Math 411 Exam 3 Fall 2013 Solutions Outline

1. Define $\bar{F}(x,y) = (ax^2 + y, x^2 + ay)$. Find all a so that the Inverse Function Theorem does not apply at $(1,-1)$. [10 pts]

   Solution:

   Calculate the derivative matrix and see where the determinant is zero.

2. Give an explicit (not a picture) example of a function $\bar{F} : \mathbb{R}^2 \to \mathbb{R}^2$ and a point (x_0, y_0) such that the derivative hypothesis for the Inverse Function Theorem does not apply at (x_0, y_0) (show it doesn't) but that $\bar{F}$ is locally invertible at (x_0, y_0) (show it is). If you can't think of such a function you can earn half credit by just doing $f : \mathbb{R} \to \mathbb{R}$ at some x_0 instead. [10 pts]

   Solution:

   Something like $\bar{F}(x,y) = (x^3, y^3)$ at the point $(0,0)$ works.

3. Suppose a shock absorption system has two inputs x and y which control two output values given by $(2xy+x, x+y^2)$. Normally the system is set at $(x,y) = (3,4)$ yielding output $(27,19)$.

   (a) Show that there is a neighborhood of $(27,19)$ such that if the output is forced to change within that neighborhood that (x,y) can change to compensate. [10 pts]

   Solution:

   Show that the Inverse Function Theorem applies.

   (b) Linearly approximate which (x,y) would yield an output of $(26, 19.5)$. You do not need to simplify matrix/vector calculations and you may leave the inverse of a matrix uncalculated. [15 pts]

   Solution:

   Use the result of the Inverse Function Theorem to get the derivative matrix of the inverse and use it to construct a linear approximation of the inverse. Then use that.

4. Define $S \subseteq \mathbb{R}^2$ by $S = \{(x,y) \mid x - y^2 + 6y = 0\}$.

   (a) Determine the single point where the derivative hypothesis of the Implicit Function Theorem does not show that S is locally a function of y. [10 pts]

   Solution:

   This was on the Spring 2013 exam, too - check there.

   (b) At that point prove that S is not locally a function of y. A well-explained picture is sufficient but the explanation is mandatory. [15 pts]

   Solution:

   This was on the Spring 2013 exam, too - check there.

5. Give an example (a clearly drawn picture!) of a graph in the plane which is neither a function of x nor a function of y but is locally both a function of x and a function of y at every point. [10 pts]

   Solution:

   Not a picture but sketch both $y = 1 - x$ and $y = 2 - x$ in the first quadrant. Together these are a graph which satisfies the requirement.

6. Suppose $\bar{F} : \mathbb{R}^{2+2} \to \mathbb{R}^2$ is such that the hypotheses of the Implicit Function Theorem are satisfied at $(x_0, y_0, z_0, w_0) = (2, 1, -1, 0)$ and moreover suppose

$$D\bar{F}(2,1,-1,0) = \begin{bmatrix} 1 & 6 & 5 & -2 \\ 0 & 1 & 1 & \beta \end{bmatrix}$$

   (a) Find the value of β for which the Implicit Function Theorem does not allow you to write y, w in terms of x, z. [5 pts]

   Solution:

   Set the determinant of the relevant submatrix (last two columns) equal to zero and solve.

(b) Let $\beta = 1$ and observe (no need to prove) that y, z can be rewritten by the Implicit Function Theorem as $(y, z) = \bar{G}(x, w)$. Find a linear approximation to $\bar{G}$ at $(2, 0)$. Simplify. [15 pts]

Solution:

Use the Implicit Function Theorem to find the derivative matrix of the implicit $\bar{G}$ and use this derivative matrix to construct a linear approximation.