1. **Introduction:** Our hope is that when \( \{f_n\} \to f \) that some properties are also passed over. We will see that this is true to a degree.

2. **Theorem (Continuity):** Suppose that \( f_n : D \to \mathbb{R} \) are all continuous, \( f : D \to \mathbb{R} \) and \( \{f_n\} \to f \).
   Then \( f \) is continuous.
   **Proof:** Let \( x_0 \in D \) and suppose \( \{x_k\} \to x_0 \). We claim \( \{f(x_k)\} \to f(x_0) \). Suppose we are given \( \epsilon > 0 \).
   - First since \( \{f_n\} \to f \) we can choose a fixed \( N \) so that \( f_N \) is within \( \frac{\epsilon}{3} \) of \( f \) everywhere. Thus \( f(x_k) \) is within \( \frac{\epsilon}{3} \) of \( f_N(x_k) \) and \( f(x_0) \) is within \( \frac{\epsilon}{3} \) of \( f_N(x_0) \).
   - Second since \( f_N \) is continuous we can choose \( K \) such that if \( k \geq K \) then \( |f_N(x_k) - f_N(x_0)| < \frac{\epsilon}{3} \).

   Then for \( k \geq K \) we have:
   \[
   |f(x_k) - f(x_0)| = |f(x_k) - f_N(x_k) + f_N(x_k) - f_N(x_0) + f_N(x_0) - f(x_0)| \\
   \leq |f(x_k) - f_N(x_k)| + |f_N(x_k) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\
   \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
   \]
3. **Theorem (Integrability)**: Suppose that \( \{f_n : [a, b] \to \mathbb{R} \} \) are all integrable, \( f : [a, b] \to \mathbb{R} \) and and \( \{f_n\} \xrightarrow{u} f \). Then \( f \) is integrable and
\[
\left\{ \int_a^b f_n \right\} \to \int_a^b f
\]

**Proof:** Given \( \epsilon > 0 \) let \( \epsilon' = \frac{\epsilon}{2(b-a)} \) since \( \{f_n\} \xrightarrow{u} f \) we can choose some \( n \in \mathbb{N} \) so that \( f_n(x) \) is within \( \epsilon' \) of \( f(x) \) everywhere. That is, for all \( x \in [a, b] \) we have:
\[
|f(x) - f_n(x)| < \epsilon' \\
-\epsilon' < f(x) - f_n(x) < \epsilon' \\
f_n(x) - \epsilon' < f(x) < f_n(x) + \epsilon'
\]

It follows by the monotonicity of the lower and upper integrals (this follows from their definition) that:
\[
\int_a^b f \leq \int_a^b [f_n + \epsilon'] = \int_a^b [f_n + \epsilon'] = \left[ \int_a^b f \right] + \epsilon'(b-a)
\]
and
\[
\int_a^b f \geq \int_a^b [f_n - \epsilon'] = \int_a^b [f_n - \epsilon'] = \left[ \int_a^b f \right] - \epsilon'(b-a)
\]

Thus
\[
\int_a^b f - \int_a^b f \leq \left[ \left[ \int_a^b f \right] + \epsilon'(b-a) \right] - \left[ \left[ \int_a^b f \right] - \epsilon'(b-a) \right] = 2\epsilon'(b-a) = \epsilon
\]

Since this holds for any \( \epsilon > 0 \) we must have
\[
\int_a^b f = \int_a^b f
\]
and hence \( f \) is integrable.

Next consider that for any \( \epsilon > 0 \) let \( \epsilon' = \frac{\epsilon}{b-a} \) then since \( \{f_n\} \xrightarrow{u} f \) we can choose some \( n \in \mathbb{N} \) so that \( f_n(x) \) is within \( \epsilon' \) of \( f(x) \) everywhere. As before we have:
\[
-\epsilon' < f_n(x) - f(x) < \epsilon' \\
\int_a^b \epsilon' < \int_a^b [f_n - f] < \int_a^b \epsilon' \\
-\epsilon'(b-a) < \int_a^b f_n - \int_a^b f < \epsilon'(b-a)
\]
\[
\left| \int_a^b f_n - \int_a^b f \right| < \epsilon'(b-a)
\]
\[
\left| \int_a^b f_n - \int_a^b f \right| < \epsilon
\]

Since this holds for any \( \epsilon \) we have
\[
\left\{ \int_a^b f_n \right\} \to \int_a^b f
\]
4. **Theorem (Differentiability):** Let $I$ be an open interval in $\mathbb{R}$. Suppose that $\{f_n : I \to \mathbb{R}\}$ is such that:

- The $f_n$ are all differentiable with continuous derivatives.
- We have $\{f_n\} \to f$ for some $f : I \to \mathbb{R}$.
- We have $\{f'_n\} \to g$ for some $g : I \to \mathbb{R}$.

Then $f$ is differentiable with a continuous derivative and for all $x \in I$ we have $f'(x) = g(x)$

**Note:** The fact that it’s not sufficient to have $\{f_n\} \to f$ with all the $f_n$ differentiable is pointed out by the example in Section 9.2 with $f_n(x) = x \tan^{-1}(nx)$ on $(-1, 1)$. This sequence does converge uniformly to $\frac{\pi}{2}|x|$ and yet differentiability is not carried over to $f$.

This example doesn’t satisfy these hypotheses because even though the sequence of derivative functions converges to some $g$, that convergence is pointwise and not uniform. The easiest way to see this is to note that for $x > 0$ since $\{f_n(x)\} \to \frac{\pi}{2}x$ we must have $\{f'_n(x)\} \to \frac{\pi}{2}$ but no matter how large $n$ is there are always points close to $x = 0$ with slope arbitrarily close to $0$ and therefore far from $\frac{\pi}{2}$.

We can see this by examining the derivative:

$$f_n(x) = \tan^{-1}(nx) + \frac{nx}{1 + n^2x^2}$$

By making $x$ very small we can make both summands small.

**Proof:** Fix $x_0 \in I$ and note that for each $x \in I$ by the First Fundamental Theorem of Calculus we have:

$$f_n(x) - f_n(x_0) = \int_{x_0}^{x} f'_n$$

Now then since $\{f'_n\} \to g$ we have:

$$\left\{\int_{x_0}^{x} f'_n\right\} \to \int_{x_0}^{x} g$$

and since $\{f_n\} \to f$ we have:

$$\{f_n(x) - f_n(x_0)\} \to f(x) - f(x_0)$$

It follows therefore that:

$$f(x_0) - f(x) = \int_{x_0}^{x} g$$

Since these are identical if the right side is differentiable then so is the left side. And indeed since the $f'_n$ are continuous and $\{f'_n\} \to g$ we know that $g$ is continuous and so by the Second Fundamental Theorem of Calculus we have:

$$\frac{d}{dx} \int_{x_0}^{x} g = g$$

and so:

$$\frac{d}{dx} [f(x) - f(x_0)] = g$$

$$f'(x) = g(x)$$

Note that since $f' = g$ and $g$ is continuous we know $f'$ has continuous derivative and since $f$ is differentiable it must be continuous.