1. **Introduction:** We know that in general the Taylor polynomial does not equal the function (other than at $x_0$) and so there is a remainder:

$$f(x) = p_n(x) + r_n(x)$$

We also have a formula for $r_n(x)$, the Lagrange Remainder Formula. However this is not the only formula for the remainder.

2. **Preliminary Definition:** Suppose $f : [a, b] \to \mathbb{R}$ is integrable. We define:

$$\int_b^a f = -\int_a^b f$$

3. **Integration by Parts:** Suppose $f, g : [a, b] \to \mathbb{R}$ are both continuous and have continuous bounded derivatives on $(a, b)$. Then

$$\int_a^b fg' = fg\bigg|_a^b - \int_a^b f'g$$

**Proof:** We have:

$$(fg)' = f'g + fg'$$

$$fg' = (fg)' - f'g$$

$$\int_a^b fg' = \int_a^b (fg)' - \int_a^b f'g$$

$$\int_a^b fg' = fg\bigg|_a^b - \int_a^b f'g$$

where the last equality holds by the First Fundamental Theorem of Calculus.

4. **The Cauchy Integral Remainder Theorem:** Let $I$ be a neighborhood of $x_0$ and let $n \in \mathbb{N}$. Suppose $f : I \to \mathbb{R}$ has $n+1$ derivatives and $f^{(n+1)} : I \to \mathbb{R}$ is continuous. Then for each $x \in I$ we have:

$$r_n(x) = \frac{1}{n!} \int_{x_0}^x f^{(n+1)}(t)(x - t)^n \, dt$$

In other words:

$$f(x) = p_n(x) + \frac{1}{n!} \int_{x_0}^x f^{(n+1)}(t)(x - t)^n \, dt$$

**Note:** The primary thing to note is that while the Lagrange Remainder Formula depends on an unknown (the $c$), the Cauchy Integral Remainder Formula does not. It does however depend on an integral that is in many cases impractical to calculate. But on a positive note this integral can be approximated using sums.
**Proof:** We proceed by induction but we will show both \( n = 0 \) and \( n = 1 \) just because they're enlightening:

For the case \( n = 0 \) observe that the right side of the above equals

\[
f(x_0) + \frac{1}{0!} \int_{x_0}^{x} f^{(0+1)}(t)(x-t)^0 \, dt = f(x_0) + \left[ f(t) \right]_{x_0}^{x} = f(x)
\]

where the First Fundamental Theorem of Calculus is used to evaluate the integral.

For the case \( n = 1 \) observe that by the First Fundamental Theorem of Calculus we have:

\[
f(x) = f(x_0) + \int_{x_0}^{x} f'(t) \, dt
\]

We proceed by rewriting the integral and integrating by parts:

\[
\int_{x_0}^{x} f'(t) \, dt = \int_{x_0}^{x} f'(t) \frac{d}{dt}(x-t) \, dt
\]

\[
= -f'(t)(x-t) \bigg|_{x_0}^{x} + \int_{x_0}^{x} f''(t)(x-t) \, dt
\]

\[
= -f'(x)(x-x) + f'(x_0)(x-x_0) + \int_{x_0}^{x} f''(t)(x-t) \, dt
\]

\[
= f'(x_0)(x-x_0) + \int_{x_0}^{x} f''(t)(x-t) \, dt
\]

Thus:

\[
f(x) = f(x_0) + f'(x_0)(x-x_0) + \int_{x_0}^{x} f''(t)(x-t) \, dt
\]

and we have our claim for \( n = 1 \).

Assume that for \( n \) we have:

\[
f(x) = p_n(x) + \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \, dt
\]

Observe that:

\[
\frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \, dt = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t) \left[ \frac{d}{dt} \frac{-1}{n+1} (x-t)^{n+1} \right] \, dt
\]

\[
= -\frac{1}{(n+1)!} \int_{x_0}^{x} f^{(n+1)}(t) \left[ \frac{d}{dt} (x-t)^{n+1} \right] \, dt
\]

\[
= -\frac{1}{(n+1)!} \left[ f^{(n+1)}(t)(x-t)^{n+1} \bigg|_{x_0}^{x} - \int_{x_0}^{x} f^{(n+2)}(t)(x-t)^{n+1} \, dt \right]
\]

\[
= -\frac{1}{(n+1)!} \left[ 0 - f^{(n+1)}(x_0)(x-x_0)^{n+1} - \int_{x_0}^{x} f^{(n+2)}(t)(x-t)^{n+1} \, dt \right]
\]

\[
= f^{(n+1)}(x_0) \frac{1}{(n+1)!} (x-x_0)^{n+1} + \frac{1}{(n+1)!} \int_{x_0}^{x} f^{(n+2)}(t)(x-t)^{n+1} \, dt
\]

So that

\[
f(x) = p_n(x) + \frac{f^{(n+1)}(x_0)}{(n+1)!} (x-x_0)^{n+1} + \frac{1}{(n+1)!} \int_{x_0}^{x} f^{(n+2)}(t)(x-t)^{n+1} \, dt
\]

\[
f(x) = p_{n+1}(x) + \frac{1}{(n+1)!} \int_{x_0}^{x} f^{(n+2)}(t)(x-t)^{n+1} \, dt
\]