Math 411 Exam 2 Spring 2013

1. Find the first-order approximation for $\bar{F}(x, y) = (x^2y, xy + y)$ at $(-1, 2)$ and use it to approximate the value of $\bar{F}(-0.9, 2.1)$.

2. (a) Given a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, what is logically incorrect about finding the matrix $[T(\bar{e}_1)\ \ldots\ T(\bar{e}_n)]$ and then using this matrix to show that the transformation is linear?

   (b) Let A be the set of all linear transformations and B be the set of all invertible transformations. Give two transformations, one which proves that $A \not\subseteq B$ and one which proves that $B \not\subseteq A$.

3. Let $f(x, y) = xy + y^2$ and $\bar{p} = (-1, 2)$. Find $\frac{\partial f}{\partial \bar{p}}(1, 1)$ using the limit definition of the directional derivative and also using the inner product calculation.

4. Suppose $\bar{F}(x, y) = (x^2y, y - 3x^2)$ and $\bar{G}(x, y) = (xy + y, y - xy)$. Use the matrix form of the chain rule to evaluate $D(\bar{F} \circ \bar{G})(x, y)$.

5. Define $f(x, y) = 2x^2 - 2xy - y^2$. Find the only critical point (x_0, y_0) and show that the Hessian at (x_0, y_0) is neither positive definite nor negative definite. Moreover show that there is at least one direction $\bar{h}_1$ in which $\left\langle \nabla^2 f(x_0, y_0)\bar{h}_1, \bar{h}_1 \right\rangle > 0$ and another direction $\bar{h}_2$ in which $\left\langle \nabla^2 f(x_0, y_0)\bar{h}_2, \bar{h}_2 \right\rangle < 0$.

6. Define

$$f(x, y) = \begin{cases} \frac{x\sqrt{x^2+y^2}}{|y|} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

   Show that f has directional derivatives in all directions at $(0, 0)$.

7. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable with $f(0, 0) = 1$ and $f(x, y) = 1$ for all $||(x, y)|| = 1$. Show that there is some point (x_0, y_0) such that $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)$.
   Hint: Use the MVT for an appropriate $\bar{h}$.