1. Prove that the subset of \( \mathbb{R} \) defined by \( \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \) has Jordan content zero.

**Solution Outline:**
For a given \( \epsilon > 0 \) use the generalized rectangle \([0, \epsilon/4]\) as well as generalized rectangles each of size \([\epsilon/(4N)]\) covering each of the finite \( N \) remaining uncovered points.

2. Define \( R = [0,1] \times [0,1] \) and \( f : R \to \mathbb{R} \) as follows.
If \( x \in \mathbb{Q} \) then put \( f(x, y) = \begin{cases} 1 & \text{if } 0 \leq y \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < y \leq 1 \end{cases} \) and if \( x \not\in \mathbb{Q} \) then put \( f(x, y) = \begin{cases} 0 & \text{if } 0 \leq y \leq \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < y \leq 1 \end{cases} \).

Find \( \int_0^1 \int_0^1 f(x, y) \, dy \, dx \).

**Solution Outline:**
In this case we can just do the iterated integral. Keep in mind that you’ll need to consider the inner integral separately for different \( x \).

3. Define \( S \subseteq \mathbb{R}^2 \) by \( S = \{(x, y) \mid x - y^2 + 6y = 0\} \).
(a) Find all points \((x_0, y_0)\) where Dini’s Theorem shows that \( S \) is locally a function of \( x \) and find the slope of that local function at \( x_0 \).

**Solution Outline:**
Defining \( f(x, y) = x - y^2 + 6y \), here we look at where \( \frac{\partial f}{\partial y}(x, y) = 0 \). This is where \(-2y + 6 = 0 \) or \( y = 3 \). This yields the point \((-9, 3)\).

(b) Show that at every point not satisfying the hypotheses of Dini’s Theorem \( S \) is not locally a function of \( x \).

**Solution Outline:**
Observe that for every \( \epsilon > 0 \) there are two solutions to \( f(-9 + \epsilon, 3) = 0 \) and hence there is no neighborhood of \((-9, 3)\) on which \( S \) is locally a function of \( x \).

4. Suppose the temperature and humidity \((t, h)\) of a storage container is set by two controls \((x, y)\) such that \( t = x^2y + 5y \) and \( h = xy + y^3 \). You set the controls at \((x, y) = (2, 3)\) to achieve the desired \((t, h) = (27, 33)\). Show that for any \((t, h)\) sufficiently close to \((27, 33)\) you can alter \((x, y)\) slightly in order to achieve this new \((t, h)\). Moreover, approximately (linearly) which \((x, y)\) would you set in order to achieve \((t, h) = (28, 34)\)?

**Solution Outline:**
First apply the Inverse Function Theorem to show that the function \( F(x, y) = (x^2y + 5y, xy + y^3) \) is locally invertible at \((x, y) = (2, 3)\). The Inverse Function Theorem then yields the derivative of the inverse which we then use to construct a linear approximation of the inverse and then plug \((28, 34)\) into this approximation.

5. Define \( D = ([0, 1] \cap \mathbb{Q}) \times ([0, 1] \cap \mathbb{Q}) \times ([0, 1] \cap \mathbb{Q}) \). Show that the volume of \( D \) is undefined.

**Solution Outline:**
If we define \( f : [0, 1] \times [0, 1] \times [0, 1] \to \mathbb{R} \) by \( f(x, y, z) = 1 \) for \((x, y, z) \in D\) and 0 otherwise. Then due to the density of \( D \) and of the complement, the upper sums are always 1 but the lower sums are always 0 so the integral and hence the volume are undefined.

6. Define the change of variables \( \Psi \) by \( \Psi(u, v) = \left(\frac{u}{u+v}, \frac{v}{u+v}\right) \) for \( u, v > 0 \).
Define \( D = \{(x, y) \mid 1 \leq x + y \leq 4 \text{ and } \frac{1}{2} \leq \frac{y}{x} \leq 3\} \). Use \( \Psi \) to rewrite \( \int_D \left(y + \frac{y^2}{x}\right) \) as an iterated integral in \( u \) and \( v \). Justify.

**Solution Outline:**
Show that \( \Psi \) is a valid change of variables and apply.
7. Define $D = [0, 1] \times [0, 1]$ and $f : D \rightarrow \mathbb{R}$ by $f(x, y) = x$. Using lower and upper sums show that $f$ is integrable over $D$ and find the integral.

**Solution Outline:**

Show that the regular partitions form an Archimedean sequence of partitions and then use either the lower or upper sum to determine the integral.