Fall 2013 Math 411 Final **Justin Wyss-Gallifent**

1. Using the definition of convergence in $\mathbb{R}^n$ prove that the sequence $\left\{\left(2+\frac{1}{n}, 1-\frac{2}{n}\right)\right\}$ converges to the point $(2,1)$.

2. Using the definition of an open set prove that the set $\{(x,y) \mid y>0\}$ is open.

3. (a) Give an example of a collection of closed intervals whose union is not closed. You do not need to prove anything, just give the intervals.

   (b) Give an example of a collection of bounded intervals whose union is not bounded. You do not need to prove anything, just give the intervals.

4. Using the definition of the directional derivative find the directional derivative of the function $f(x,y)=xy-3x$ at $(1,1)$ in the direction of $(-2,3)$.

5. Let $f(x,y)=xy^3$. Find the second-order approximation for f at $(1,2)$ and use it to approximate $f(1.01,1.97)$. Simplify matrix calculations only.

6. Find the value of θ guaranteed by the Mean Value Theorem for $f(x,y)=3xy+y^3$ at the point $\bar{x}=(1,-2)$ with $\bar{h}=(2,3)$.

7. Find all values of a so that the function $F(x,y)=(x-ay^2, ax^2-y)$ satisfies the hypotheses of the Inverse Function Theorem at $(1,1)$. For an unknown a satisfying these hypotheses, find $D(F^{-1})(f(1,1))$.

8. Define $F:\mathbb{R}^{1+2}\to\mathbb{R}^2$ by $F(x,y,z)=(x-yz, x^2y+z)$.

   (a) Find $DF(1,-1,-1)$.

   (b) Prove that the hypotheses for the Implicit Function theorem do not apply at $(1,-1,-1)$ for writing $(y,z)=H(x)$ near $(1,-1,-1)$.

   (c) Prove that the hypotheses for the Implicit Function Theorem hold at $(1,-1,-1)$ for writing $(x,z)=G(y)$ for (x,y,z) near $(1,-1,-1)$. Write down a first-order approximation for G at $(1,-1,-1)$.

9. Define $I=[0,1]\times[0,1]$ and define $f:I\to\mathbb{R}$ by

$$f(x,y)=\begin{cases}1 & \text{if } y\le x\\ 0 & \text{if } y>x\end{cases}$$

   Use the Archimedes Riemann Theorem with an appropriate Archimedian sequence of partitions to evaluate $\int_I f$.

10. Prove rigorously that the boundary of the rectangle $[0,1]\times[2,4]$ has Jordan content zero.

11. Let Σ be the surface parametrized by $\bar{r}(u,v)=(u-2uv, 4u^2)$ for $0\le u\le 3$, $-1\le v\le 2$.

    (a) Explain informally why $\int_\Sigma f=0$.

    (b) Evaluate $\int_\Sigma f\ dS$ until you have an iterated integral then stop.

12. Apply the change of variables formula to the integral $\int_D x+y\ d(x,y)$ where D is the region in the plane bounded by the parabolas $y=x^2$, $y=x^2+1$, $y=1-x^2$ and $y=2-x^2$. Use Ψ which has $\Psi^{-1}(x,y)=(y-x^2, y+x^2)$. Be rigorous and clear about your Ψ, $D\Psi$, $\Psi^{-1}(D)$ (draw a picture) and especially $f(\Psi(x))$. Simplify but do not evaluate.