MATH 411 (JWG) Exam 1 Spring 2021 Solutions

Due by Thu Feb 25 at 12:00pm

Exam Logistics:

1. From the moment you download this exam you have three hours to take the exam and submit to Gradescope. This includes the entire upload and tag procedure so do not wait until the last minute to do these things.
2. Tag your problems! Please! Pretty please!
3. You may print the exam, write on it, scan and upload.
4. Or you may just write on it on a tablet and upload.
5. Or you are welcome to write the answers on separate pieces of paper if other options don't appeal to you, then scan and upload.

Exam Rules:

1. You may ask for clarification on questions but you may not ask for help on questions!
2. You are permitted to use official class resources which means your own written notes, class Panopto recordings and the textbook.
3. You are not permitted to use other resources. Thus no friends, internet, calculators, Wolfram Alpha, etc.
4. By taking this exam you agree that if you are found in violation of these rules that the minimum penalty will be a grade of 0 on this exam.

Exam Work:

1. Show all work as appropriate for and using techniques learned in this course.
2. Any pictures, work and scribbles which are legible and relevant will be considered for partial credit.
3. Arithmetic calculations do not need to be simplified unless specified.

1. Prove from the definition of convergence in $\mathbb{R}^2$ that $\left\{(2+1/k, 4-1/k^2)\right\}$ converges to $(2,4)$. [10 pts]

Solution:

Observe that:

$$\begin{aligned}
||(2+1/k, 4-1/k^2) - (2,4)|| &= ||(1/k, -1/k^2)|| \\
&= \sqrt{(1/k)^2 + (-1/k^2)^2} \\
&= \sqrt{1/k^2 + 1/k^4} \\
&= \frac{1}{k}\sqrt{1+1/k^2} \\
&< \frac{1}{k}\sqrt{2} \\
&< \frac{\sqrt{2}}{k}
\end{aligned}$$

For any ϵ then we choose K so that $\frac{\sqrt{2}}{K} < \epsilon$ then if $k \geq K$ then $||(2+1/k, 4-1/k^2)-(2,4)|| < \frac{\sqrt{2}}{k} < \frac{\sqrt{2}}{K} < \epsilon$.

2. Suppose that $\{\bar{u}_k\} \to \bar{u}$ and $\{\bar{v}_k\} \to \bar{v}$.
Note: For what follows, $\perp$ means perpendicular to and $\not\perp$ means not perpendicular to.

(a) Prove that if $\forall k$, $\bar{u}_k \perp \bar{v}_k$ then $\bar{u} \perp \bar{v}$. [10 pts]

Solution:

By the componentwise convergence criterion we have:

$$\{p_i(u_k)\} \to p_i(\bar{u}) \text{ and } \{p_i(v_k)\} \to p_i(\bar{v}) \text{ for all } 1 \leq i \leq n$$

In addition we know that for all k:

$$0 = \langle u_k, v_k \rangle = p_1(u_k)p_1(v_k) + ... + p_n(u_k)p_n(v_k)$$

Since convergence is preserved over projections, addition and multiplication we then have:

$$\{\langle u_k, v_k \rangle\} = \{p_1(u_k)p_1(v_k) + ... + p_n(u_k)p_n(v_k)\} \to p_1(\bar{u})p_1(\bar{v}) + ... + p_n(\bar{u})p_n(\bar{v}) = \langle \bar{u}, \bar{v} \rangle$$

Thus since $\{\langle u_k, v_k \rangle\} = \{0\} \to 0$ we know that $\langle \bar{u}, \bar{v} \rangle = 0$.

(b) Give an example which shows that we could have $\bar{u} \perp \bar{v}$ even if $\forall k$, $\bar{u}_k \not\perp \bar{v}_k$. You do not need to prove your example works. Reasonably descriptive pictures suffice. [5 pts]

Solution:

For example in $\mathbb{R}^2$ consider $\bar{u}_k = (1, 0)$ and $\bar{v}_k = (0, 1 + 1/k)$.

3. Prove from the definition of an open set that the set $\{(x, y) \in \mathbb{R}^2 \mid x > 0\}$ is open in $\mathbb{R}^2$. [10 pts]

Solution:

For any $(x, y) \in A$ let $r = x$ and then observe that $B_r(x, y) \subseteq A$ so that (x, y) is an interior point and so A is open.

4. Give an example of each of the following.

(a) A collection of open sets in $\mathbb{R}^2$ whose intersection is not open. [5 pts]

Solution:

For example for each $k = 1, 2, 3, \ldots$ use $B_{1/k}((0,0))$ becaues then the intersection is just $(0,0)$ which is not open.

(b) An path-connected set A and a convex set B with $A \cap B$ not convex. Clear pictures will suffice. [5 pts]

Solution:

Let B be the letter U and let A be a big open ball containing U.

(c) A collection of path-connected sets such that each set overlaps at least one other set but the union of all the sets is not path-connected. Clear pictures will suffice. [5 pts]

Solution:

Let A, B, C, D be four disks such that A and B overlap and C and D overlap.

5. Let $S = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Z} \,\text{or}\, y \in \mathbb{Z} \,(\text{or both})\}$. Prove that S is path-connected. [10 pts]

Solution:

Let $(x, y) \in S$.

If $x \in \mathbb{Z}$ then the path $\phi : [0, 2] \to S$ given by:

$$\phi(t) = \begin{cases} (x, (1-t)y) & \text{for } t \in [0, 1] \\ ((2-t)x, 0) & \text{for } t \in [1, 2] \end{cases}$$

Then ϕ is a path in S from (x, y) to $(x, 0)$ to $(0, 0)$.

For any two points we can simply find a path from one of them to the origin and then reverse the path from the origin to the other one.

6. Prove that $S = \{(x, y) \mid y - x^2 > 1 \text{ and } y < x + 4\}$ is open in $\mathbb{R}^2$. [10 pts]

Solution:

Define $f_1(x, y) = y - x^2$ and $f_2(x, y) = y - x$. Both are continuous.

Then observe that $S = S_1 \cap S_2$ where:

$$S_1 = \{(x, y) \mid f_1(x, y) > 1\}$$
$$S_2 = \{(x, y) \mid f_2(x, y) < 4\}$$

Both S_1 and S_2 are open since they are the inverse image of an open interval under a continuous map (corollary from the notes) and the intersection of two open sets is open (from class).

7. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by [10 pts]

$$f(x,y) = \begin{cases} \frac{2x^3\sin(x)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Prove that f is continuous.

Solution:

There are two cases:

- For $(x,y) \neq (0,0)$ suppose $\{(x_k,y_k)\} \to (0,0)$ with $(x_k,y_k) \neq (x,y)$. By choosing k large enough we can keep (x_k,y_k) close to (x,y) and away from $(0,0)$ so for those k we have:

  $$f(x_k,y_k) = \frac{2x_k^3\sin(x_k)}{x_k^2+y_k^2}$$

  Then since f is constructed in a continuous manner from continuous functions it is continuous and hence $\{f(x_k,y_k)\} \to f(x,y)$.

  Note: The issue of keeping (x_k,y_k) away from $(0,0)$ is subtle but formally necessary since the continuous construction is only valid away from $(0,0)$ but the sequence may hit $(0,0)$ early on. I did not deduct points if you missed this nuance though.

- For $(x,y) = (0,0)$ suppose $\{x_k,y_k)\} \to (x,y)$ with $(x_k,y_k) \neq (0,0)$. Then for any k:
  - If $x_k = 0$ then $y_k \neq 0$ and so:

    $$\left|\frac{2x_k^3\sin(x_k)}{x_k^2+y_k^2}\right| = 0 \leq 2|x_k|$$

  - If $x_k \neq 0$ then:

    $$\left|\frac{2x_k^3\sin(x_k)}{x_k^2+y_k^2}\right| \leq \left|\frac{2x_k^3}{x_k^2}\right| = 2|x_k|$$

  It follows from the Comparison Lemma since $\{x_k\} \to 0$ that:

  $$\{f(x_k,y_k)\} = \left\{\frac{2x_k^3\sin(x_k)}{x_k^2+y_k^2}\right\} \to 0 = f(0,0)$$

8. Suppose $S \subset \mathbb{R}^3$ is a path-connected set containing the points $(1, 3, -2)$ and $(0, 1, 0)$.

(a) Prove that S must contains a point with second component equal to 2. [5 pts]

Solution:

Define $f : S \to \mathbb{R}$ by $f(x, y, z) = y$. Since f is continuous and S is path-connected and hence has the IVP and since $f(1, 3, -2) = 3$ and $f(0, 1, 0) = 1$ and $1 \leq 2 \leq 3$ there is some point $\bar{x} \in S$ with $f(\bar{x}) = 2$.

(b) Prove that S must contain a point whose distance from the origin is 3. [5 pts]

Solution:

Define $f : S \to \mathbb{R}$ by $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Since f is continuous and S is path-connected and hence has the IVP and since $f(1, 3, -2) = \sqrt{14}$ and $f(0, 1, 0) = 1$ and $1 \leq 3 \leq \sqrt{14}$ there is some point $\bar{x} \in S$ with $f(\bar{x}) = 3$.

9. Let $A \subseteq \mathbb{R}^n$ with $A \neq \emptyset$ and $\partial A \neq \emptyset$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by [10 pts]

$$f(\bar{u}) = \begin{cases} 1 & \text{if } \bar{u} \in A \\ 0 & \text{if } \bar{u} \notin A \end{cases}$$

Show that f is not continuous at any point in ∂A.

Solution:

Suppose f is continuous at some $\bar{x} \in \partial A$.

By definition of the boundary, for all $k = 1, 2, 3, ...$ there is a point $\bar{u}_k \in B_{1/k}(\bar{x}) \cap A$ and a point $\bar{v}_k \in B_{1/k}(\bar{x}) \cap A^C$.

We have $\{\bar{u}_k\} \to \bar{u}$ and hence by continuity $\{f(\bar{u}_k)\} \to f(\bar{x})$ and $\forall k, f(\bar{u}_k) = 1$ so $f(\bar{x}) = 1$.

We have $\{\bar{v}_k\} \to \bar{u}$ and hence by continuity $\{f(\bar{v}_k)\} \to f(\bar{x})$ and $\forall k, f(\bar{v}_k) = 0$ so $f(\bar{x}) = 0$.

This is a contradiction.