1. **Introduction:** We started this chapter by taking a function and creating the series of Taylor Polynomials from the function. Now we will go the other way, we’ll start with a series which converges and use it to define a function.

2. **Theorem (Ratio Test):** Consider the series:

\[ \sum_{n=0}^{\infty} a_n \]

Suppose that:

\[ \left\{ \frac{|a_{n+1}|}{a_n} \right\} \rightarrow L \]

Then:

- If \( L < 1 \) then the series converges (absolutely).
- If \( L > 1 \) then the series diverges.

**Proof:** Omit.

3. **Definition:** Given a sequence \( \{c_n\} \) we define the domain of convergence of the series:

\[ \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + ... \]

to be the set \( D \) of all \( x \in \mathbb{R} \) such that the series converges. Note that \( D \) is nonempty since 0 \( \in D \). Then we can define \( f : D \rightarrow \mathbb{R} \) by:

\[ f(x) = \sum_{n=0}^{\infty} c_n x^n \]

and we say that the series is a power series expansion of \( f(x) \).

**Example:** Consider the series:

\[ \sum_{n=0}^{\infty} \frac{2^n x^n}{n+1} \]

We define:

\[ a_n = \frac{2^n x^n}{n+1} \]

and observe that:

\[ \left\{ \frac{a_{n+1}}{a_n} \right\} = \left\{ \frac{2^{n+1} x^{n+1}}{(n+2)} \cdot \frac{n+1}{2^n x^n} \right\} = \left\{ 2|x| \cdot \frac{n+1}{n+2} \right\} \rightarrow 2|x| \]

It follows that the series converges (absolutely) when \( 2|x| < 1 \) which is when \( |x| < \frac{1}{2} \).
4. **Theorem:**
Consider the power series:
\[ \sum_{n=0}^{\infty} c_n x^n \]
If \( r \neq 0 \) is in the domain of convergence of the power series then so is the entire interval \((-|r|, |r|)\).
In addition the power series converges uniformly on this interval.

**Proof:** Omit.

**Meaning:** For example if \( x = 5 \) is in the domain of convergence then the domain of convergence contains all of \((-5, 5)\).

**Corollary:** The domain of convergence of a power series always has one of the forms \( \{0\} \), \((r, r)\), \([r, r)\) or \([r, r]\). We can have \( r = \infty \) in the parenthetical cases.

5. **Theorem (Differentiation):**
Consider the power series:
\[ \sum_{n=0}^{\infty} c_n x^n \]
Suppose \((-r, r)\) is in the domain of convergence then the function \( f : (-r, r) \to \mathbb{R} \) defined by this power series has derivatives of all order and the derivatives may be calculated on a term-by-term basis. In other words:
\[
\frac{d^n}{dx^n} \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{d^n}{dx^n} [c_n x^n]
\]
And in fact all of these derivatives also converge on \((-r, r)\).

**Proof:** Omit.

**Example:** The earlier example yielded \( f : (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R} \) defined by:
\[
f(x) = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}
\]
It follows that the function \( f \) is differentiable and
\[
f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} \left[ \frac{2^n x^n}{n!} \right]
\]
Notice we need to be careful if we rewrite this as a sum because the 0\textsuperscript{th} term vanishes as it’s constant. The result is therefore:
\[
f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} \left[ \frac{2^n x^n}{n!} \right] = \sum_{n=1}^{\infty} \frac{2^n n x^{n-1}}{n!}
\]
6. **Differential Equations:** Functions defined through power series can be useful when dealing with differential equations. Custom-construction of power series to solve differential equations is beyond the scope of the course but we can at the very least consider the following.

**Example:** Consider the power series:

\[
\sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} = \frac{1}{(3(0))!} + \frac{x^{3(1)}}{(3(1))!} + \frac{x^{3(2)}}{(3(2))!} + \ldots = 1 + \frac{1}{6}x^3 + \frac{1}{720}x^6 + \ldots
\]

The Ratio Test shows that the power series converges for all \(x\) and so defines a function \(f: \mathbb{R} \rightarrow \mathbb{R}\). It follows that:

\[
\begin{align*}
  f(x) &= \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} \\
  f'(x) &= \sum_{n=1}^{\infty} \frac{3nx^{3n-1}}{(3n)!} = \sum_{n=1}^{\infty} \frac{x^{3n-1}}{(3n-1)!} \\
  f''(x) &= \sum_{n=1}^{\infty} \frac{(3n-1)x^{3n-2}}{(3n-1)!} = \sum_{n=1}^{\infty} \frac{x^{3n-2}}{(3n-2)!} \\
  f'''(x) &= \sum_{n=1}^{\infty} \frac{(3n-2)x^{3n-3}}{(3n-2)!} = \sum_{n=1}^{\infty} \frac{x^{3n-3}}{(3n-3)!} = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} = f(x)
\end{align*}
\]

It follows that this \(f(x)\) satisfies the differential equation:

\[
f'''(x) = f(x)
\]

This is interesting because we are familiar with a function which equals its derivative (for example \(y = e^x\)) and its second derivative (for example \(y = e^{-x}\)) and even its fourth derivative (for example \(y = \sin x\) and \(y = \cos x\)) but not its third derivative.

**Note:** In this example \(f(0) = 1\). Since any multiple of \(f\) also satisfies this differential equation we can multiply by the power series by any \(c\) to force \(f(0) = c\). For example the function defined by the power series:

\[
f(x) = \sum_{n=0}^{\infty} \frac{17x^{3n}}{(3n)!}
\]

satisfies the same differential equation and has \(f(0) = 17\).

**Note:** In this example \(f'(0) = 0\). Changing this is trickier. One approach is to take a term-by-term antiderivative of \(f\) which will then satisfy the same differential equation and adjust it accordingly. For example to get \(f(0) = 17\) and \(f'(0) = 42\) we could do:

\[
f(x) = \sum_{n=0}^{\infty} \left[ \frac{17x^{3n}}{(3n)!} + \frac{42x^{3n+1}}{(3n+1)!} \right]
\]

Observe that the way we have written this is not in a standard power-series way but it’s possible to rewrite it.