Math 411 Exam 1 Spring 2013

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1. Define the set $A \subseteq \mathbb{R}^2$ by

   $$A = A_1 \cup A_2 \text{ where } A_1 = \{(0,y) \mid -1 \leq y \leq 1\} \text{ and } A_2 = \{(x, \sin(1/x)) \mid 0 < x \leq 1\}$$

   Define $f : A \to \mathbb{R}$ by

   $$f(x,y) = \begin{cases} 1 & \text{for } (x,y) \in A_1 \\ 2 & \text{for } (x,y) \in A_2 \end{cases}$$

   Prove that f is not continuous on A.

2. Prove that if $S \subseteq \mathbb{R}^n$ has the Intermediate Value Property then it is connected.

3. Prove using the definition of open that the set $S = \{(x,y) \mid x > 0, y > 0\}$ is open in $\mathbb{R}^2$.

4. Suppose $a, b, c, d \in \mathbb{R}$ with $a < b \leq c < d$. Prove using the definition of pathwise-connected that the subset of $\mathbb{R}$ given by $S = [a,b] \cup [c,d]$ is pathwise-connected iff $b = c$.

5. Give a specific example to show that it's possible to have nonzero sequence $\{\bar{u}_k\}$ in $\mathbb{R}^2$ and nonzero $\bar{u}, \bar{v} \in \mathbb{R}^2$ with $\{\bar{u}_k\}$ converging to $\bar{u}$ with $\bar{u} \perp \bar{v}$ but $\forall k,\ \bar{u}_k \not\perp \bar{v}$. Make sure to prove your claims on perpendicularity and convergence.

6. Show that $\mathbb{Q} \cap [0,1]$ is a connected subset of $\mathbb{R}$.

7. Give an example of each of the following. No justification is required.

   (a) A collection of closed subsets of $\mathbb{R}^2$ whose union is not closed in $\mathbb{R}^2$.

   (b) A sequence in $\mathbb{R}^2$ which does not converge but whose magnitude does converge.

   (c) A subset of $\mathbb{R}$ which is neither open nor closed in $\mathbb{R}$.

   (d) A function $f : \mathbb{R} \to \mathbb{R}$ and an open set $A \subseteq \mathbb{R}$ such that $f^{-1}(A)$ is not open.

8. Let $A \subseteq \mathbb{R}^n$. Prove that A is open in $\mathbb{R}^n$ iff $A \cap \partial A = \emptyset$.