a^b g$$

**Proof:** Take an Archimedean sequence of partitions for $f$ and one for $g$. For each $n$ take the union $P_n$ of the corresponding partitions. By the above lemma the resulting $\{P_n\}$ is an Archimedean sequence of partitions for both $f$ and $g$. From here we get:

$$\{U(g, P_n) - U(f, P_n)\} \to \int_a^b g - \int_a^b f$$

However since $f(x) \leq g(x)$ we have $U(g, P_n) - U(f, P_n) \geq 0$ and therefore since $[0, \infty)$ is closed we have

$$\int_a^b g - \int_a^b f \geq 0$$

5. **Theorem (Linearity):** Suppose $f, g : [a, b] \to \mathbb{R}$ are integrable and $\alpha, \beta \in \mathbb{R}$. Then the function $\alpha f + \beta g$ is integrable on $[a, b]$ and

$$\int_a^b \alpha f + \beta g = \alpha \int_a^b f + \beta \int_a^b g$$

**Proof:** Omit (several pages).