MATH 411 (JWG) Exam 3 Spring 2021 Solutions

Due by Tue May 11 at 10:00pm

Exam Logistics:

1. From the moment you download this exam you have three hours to take the exam and submit to Gradescope. This includes the entire upload and tag procedure so do not wait until the last minute to do these things.
2. Tag your problems! Please! Pretty please!
3. You may print the exam, write on it, scan and upload.
4. Or you may just write on it on a tablet and upload.
5. Or you are welcome to write the answers on separate pieces of paper if other options don't appeal to you, then scan and upload.

Exam Rules:

1. You may ask for clarification on questions but you may not ask for help on questions!
2. You are permitted to use official class resources which means your own written notes, class Panopto recordings and the textbook.
3. You are not permitted to use other resources. Thus no friends, internet, calculators, Wolfram Alpha, etc.
4. By taking this exam you agree that if you are found in violation of these rules that the minimum penalty will be a grade of 0 on this exam.

Exam Work:

1. Show all work as appropriate for and using techniques learned in this course.
2. Any pictures, work and scribbles which are legible and relevant will be considered for partial credit.
3. Arithmetic calculations do not need to be simplified unless specified.

1. In the proof of Fubini's Theorem we proved the $U(A, P^X) \leq U(f, P)$ part for any partition $P = (P^X, P^Y)$ of I. State and prove the $L(f, P^X) \geq L(A, P)$ part. [10pts]

Solution:

Suppose $P = (P^X, P^Y)$ is a partition of I. For a particular i, j put:

$$\begin{aligned} m_{ij} &= \inf\{f(x,y) \,|\, (x,y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]\} \\ m_i &= \inf\{A(x) \,|\, x \in [x_{i-1}, x_i]\} \end{aligned}$$

For a given i, within the i^{th} column for every x we have:

$$A(x) = \int_c^d f(x,y)\,dy = \sum_j \left[\int_{y_{j-1}}^{y_j} f(x,y)\,dy\right] \geq \sum_j m_{ij}(y_j - y_{j-1})$$

Since it this is true for every $x \in [x_{i-1}, x_i]$ we have:

$$m_i = \inf\{A(x) \,|\, x \in [x_{i-1}, x_i]\} \geq \sum_j m_{ij}(y_j - y_{j-1})$$

Multiplying through by $(x_i - x_{i-1})$ and summing over all j yields:

$$\sum_i m_i(x_i - x_{i-1}) \geq \sum_{i,j} m_{ij}(x_i - x_{i-1})(y_j - y_{j-1})$$

This is exactly as desired:

$$L(A, P^X) \geq L(f, P)$$

2. Define $\bar{F} : \mathbb{R}^{2+2} \to \mathbb{R}^2$ by:

$$\bar{F}(x, y, z, w) = (x^3 w - 2y - z, xy^2 z - 2w)$$

(a) Prove that the Implicit Function Theorem applies at the point $(1, 2, 4, 8)$. [10pts]

Solution:

First observe that if $X = (x, y)$ and $Y = (z, w)$ then:

$$D\bar{F} = \begin{bmatrix} 3x^2 w & -2 & -1 & x^3 \\ \underbrace{y^2 z \quad 2xyz}_{D_X \bar{F}} & & \underbrace{xy^2 \quad -2}_{D_Y \bar{F}} & \end{bmatrix}$$

Then observe that:

$$\det D_Y \bar{F}(1, 2, 4, 8) = \det \begin{bmatrix} -1 & 1 \\ 4 & -2 \end{bmatrix} = -2 \neq 0$$

Thus the Implicit Function Theorem applies.

(b) Use the Implicit Function Theorem to construct a linear approximation to the associated $\bar{G} : \mathbb{R}^2 \to \mathbb{R}^2$ and use it to approximate the values z, w such that $(1.01, 1.95, z, w)$ satisfies $\bar{F}(1.01, 1.95, z, w) = (0, 0)$. [10pts]

Solution:

The Implicit Function Theorem states that there is a neighborhood of $(1, 2, 4, 8)$ and a function $\bar{G} : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

$$G(x, y) = (z, w) \quad \text{iff} \quad f(x, y, z, w) = (0, 0)$$

and:

$$D_X\bar{F}(x, y, z, w) + D_Y\bar{F}(x, y, z, w)D\bar{G}(x, y) = 0$$

We know that $\bar{G}(1, 2) = (4, 8) = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ and:

$$\begin{bmatrix} 24 & -2 \\ 16 & 16 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 4 & -2 \end{bmatrix} D\bar{G}(1, 2) = 0$$

And so:

$$D\bar{G}(1, 2) = -\begin{bmatrix} -1 & 1 \\ 4 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 24 & -2 \\ 16 & 16 \end{bmatrix} = -\begin{bmatrix} 32 & 6 \\ 56 & 4 \end{bmatrix}$$

Our linear approximation is then:

$$(z, w) = \bar{G}(x, y) \approx \begin{bmatrix} 4 \\ 8 \end{bmatrix} - \begin{bmatrix} 32 & 6 \\ 56 & 4 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$$

Thus:

$$(z, w) = \bar{G}(1.01, 1.95) \approx \begin{bmatrix} 4 \\ 8 \end{bmatrix} - \begin{bmatrix} 32 & 6 \\ 56 & 4 \end{bmatrix} \left(\begin{bmatrix} 1.01 \\ 1.95 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$$

3. Show from the definition of Jordan Content that the graph of $f(x) = x^2$ on the interval $[0, 1]$ has Jordan Content zero. [15pts]

Solution:

Suppose $n \in \mathbb{N}$. For $i = 1, ..., n$ consider the rectangle with lower-left corner $\left(\frac{i-1}{n}, \left(\frac{i-1}{n}\right)^2\right)$ and upper-right corner $\left(\frac{i}{n}, \left(\frac{i}{n}\right)^2\right)$. Such a rectangle has area:

$$\frac{1}{n}\left[\left(\frac{i}{n}\right)^2 - \left(\frac{i-1}{n}\right)^2\right]$$

Hence the sum of the areas is:

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 - \left(\frac{i-1}{n}\right)^2\right] &= \frac{1}{n^3}\sum_{i=1}^{n}(i^2 - (i-1)^2) \\
&= \frac{1}{n^3}\left[(1^2 - 0^2) + (2^2 - 1^2) + (3^2 - 2^2) + ... + (n^2 - (n-1)^2)\right] \\
&= \frac{1}{n^3}n^2 \\
&= \frac{1}{n}
\end{aligned}$$

The union of the rectangles covers the graph completely so for any $\epsilon > 0$ given choose n so $\frac{1}{n} < \epsilon$ and then the union all the rectangles will have area less than epsilon.

4. Define $\bar{F} : \{(x, y) \mid y > 0\} \to \mathbb{R}^2$ by $\bar{F}(x, y) = \left(\frac{x}{y}, x - y\right)$.

(a) Categorize the points (x, y) in the domain for which the Inverse Function Theorem applies and for which the Inverse Function Theorem does not apply. [10pts]

Solution:

We have:

$$\det D\bar{f}(x,y) = \begin{bmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ 1 & -1 \end{bmatrix} = -\frac{1}{y} + \frac{x}{y^2} = -\frac{y}{y^2} + \frac{x}{y^2} = \frac{x-y}{y^2}$$

This will be zero precisely when $y = x$.

(b) Show that the function is really not locally invertible at the points where the Inverse Function Theorem does not apply. [10pts]

Solution:

For a given point (x_0, x_0) every neighborhood will contain infinitely many points of the form (x, x) and we have $f(x, x) = (1, 0)$. Thus it is not possible to have a neighborhood on which f is 1-1.

5. Define $f(x, y) = y$ on $I = [0, 2] \times [0, 1]$. Prove that the following sequence of partitions $\{P_k\}$ is an Archimedian sequence of partitions and use this sequence to prove that f is integrable on I and to calculate $\int_I f$: [15pts]

$$P_k = \left[\frac{2(i-1)}{k}, \frac{2i}{k}\right] \times \left[\frac{j-1}{k}, \frac{j}{k}\right] \quad \text{for} \quad i, j = 1, 2, ..., k$$

Solution:

Note that the area of each sub-rectangle is

$$\left(\frac{2}{k}\right)\left(\frac{1}{k}\right) = \frac{2}{k^2}$$

Since $f(x, y) = y$ we know the infimum m_{ij} of $f(x, y)$ on each sub-rectangle occurs at the lower edge and the supremum M_{ij} of $f(x, y)$ on each sub-rectangle occurs at the upper edge. Hence $m_{ij} = (j-1)/k$ and $M_{ij} = j/k$. Thus we have:

$$U(f, P_k) - L(f, P_k) = \sum_{i=1}^{k}\sum_{j=1}^{k}(M_{ij} - m_{ij})\frac{2}{k^2} = \sum_{i=1}^{k}\sum_{j=1}^{k}\left(\frac{j}{k} - \frac{j-1}{k}\right)\frac{2}{k^2} = \sum_{i=1}^{k}\sum_{j=1}^{k}\frac{2}{k^3} = \frac{2}{k}$$

Since $\left\{\frac{2}{k}\right\} \to 0$ we know that $\{P_k\}$ is an Archimedian sequence of partitions. Then:

$$\begin{aligned}
\int_I f &= \lim_{k\to\infty} U(f, P_k) \\
&= \lim_{k\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{k} M_{ij}\frac{2}{k^2} \\
&= \lim_{k\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{k}\left(\frac{j}{k}\right)\frac{2}{k^2} \\
&= \lim_{k\to\infty} \frac{2}{k^3}\sum_{i=1}^{k}\sum_{j=1}^{k} j \\
&= \lim_{k\to\infty} \frac{2}{k^3}\sum_{i=1}^{k}\frac{k(k+1)}{2} \\
&= \lim_{k\to\infty} \frac{2}{k^3}\frac{k^2(k+1)}{2} \\
&= \lim_{k\to\infty} \frac{k^3+k^2}{k^3} \\
&= 1
\end{aligned}$$

6. The parabolic coordinate system on $\mathbb{R}^2$ is defined by a pair σ, τ and has the associated change of variables:

$$\Psi : \mathbb{R}^2_{(\sigma,\tau)} \to \mathbb{R}^2_{(x,y)}$$

given by:

$$(x,y) = \Psi(\sigma,\tau) = \left(\sigma\tau, \frac{1}{2}(\tau^2 - \sigma^2)\right)$$

(a) The equations $\sigma = 0, 1$ and $\tau = 0, 1$ form "lines" in the parabolic coordinate system. Convert these to lines in the rectangular coordinate system and plot together. [10pts]

Solution:

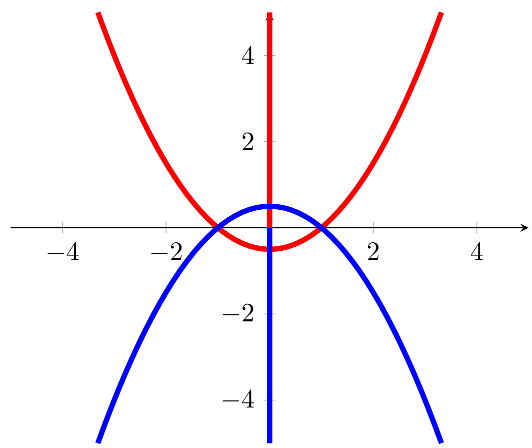
When $\sigma = 0$ we have $x = 0$ and $y = \frac{1}{2}\tau^2$. Since τ can be anything, y can be any nonnegative number.

When $\sigma = 1$ we have $x = \tau$ and $y = \frac{1}{2}(\tau^2 - 1) = \frac{1}{2}(x^2 - 1)$.

When $\tau = 0$ we have $x = 0$ and $y = -\frac{1}{2}\sigma^2$ Since σ can be anything, y can be any nonpositive number.

When $\tau = 1$ we have $x = \sigma$ and $y = -\frac{1}{2}(1 - \sigma^2) = \frac{1}{2}(1 - x^2)$.

The plot is as follows. The first two are red and the second two are blue.



(b) Show that Ψ is a smooth change of variables when restricted to $\sigma > 0$ and $\tau > 0$ and write down its corresponding integral transformation. [10pts]

Solution:

Observe that Ψ is continuously differentiable since the component functions are.

To show it is 1-1 suppose $\Psi(\sigma_1, \tau_1) = \Phi(\sigma_2, \tau_2)$. Then $\sigma_1\tau_1 = \sigma_2\tau_2$ and $\frac{1}{2}(\sigma_1^2 - \tau_1^2) = \sigma_2^2 - \tau_2^2$. The second yields

$$\sigma_1^2 - \sigma_2^2 = \tau_1^2 - \tau_2^2$$

and if we multiply through by τ_2^2 we get

$$\sigma_1^2\tau_2^2 - \sigma_2^2\tau_2^2 = \tau_1^2\tau_2^2 - \tau_2^4$$

which then becomes

$$\sigma_1^2\tau_2^2 - \sigma_1^2\tau_1^2 = \tau_1^2\tau_2^2 - \tau_2^4$$

or

$$\sigma_1^2(\tau_2^2 - \tau_1^2) = \tau_2^2(\tau_1^2 - \tau_2^2)$$

or

$$(\sigma_1^2 + \tau_2^2)(\tau_2^2 - \tau_1^2) = 0$$

Since $\sigma_1^2 + \tau_2^2 > 0$ and $\tau_1, \tau_2 > 0$ we have $\tau_1 = \tau_2$ and then $\sigma_1 = \sigma_2$.

The derivative matrix has:

$$\det D\Psi(\sigma, \tau) = \det \begin{bmatrix} \tau & -\sigma \\ \sigma & \tau \end{bmatrix} = \tau^2 + \sigma^2 \neq 0$$

Thus Ψ is a smooth change of variables and the corresponding integral transformation is:

$$\int_{\Psi(K)} f(x, y)\, d(x, y) = \int_K f(\Psi(\sigma, \tau))|\det D\Psi(\sigma, \tau)|\, d(\sigma, \tau)$$

Which is then:

$$\int_{\Psi(K)} f(x, y)\, d(x, y) = \int_K f\left(\sigma\tau, \frac{1}{2}(\tau^2 - \sigma^2)\right)(\tau^2 + \sigma^2)\, d(\sigma, \tau)$$