MATH 411 (JWG) Exam 2 Spring 2021 Solution Outlines

1. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by:

$$f(x,y) = \begin{cases} \frac{2x^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 2 & \text{if } (x,y) = (0,0) \end{cases}$$

   (a) Prove using the limit definition that f only has one partial derivative at $(0,0)$. [10 pts]

   Solution:

   Direct calculation show that $\frac{\partial f}{\partial x}(0,0) = 0$ but $\frac{\partial f}{\partial y}(0,0)$ DNE.

   (b) Which value would need to replace 2 so that the other partial derivative exists instead? Explain. [5 pts]

   Solution:

   If we replace 2 by 0 then $\frac{\partial f}{\partial x}(0,0)$ DNE but $\frac{\partial f}{\partial y}(0,0) = 0$.

2. If $f(x,y)$ gives the temperature in celsius of the plane at the point (x,y) in meters and if $\bar{p} = (1,1)$ explain in your own words the meaning of the following expression. [10 pts]

$$\langle \nabla \langle \nabla f(x,y), \bar{e}_i \rangle, \bar{p} \rangle$$

Solution:

The inside inner product gives the instantaneous change in temperature in the i-direction, for example if $i = 1$ this is the instantaneous change in the x-direction and if $i = 2$ this is the instantaneous change in the y-direction. The outer inner product finds the instantaneous change in that quantity at a point (x,y) as we move in the $\bar{p}$ direction.

3. Let $f(x,y) = xy + x$ and $\bar{p} = (-1,2)$.

   (a) Show that the limit definition of $\frac{\partial f}{\partial \bar{p}}(\bar{x})$ and the convenient $\langle \nabla f(\bar{x}), \bar{p} \rangle$ yield the same result for any $\bar{x}$. [10 pts]

   Solution:

   This is just direct calculation.

   (b) Given $\bar{x} = (1,-3)$, find the specific value of θ satisfying the conditions of the Mean Value Theorem. [10 pts]

   Solution:

   This is just direct calculation yielding $\theta = \frac{1}{2}$.

4. Given the function $\bar{F} : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\bar{F}(x, y, z) = (x^2y + yz^2, xyz)$.

   (a) Find the first-order approximation (in the $\bar{F}(x, y, z) \approx$ ... form) of $\bar{F}$ at the point $(x_0, y_0, z_0) = (1, 2, 3)$. You can leave this as a matrix expression. [10 pts]

   Solution:

   Calculation yields $f(x, y, z) \approx \begin{bmatrix} 20 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 10 & 12 \\ 6 & 3 & 2 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 2 \\ z - 3 \end{bmatrix}$.

   (b) Use your answer to (a) to approximate $\bar{F}(1.1, 1.9, 3.01)$. This should be simplified. [5 pts]

   Solution:

   Plug into the above.

5. Given the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^3y - xy^3 + x^2y^2$.

   (a) Find the second-order approximation (in the $f(x, y) \approx$ ... form) of f at the point $(x_0, y_0) = (2, 3)$. You can leave this as a matrix expression. [10 pts]

   Solution:

   Calculation yields $f(x, y) \approx 6 + \begin{bmatrix} 45 & -22 \end{bmatrix} \begin{bmatrix} x - 2 \\ x - 3 \end{bmatrix} + \frac{1}{2} \left\langle \begin{bmatrix} 54 & 9 \\ 9 & -28 \end{bmatrix} \begin{bmatrix} x - 2 \\ x - 3 \end{bmatrix}, \begin{bmatrix} x - 2 \\ x - 3 \end{bmatrix} \right\rangle$.

   (b) Is the Hessian matrix at $(2, 3)$ positive definite, negative definite, or neither? Justify. [10 pts]

   Solution:

   Neither because $b^2 - ac < 0$.

6. Supppose $\phi : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable and define $f(x, y) = \phi(x+y, x-y)$. Express $\nabla f(x, y)$ as a vector involving combinations of $\frac{\partial \phi}{\partial x_1}(x + y, x - y)$ and $\frac{\partial \phi}{\partial x_2}(x + y, x - y)$. [10 pts]

   Solution:

   Define $h(x, y) = (x + y, x - y)$ and then $f(x, y) = (\phi \circ h)(x, y)$ and apply the chain rule.