Math 410 Section 6.2: Integrability and the Archimedes-Riemann Theorem

1. **Introduction:** So far we know what the upper and lower Darboux integrals are but there are two things to settle: What it means for a function to be integrable and if there's a nicer way to actually find this integral.

2. **Integrability:**

   (a) **Definition:** Suppose $f : [a, b] \to \mathbb{R}$ is bounded. We say that f is integrable on $[a, b]$ if the upper and lower Darboux integrals are equal. That is, if

   $$\underline{\int_a^b} f = \overline{\int_a^b} f$$

   If this is the case we denote the value by

   $$\int_a^b f$$

   (b) **Note:** We saw before that $\underline{\int_a^b} f \leq \overline{\int_a^b} f$ always, integrability just means they're equal.

   (c) **Example:** We saw that $f : [0, 2] \to \mathbb{R}$ defined by $f(x) = 3$ has $\underline{\int_0^2} 3 = 6$ and $\overline{\int_0^2} 3 = 6$ therefore $f(x) = 3$ is integrable on $[0, 2]$ and $\int_0^2 3 = 6$.

   (d) **Example:** We saw that $f : [0, 1] \to \mathbb{R}$ defined by by $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise has $\underline{\int_0^1} f = 0$ and $\overline{\int_0^1} f = 1$ therefore f is not integrable on $[0, 1]$ and $\int_0^1 f$ is undefined.

3. **The Archimedes Riemann Theorem**

   (a) **Introduction:** The AR-Theorem provides a more convenient way of determining if a function is integrable without worrying about sup and inf. Loosely speaking it says that to prove integrability all we need to do is obtain a sequence of partitions for which the lower sums increase and the upper sums decrease and these converge to the same value and that value will be the integral.

   (b) **Lemma:** Suppose $f : [a, b] \to \mathbb{R}$ is bounded and P is a partition of $[a, b]$. Then

   $$L(f, P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f, P)$$

   Proof: The left and right inequalities follow from the definition of the Darboux integrals and the middle inequality was proved.

(c) **Archimedes-Riemann Theorem:** Suppose $f : [a,b] \to \mathbb{R}$ is bounded. Then f is integrable on $[a,b]$ iff there is a sequence of partition $\{P_n\}$ such that

$$\{U(f,P_n) - L(f,P_n)\} \to 0$$

Moreover if $\{P_n\}$ is such a sequence then

$$\{L(f,P_n)\} \to \int_a^b f \text{ and } \{U(f,P_n)\} \to \int_a^b f$$

Such a sequence of partition is then called an Archimedian sequence of partitions.
Proof: There are three parts to this:

- Suppose we have such a sequence of partitions $\{P_n\}$. Then observe that for each n we have:

$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le U(f,P_n) - L(f,P_n)$$

  Taking the limit as $n \to \infty$ then tells us that the right side goes to zero and so the middle must be zero so

$$\overline{\int_a^b} f = \underline{\int_a^b} f$$

- Suppose on the other hand that f is integrable on $[a,b]$. Let $n \in \mathbb{N}$. Since $\int_a^b f = \underline{\int_a^b} f$ is the least upper bound on the set L we know that $\int_a^b f - \frac{1}{n}$ is not an upper bound so there is some partition P_L with

$$L(f,P_L) > \int_a^b f - \frac{1}{n}$$

  Similiarly there is some partition P_U with

$$U(f,P_U) < \int_a^b f + \frac{1}{n}$$

  Let P_n be the union of the two partitions which is therefore a refinement of both. Recalling that when we refine a partition lower sums go up and upper sums go down we get

$$L(f,P_n) \ge L(f,P_L) > \int_a^b f - \frac{1}{n}$$

  and

$$L(f,P_n) \le U(f,P_U) < \int_a^b f + \frac{1}{n}$$

  It then follows that:

$$0 \le U(f,P_n) - L(f,P_n) < \left[\int_a^b f + \frac{1}{n}\right] - \left[\int_a^b f - \frac{1}{n}\right] = \frac{2}{n}$$

  So then $\{U(f,P_n) - L(f,P_n)\} \to 0$ by the Comparison Lemma.
- Lastly suppose $\{P_n\}$ is a sequence of partitions satisfying $\{U(f,P_n) - L(f,P_n)\} \to 0$. Observing that from the Lemma and integrability we get

$$0 \le U(f,P_n) - \int_a^b f \le U(f,P_n) - L(f,P_n)$$

  and so $\{U(f,P_n)\} \to \int_a^b f$ by the Comparison Lemma. A similar argument holds for $\{L(f,P_n)\}$.

(d) **The Regular Partition:** The most useful sequence of partitions is to just set P_n to divide $[a,b]$ into n intervals of equal size. That is:

$$P_n = \left\{a, a + \frac{b-a}{n}, a + 2\left(\frac{b-a}{n}\right), ..., b\right\}$$

(e) **Example:** Consider $f : [0,4] \to \mathbb{R}$ defined by $f(x) = 6x$. If $\{P_n\}$ is the regular sequence of partitions then for each n we have:

$$\begin{aligned}
L(f,P_n) &= f(0)\frac{1}{n} + f\left(\frac{4-0}{n}\right)\frac{4}{n} + f\left(2\cdot\frac{4-0}{n}\right)\frac{4}{n} + f\left(3\cdot\frac{4-0}{n}\right)\frac{4}{n} + ... + +f\left((n-1)\cdot\frac{4-0}{n}\right)\frac{4}{n} \\
&= \frac{4}{n}\left[6(0) + 6\left(\frac{4-0}{n}\right) + 6\left(2\cdot\frac{4-0}{n}\right) + 6\left(3\cdot\frac{4-0}{n}\right) + ... + +6\left((n-1)\cdot\frac{4-0}{n}\right)\right] \\
&= \frac{96}{n^2}\left[0 + 1 + 2 + ... + (n-1)\right] \\
&= \frac{96}{n^2}\left[\frac{(n-1)(n)}{2}\right] \\
&= \frac{48(n-1)}{n}
\end{aligned}$$

A similar argument shows that

$$U(f,P_n) = \frac{48(n+1)}{n}$$

Therefore

$$\{U(f,P_n) - L(f,P_n)\} = \left\{\frac{48(n+1)}{n} - \frac{48(n-1)}{n}\right\} = \left\{\frac{96}{n}\right\} \to 0$$

so f is integrable on $[0,4]$ and:

$$\{U(f,P_n)\} = \left\{\frac{48(n+1)}{n}\right\} = \left\{48 + \frac{48}{n}\right\} \to 48 = \int_0^4 6x$$

4. **Some Integrable Functions**

(a) **Theorem (Mononotone Functions are Integrable):** If $f : [a,b] \to \mathbb{R}$ is monotone then f is integrable.
Proof: The proof of this is similar to the previous example done more generally.

(b) **Theorem (Step Functions are Integrable):** A step function $f : [a,b] \to \mathbb{R}$ is a function defined by choosing a fixed partition $\{a = x_0, x_1, ..., x_k = b\}$ and insisting that f is constant on each (x_{i-1}, x_i). If $f : [a,b] \to \mathbb{R}$ is a step function then f is integrable.
Proof: Omitted.