MATH 411 (JWG) Exam 3 Spring 2021

Exam Logistics:

1. From the moment you download this exam you have three hours to take the exam and submit to Gradescope. This includes the entire upload and tag procedure so do not wait until the last minute to do these things.
2. Tag your problems! Please! Pretty please!
3. You may print the exam, write on it, scan and upload.
4. Or you may just write on it on a tablet and upload.
5. Or you are welcome to write the answers on separate pieces of paper if other options don't appeal to you, then scan and upload.

Exam Rules:

1. You may ask for clarification on questions but you may not ask for help on questions!
2. You are permitted to use official class resources which means your own written notes, class Panopto recordings and the textbook.
3. You are not permitted to use other resources. Thus no friends, internet, calculators, Wolfram Alpha, etc.
4. By taking this exam you agree that if you are found in violation of these rules that the minimum penalty will be a grade of 0 on this exam.

Exam Work:

1. Show all work as appropriate for and using techniques learned in this course.
2. Any pictures, work and scribbles which are legible and relevant will be considered for partial credit.
3. Arithmetic calculations do not need to be simplified unless specified.

1. In the proof of Fubini's Theorem we proved the $U(A, P^X) \leq U(f, P)$ part for any partition $P = (P^X, P^Y)$ of I. State and prove the $L(A, P^X) \geq L(f, P)$ part. [10pts]

2. Define $\bar{F} : \mathbb{R}^{2+2} \to \mathbb{R}^2$ by:

$$\bar{F}(x, y, z, w) = (x^3 w - 2y - z, xy^2 z - 2w)$$

   (a) Prove that the Implicit Function Theorem applies at the point $(1, 2, 4, 8)$. [10pts]

   (b) Use the Implicit Function Theorem to construct a linear approximation to the associated $\bar{G} : \mathbb{R}^2 \to \mathbb{R}^2$ and use it to approximate the values z, w such that $(1.01, 1.95, z, w)$ satisfies $\bar{F}(1.01, 1.95, z, w) = (0, 0)$. [10pts]

3. Show from the definition of Jordan Content that the graph of $f(x) = x^2$ on the interval $[0, 1]$ has Jordan Content zero. [15pts]

4. Define $\bar{F} : \{(x, y) \,|\, y > 0\} \to \mathbb{R}^2$ by $\bar{F}(x, y) = \left(\frac{x}{y}, x - y\right)$.

   (a) Categorize the points (x, y) in the domain for which the Inverse Function Theorem applies and for which the Inverse Function Theorem does not apply. [10pts]

   (b) Show that the function is really not locally invertible at the points where the Inverse Function Theorem does not apply. [10pts]

5. Define $f(x, y) = y$ on $I = [0, 2] \times [0, 1]$. Prove that the following sequence of partitions $\{P_k\}$ is an Archimedean sequence of partitions and use this sequence to prove that f is integrable on I and to calculate $\int_I f$: [15pts]

$$P_k = \left[\frac{2(i-1)}{k}, \frac{2i}{k}\right] \times \left[\frac{j-1}{k}, \frac{j}{k}\right] \quad \text{for} \quad i, j = 1, 2, ..., k$$

6. The parabolic coordinate system on $\mathbb{R}^2$ is defined by a pair σ, τ and has the associated change of variables:

$$\Psi : \mathbb{R}^2_{(\sigma,\tau)} \to \mathbb{R}^2_{(x,y)}$$

   given by:

$$(x, y) = \Psi(\sigma, \tau) = \left(\sigma\tau, \frac{1}{2}(\tau^2 - \sigma^2)\right)$$

   (a) The equations $\sigma = 0, 1$ and $\tau = 0, 1$ form "lines" in the parabolic coordinate system. Convert these to lines in the rectangular coordinate system and plot together. [10pts]

   (b) Show that Ψ is a smooth change of variables when restricted to $\sigma > 0$ and $\tau > 0$ and write down its corresponding integral transformation. [10pts]