MATH 411 (JWG) Exam 1 Spring 2021

Due by Thu Feb 25 at 12:00pm

Exam Logistics:

1. From the moment you download this exam you have three hours to take the exam and submit to Gradescope. This includes the entire upload and tag procedure so do not wait until the last minute to do these things.
2. Tag your problems! Please! Pretty please!
3. You may print the exam, write on it, scan and upload.
4. Or you may just write on it on a tablet and upload.
5. Or you are welcome to write the answers on separate pieces of paper if other options don't appeal to you, then scan and upload.

Exam Rules:

1. You may ask for clarification on questions but you may not ask for help on questions!
2. You are permitted to use official class resources which means your own written notes, class Panopto recordings and the textbook.
3. You are not permitted to use other resources. Thus no friends, internet, calculators, Wolfram Alpha, etc.
4. By taking this exam you agree that if you are found in violation of these rules that the minimum penalty will be a grade of 0 on this exam.

Exam Work:

1. Show all work as appropriate for and using techniques learned in this course.
2. Any pictures, work and scribbles which are legible and relevant will be considered for partial credit.
3. Arithmetic calculations do not need to be simplified unless specified.

1. Prove from the definition of convergence in $\mathbb{R}^2$ that $\left\{(2+1/k, 4-1/k^2)\right\}$ converges to $(2,4)$. [10 pts]

Solution:

2. Suppose that $\{\bar{u}_k\} \to \bar{u}$ and $\{\bar{v}_k\} \to \bar{v}$.
Note: For what follows, $\perp$ means perpendicular to and $\not\perp$ means not perpendicular to.

   (a) Prove that if $\forall k$, $\bar{u}_k \perp \bar{v}_k$ then $\bar{u} \perp \bar{v}$. [10 pts]
   Solution:

   (b) Give an example which shows that we could have $\bar{u} \perp \bar{v}$ even if $\forall k$, $\bar{u}_k \not\perp \bar{v}_k$. You do not need to prove your example works. Reasonably descriptive pictures suffice. [5 pts]
   Solution:

3. Prove from the definition of an open set that the set $\{(x,y) \in \mathbb{R}^2 \mid x > 0\}$ is open in $\mathbb{R}^2$. [10 pts]

4. Give an example of each of the following.

   (a) A collection of open sets in $\mathbb{R}^2$ whose intersection is not open. [5 pts]

   Solution:

   (b) An path-connected set A and a convex set B with $A \cap B$ not convex. Clear pictures will suffice. [5 pts]

   Solution:

   (c) A collection of path-connected sets such that each set overlaps at least one other set but the union of all the sets is not path-connected. Clear pictures will suffice. [5 pts]

   Solution:

5. Let $S = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Z} \,\text{or}\, y \in \mathbb{Z}\, \text{(or both)}\}$. Prove that S is path-connected. [10 pts]

Solution:

6. Prove that $S = \{(x, y) \,|\, y - x^2 > 1 \text{ and } y < x + 4\}$ is open in $\mathbb{R}^2$. [10 pts]

Solution:

7. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by [10 pts]

$$f(x, y) = \begin{cases} \frac{2x^3 \sin(x)}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Prove that f is continuous.

Solution:

8. Suppose $S \subset \mathbb{R}^3$ is a path-connected set containing the points $(1, 3, -2)$ and $(0, 1, 0)$.

(a) Prove that S must contains a point with second component equal to 2. [5 pts]

(b) Prove that S must contain a point whose distance from the origin is 3. [5 pts]

Solution:

9. Let $A \subseteq \mathbb{R}^n$ with $A \neq \emptyset$ and $\partial A \neq \emptyset$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by [10 pts]

$$f(\bar{u}) = \begin{cases} 1 & \text{if } \bar{u} \in A \\ 0 & \text{if } \bar{u} \notin A \end{cases}$$

Show that f is not continuous at any point in ∂A.

Solution: