Math 411 Exam 1 Spring 2013 Solution Outlines

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1. Define the set $A \subseteq \mathbb{R}^2$ by

$$A = A_1 \cup A_2 \text{ where } A_1 = \{(0, y) \mid -1 \leq y \leq 1\} \text{ and } A_2 = \{(x, \sin(1/x)) \mid 0 < x \leq 1\}$$

Define $f : A \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 1 & \text{for } (x, y) \in A_1 \\ 2 & \text{for } (x, y) \in A_2 \end{cases}$$

Prove that f is not continuous on A.

Outline of Solution:

The space is connected hence has the IVP. If f were continuous then the image would be an interval but it isn't.

2. Prove that if $S \subseteq \mathbb{R}^n$ has the Intermediate Value Property then it is connected.

Outline of Solution:

This is from the notes and book.

3. Prove using the definition of open that the set $S = \{(x, y) \mid x > 0, y > 0\}$ is open in $\mathbb{R}^2$.

Outline of Solution:

For any $(x_0, y_0) \in S$ let $r = \min(x_0, y_0)$ and show that $B_r(x_0, y_0) \subseteq S$.

4. Suppose $a, b, c, d \in \mathbb{R}$ with $a < b \leq c < d$. Prove using the definition of pathwise-connected that the subset of $\mathbb{R}$ given by $S = [a, b] \cup [c, d]$ is pathwise-connected iff $b = c$.

Outline of Solution:

If $b = c$ then $[a, b] \cup [c, d] = [a, d]$ is an interval and it's easy to show.

If $b < c$ then show that if S were path connected then every point between b and c would need to be in S, a constradiction.

5. Give a specific example to show that it's possible to have nonzero sequence $\{\bar{u}_k\}$ in $\mathbb{R}^2$ and nonzero $\bar{u}, \bar{v} \in \mathbb{R}^2$ with $\{\bar{u}_k\}$ converging to $\bar{u}$ with $\bar{u} \perp \bar{v}$ but $\forall k,\ \bar{u}_k \not\perp \bar{v}$. Make sure to prove your claims on perpendicularity and convergence.

Outline of Solution:

For example $\{\bar{u}_k\} = \left\{\left(1 + \frac{1}{k}, 1\right)\right\}$ and $\bar{v} = (1, 0)$.

6. Show that $\mathbb{Q} \cap [0, 1]$ is a connected subset of $\mathbb{R}$.

Outline of Solution:

This problem had an error in that $\mathbb{Q} \cap [0, 1]$ is not a connected subset of $\mathbb{R}$. It is not connected because for example the open sets $(-\inf, \sqrt{2}/2)$ and $(\sqrt{2}/2, \inf)$ separate it.

7. Give an example of each of the following. No justification is required.

   (a) A collection of closed subsets of $\mathbb{R}^2$ whose union is not closed in $\mathbb{R}^2$.

   Outline of Solution:

   For each n let S_n be the closed ball of radius $1 - \frac{1}{n}$ centered at the origin. Then the union of these is $B_1(0,0)$ which is not closed.

   (b) A sequence in $\mathbb{R}^2$ which does not converge but whose magnitude does converge.

   Outline of Solution:

   $\left\{\left(\left(1-\frac{1}{k}\right)\cos(k), \left(1-\frac{1}{k}\right)\sin(k)\right)\right\}$

   (c) A subset of $\mathbb{R}$ which is neither open nor closed in $\mathbb{R}$.

   Outline of Solution:

   $[1, 2)$

   (d) A function $f : \mathbb{R} \to \mathbb{R}$ and an open set $A \subseteq \mathbb{R}$ such that $f^{-1}(A)$ is not open.

   Outline of Solution:

   You'll need to pick a non-continuous f.

8. Let $A \subseteq \mathbb{R}^n$. Prove that A is open in $\mathbb{R}^n$ iff $A \cap \partial A = \emptyset$.

   Outline of Solution:

   Suppose A is open. By way of contradiction suppose $A \cap \partial A \neq \emptyset$ and thus $\exists\, \bar{x} \in A \cap \partial A$. Since $\bar{x} \in \partial A$ every $B_r(\bar{x})$ intersects A' which contradicts $\bar{x} \in A$ with A open.

   Suppose $A \cap \partial A = \emptyset$. By way of contradiction suppose A is not open and so $\exists\, \bar{x} \in A$ such that no $B_r(\bar{x}) \subset A$, meaning every $B_r(\bar{x})$ intersects A' which means that $\bar{x} \in \partial A$ which contradicts $A \cap \partial A = \emptyset$.