Math 411 Exam 1 Fall 2013 Solution Outlines

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1. Prove that $\mathbb{R}$ is connected.

   Solution:

   Prove it's path connected, which is easier.

2. Suppose $\bar{u} \in \mathbb{R}^n$ with $||\bar{u}|| < 1$. Show that if $\bar{v} \in \mathbb{R}^n$ and $||\bar{v} - \bar{u}|| < 1 - ||\bar{u}||$ then $||\bar{v}|| < 1$.

   Solution:

   You'll need to use the triangle inequatlity in a sneaky way. Note that $||\bar{u}|| = ||\bar{u} - \bar{v} + \bar{v}||$.

3. Prove from the definition that the sequence $\left\{2 + \frac{1}{k}, 1 - \frac{2}{k^2}\right\}$ converges to $(2, 1)$.

   Solution:

   No outline - just go straight to the definition.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous such that $f(x) \geq 0$ for $x \in \mathbb{Q}$. Prove that $f(x) \geq 0$ for all $x \in \mathbb{R}$.

   Solution:

   Proceed by contradiction and use the density of $\mathbb{Q}$. This is actually a 410 question.

5. Prove from the definition that the closed ball $S = \left\{(x, y) \mid x^2 + y^2 \leq 4\right\}$ is path connected. Make sure you show that your path is actually inside the set.

   Solution:

   There are various way to construct a path. If you use a straight line then the triangle inequality can help you show it's inside the set. Alternatively your path can go from the first point straight in to the origin and then back out to the second point. Then showing it's inside is easier.

6. Prove from the definitions that if $A \subseteq \mathbb{R}^n$ is open in $\mathbb{R}^n$ then $\mathbb{R}^n - A$ is closed in $\mathbb{R}^n$.

   Solution:

   Suppose $A \subseteq \mathbb{R}^n$ is open. Let $\bar{x} \in \mathbb{R}^n - A$ and let $\{\bar{x}_k\}$ be in $\mathbb{R}^n - A$ and converging to $\bar{x}$. If $\bar{x} \notin \mathbb{R}^n - A$ then $\bar{x} \in A$ and then $\exists\, B_r(\bar{x}) \subset A$. But then by the definition of convergence eventually $\{\bar{x}_k\}$ is in $B_r(\bar{x})$ which is a contradiction.

7. Give an example of each of the following. No justification is required. Pictures suffice if you wish.

   (a) A collection of open subsets of $\mathbb{R}$ whose intersection is not open.

   Solution:

   This was on the Spring 2013 exam too - look there.

   (b) Two pathwise connected sets whose intersection is not pathwise connected.

   Solution:

   Take a $\cap$ shape and a $\cup$ shape and overlap their endpoints so the intersection is two disjoint sets.

   (c) A non-convex pathwise connected set.

   Solution:

   A $\cup$ shape for example.