Math 411 Exam 3 Spring 2013 Solution Outlines

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1. Prove that the subset of $\mathbb{R}$ defined by $\{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ has Jordan content zero.

   Solution Outline:

   For a given $\epsilon > 0$ use the generalized rectangle $[0, \epsilon/4]$ as well as generalized rectangles each of size $[\epsilon/(4N)]$ covering each of the finite N remaining uncovered points.

2. Define $R = [0,1] \times [0,1]$ and $f : R \to \mathbb{R}$ as follows.

   If $x \in \mathbb{Q}$ then put $f(x,y) = \begin{cases} 1 & \text{if } 0 \le y \le \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < y \le 1 \end{cases}$ and if $x \notin \mathbb{Q}$ then put $f(x,y) = \begin{cases} 0 & \text{if } 0 \le y \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < y \le 1 \end{cases}$.

   Find $\int_0^1 \int_0^1 f(x,y)\, dy\, dx$.

   Solution Outline:

   In this case we can just do the iterated integral. Keep in mind that you'll need to consider the inner integral separately for different x.

3. Define $S \subseteq \mathbb{R}^2$ by $S = \left\{(x,y) \mid x - y^2 + 6y = 0\right\}$.

   (a) Find all points (x_0, y_0) where Dini's Theorem shows that S is locally a function of x and find the slope of that local function at x_0.

   Solution Outline:

   Defining $f(x,y) = x - y^2 + 6y$, here we look at where $\frac{\partial f}{\partial y}(x,y) = 0$. This is where $-2y + 6 = 0$ or $y = 3$, This yields the point $(-9, 3)$.

   (b) Show that at every point not satisfying the hypotheses of Dini's Theorem S is not locally a function of x.

   Solution Outline:

   Observe that for every $\epsilon > 0$ there are two solutions to $f(-9+\epsilon, 3) = 0$ and hence there is no neighborhood of $(-9,3)$ on which S is locally a function of x.

4. Suppose the temperature and humidity (t,h) of a storage container is set by two controls (x,y) such that $t = x^2y + 5y$ and $h = xy + y^3$. You set the controls at $(x,y) = (2,3)$ to achieve the desired $(t,h) = (27,33)$. Show that for any (t,h) sufficiently close to $(27,33)$ you can alter (x,y) slightly in order to achieve this new (t,h). Moreover, approximately (linearly) which (x,y) would you set in order to achieve $(t,h) = (28,34)$?

   Solution Outline:

   First apply the Inverse Function Theorem to show that the function $\bar{F}(x,y) = (x^2y + 5y, xy + y^3)$ is locally invertible at $(x,y) = (2,3)$. The Inverse Function Theorem then yields the derivative of the inverse which we then use to construct a linear approximation of the inverse and then plug $(28,34)$ into this approximation.

5. Define $D = ([0,1] \cap \mathbb{Q}) \times ([0,1] \cap \mathbb{Q}) \times ([0,1] \cap \mathbb{Q})$. Show that the volume of D is undefined.

   Solution Outline:

   If we define $f : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}$ by $f(x,y,z) = 1$ for $(x,y,z) \in D$ and 0 otherwise. Then due to the density of D and of the complement, the upper sums are always 1 but the lower sums are always 0 so the integral and hence the volume are undefined.

6. Define the change of variables Ψ by $\Psi(u,v) = \left(\frac{u}{v+1}, \frac{uv}{v+1}\right)$ for $u, v > 0$.

   Define $D = \left\{(x,y) \mid 1 \le x + y \le 4 \text{ and } \frac{1}{2} \le \frac{y}{x} \le 3\right\}$. Use Ψ to rewrite $\int_D \left(y + \frac{y^2}{x}\right)$ as an iterated integral in u and v. Justify.

   Solution Outline:

   Show that Ψ is a valid change of variables and apply.

7. Define $D = [0, 1] \times [0, 1]$ and $f : D \to \mathbb{R}$ by $f(x, y) = x$. Using lower and upper sums show that f is integrable over D and find the integral.

   Solution Outline:

   Show that the regular partitions form an Archimedean sequence of partitions and then use either the lower or upper sum to determine the integral.