Math 411 Exam 2 Spring 2013 Solution Outline

1. Find the first-order approximation for $\bar{F}(x,y) = (x^2y, xy + y)$ at $(-1,2)$ and use it to approximate the value of $\bar{F}(-0.9, 2.1)$.

   Solution Outline:

   Just use the formula $F(x,y) = F(x_0,y_0) + DF(x_0,y_0)(x - x_0, y - y_0)$.

2. (a) Given a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, what is logically incorrect about finding the matrix $[T(\bar{e}_1) \ ... \ T(\bar{e}_n)]$ and then using this matrix to show that the transformation is linear?

   Solution Outline:

   There's no guarantee that the matrix you get actually corresponds to the original transformation. The original transformation must be known to be linear in order to construct the matrix for it.

   (b) Let A be the set of all linear transformations and B be the set of all invertible transformations. Give two transformations, one which proves that $A \not\subseteq B$ and one which proves that $B \not\subseteq A$.

   Solution Outline:

   For the first, something like $T(x,y) = (x,0)$. For the second, something like $T(x,y) = (x^3, y)$.

3. Let $f(x,y) = xy + y^2$ and $\bar{p} = (-1,2)$. Find $\frac{\partial f}{\partial \bar{p}}(1,1)$ using the limit definition of the directional derivative and also using the inner product calculation.

   Solution Outline:

   For the first calculate explicitly:

   $$\lim_{t \to 0} \frac{f((1,1) + t(-1,2)) - f((1,1))}{t}$$

   For the second calculate explicitly:

   $$\langle \nabla f((1,1)), (-1,2) \rangle$$

4. Suppose $\bar{F}(x,y) = (x^2y, y - 3x^2)$ and $\bar{G}(x,y) = (xy + y, y - xy)$. Use the matrix form of the chain rule to evaluate $D(\bar{F} \circ \bar{G})(x,y)$.

   Solution Outline:

   This is basically just brute force calculation:

   $$D(\bar{F} \circ \bar{G})(x,y) = D\bar{F}(\bar{G}(x,y)) D\bar{G}(x,y)$$

   The one thing to be careful of is making sure that when you do $D\bar{F}$ the result will have x, y in it but you need to make sure you also plug $\bar{G}(x,y)$ into it appropriately.

5. Define $f(x,y) = 2x^2 - 2xy + y^2$. Find the only critical point (x_0,y_0) and show that the Hessian at (x_0,y_0) is neither positive definite nor negative definite. Moreover show that there is at least one direction $\bar{h}_1$ in which $\left\langle \nabla^2 f(x_0,y_0)\bar{h}_1, \bar{h}_1 \right\rangle > 0$ and another direction $\bar{h}_2$ in which $\left\langle \nabla^2 f(x_0,y_0)\bar{h}_2, \bar{h}_2 \right\rangle < 0$.

   Solution Outline:

   Finding the critical point is basically precalculus. To show the Hessian is neither positive nor negative definite calculate the Hessian $\nabla^2 f(x_0,y_0)$ and then find some (x,y) with $\left\langle \nabla^2 f(x_0,y_0)(x,y), (x,y) \right\rangle < 0$ (which ensures it's not positive definite) and then find some (x,y) with $\left\langle \nabla^2 f(x_0,y_0)(x,y), (x,y) \right\rangle > 0$ (which ensures it's not negative definite).

6. Define

$$f(x,y) = \begin{cases} \frac{x\sqrt{x^2+y^2}}{|y|} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

Show that f has directional derivatives in all directions at $(0,0)$.

Solution Outline:

Just work out the calculation for an arbitrary $\bar{p}$. This may sound overwhelming but the calculation is simple. Let $\bar{p} = (p_1, p_2)$ and calculate:

$$\lim_{t \to 0} \frac{f((0,0) + t(p_1, p_1)) - f(0,0)}{t}$$

The numerator is just $f(tp_1, tp_2)$ so you'll need one case where $p_2 \neq 0$ and one case where $p_2 = 0$. The first case is basic algebra and the second case is trivial.

7. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable with $f(0,0) = 1$ and $f(x,y) = 1$ for all $||(x,y)|| = 1$. Show that there is some point (x_0, y_0) such that $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)$.
Hint: Use the MVT for an appropriate $\bar{h}$.

Solution Outline:

Note that the conditions state that $f(x,y) = 1$ at the origin and on the circle of radius 1 centered at the origin. This means somewhere along the vector $\bar{h} = (1, -1)$ anchored at the origin there must be a place where the directional derivative is zero. Use the MVT with $\bar{h}$.