Math 411 Exam 3 Fall 2013 Solutions Outline

1. Define $\bar{F}(x, y) = (ax^2 + y, x^2 + ay)$. Find all $a$ so that the Inverse Function Theorem does not apply at $(1, -1)$.

   **Solution:**
   Calculate the derivative matrix and see where the determinant is zero.

2. Give an explicit (not a picture) example of a function $\bar{F}: \mathbb{R}^2 \to \mathbb{R}^2$ and a point $(x_0, y_0)$ such that the derivative hypothesis for the Inverse Function Theorem does not apply at $(x_0, y_0)$ (show it doesn’t) but that $\bar{F}$ is locally invertible at $(x_0, y_0)$ (show it is). If you can’t think of such a function you can earn half credit by just doing $f: \mathbb{R} \to \mathbb{R}$ at some $x_0$ instead.

   **Solution:**
   Something like $\bar{F}(x, y) = (x^3, y^3)$ at the point $(0, 0)$ works.

3. Suppose a shock absorption system has two inputs $x$ and $y$ which control two output values given by $(2xy + x + xy^2)$. Normally the system is set at $(x, y) = (3, 4)$ yielding output $(27, 19)$.

   (a) Show that there is a neighborhood of $(27, 19)$ such that if the output is forced to change within that neighborhood then $(x, y)$ can change to compensate.

   **Solution:**
   Show that the Inverse Function Theorem applies.

   (b) Linearly approximate which $(x, y)$ would yield an output of $(26, 19.5)$. You do not need to simplify matrix/vector calculations and you may leave the inverse of a matrix uncalculated.

   **Solution:**
   Use the result of the Inverse Function Theorem to get the derivative matrix of the inverse and use it to construct a linear approximation of the inverse. Then use that.

4. Define $S \subseteq \mathbb{R}^2$ by $S = \{(x, y) \mid x - y^2 + 6y = 0\}$.

   (a) Determine the single point where the derivative hypothesis of the Implicit Function Theorem does not show that $S$ is locally a function of $y$.

   **Solution:**
   This was on the Spring 2013 exam, too - check there.

   (b) At that point prove that $S$ is not locally a function of $y$. A well-explained picture is sufficient but the explanation is mandatory.

   **Solution:**
   This was on the Spring 2013 exam, too - check there.

5. Give an example (a clearly drawn picture!) of a graph in the plane which is neither a function of $x$ nor a function of $y$ but is locally both a function of $x$ and a function of $y$ at every point.

   **Solution:**
   Not a picture but sketch both $y = 1 - x$ and $y = 2 - x$ in the first quadrant. Together these are a graph which satisfies the requirement.

6. Suppose $\bar{F}: \mathbb{R}^{2+2} \to \mathbb{R}^2$ is such that the hypotheses of the Implicit Function Theorem are satisfied at $(x_0, y_0, z_0, w_0) = (2, 1, -1, 0)$ and moreover suppose

   $D\bar{F}(2, 1, -1, 0) = \begin{bmatrix} 1 & 6 & 5 & -2 \\ 0 & 1 & 1 & \beta \end{bmatrix}$

   (a) Find the value of $\beta$ for which the Implicit Function Theorem does not allow you to write $y, w$ in terms of $x, z$.

   **Solution:**
   Set the determinant of the relevant submatrix (last two columns) equal to zero and solve.
(b) Let $\beta = 1$ and observe (no need to prove) that $y, z$ can be rewritten by the Implicit Function Theorem as $(y, z) = \bar{G}(x, w)$. Find a linear approximation to $\bar{G}$ at $(2, 0)$. 
Simplify.

**Solution:**
Use the Implicit Function Theorem to find the derivative matrix of the implicit $\bar{G}$ and use this derivative matrix to construct a linear approximation.