1. Let $\bar{x}_0 \in \mathbb{R}^n$ and $r \in \mathbb{R}^+$. Prove from the definition of closed that $S = \{\bar{x} \in \mathbb{R}^n \mid ||\bar{x} - \bar{x}_0|| < r\}$ is not closed.

2. Let $A, B \subseteq \mathbb{R}$ be sequentially compact. Prove that $\{(x,y) | x \in A, y \in B\} \subseteq \mathbb{R}^2$ is sequentially compact.

3. Let $\mathcal{O} \subseteq \mathbb{R}^n$ be open and $\bar{x} \in \mathcal{O}$. Suppose $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable and $\nabla f(\bar{x}) \neq \bar{0}$. Prove that the direction of norm 1 at $\bar{x}$ in which $f$ increases the fastest is the direction $\nabla f(\bar{x})/||\nabla f(\bar{x})||$.

4. Let $\mathcal{O} = \left\{(x,y,z) \in \mathbb{R}^3 \mid xyz > -1\right\}$ and define $g : \mathcal{O} \to \mathbb{R}$ by $g(x,y,z) = \sqrt{1 + xyz}$.
   (a) Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(t) = g(3t, 1-t, t)$. Calculate $\phi''(0)$ directly.
   (b) Find the Hessian for $g$ at $(0,1,0)$ and use this to calculate $\phi''(0)$.

5. Let $a, b \in \mathbb{R}$ be fixed nonzero constants with $a^2 + b^2 = 1$. Define $P : \mathbb{R}^2 \to \mathbb{R}$ to be the projection function to the line $ax + by = 0$. Explicitly calculate $P(x,y)$ for any $(x,y)$. Prove that $P$ is linear and determine the matrix of the transformation.

6. Define $f(x,y) = (2xy + y - x + \sin(x+y), x^2 + y - e^x)$.
   (a) Show that the Inverse Function Theorem applies at $(0,0)$.
   (b) Find $Df^{-1}(0,-1)$.
   (c) Can we find solutions to $f(x,y) = (0,-1)$ arbitrarily close to $(0,0)$? Justify.

7. Suppose your thermostat has two dials to set, one labeled $x$ and the other labeled $y$. The resulting temperature is $T(x,y) = x^2y + x^3y^3$. Suppose you set $x = 3$ and $y = 2$. Use Dini's Theorem to prove that if $x$ changes slightly that you can alter $y$ to maintain the same temperature. Make sure you explain how Dini's Theorem is applying to the problem.

8. Suppose $S$ is a bounded countably infinite set of points in $\mathbb{R}^n$. Prove that an infinite subset may be chosen with Jordan content zero.

9. Define $I = [0,1] \times [0,1]$ and $f : I \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 0 & \text{for } y \in \mathbb{Q} \\ 2x - 1 & \text{for } y \notin \mathbb{Q} \end{cases}$$

   Of the following three, which exist and which do not? Justify informally but clearly.

   (a) $\int_I f$ (b) $\int_0^1 \int_0^1 f(x,y)\ dx\ dy$ (c) $\int_0^1 \int_0^1 f(x,y)\ dy\ dx$

10. Prove Green's Theorem for a generalized rectangle. That is, prove that if $I$ is a generalized rectangle with boundary $C$ and $M, N : I \to \mathbb{R}$ are continuously differentiable then we have $\int_C M\ dx + N\ dy = \int_I N_x - M_y$.

11. Let $C$ consist of the two straight line segments from $(0,1)$ to $(0,0)$ to $(1,0)$. Prove from the definition that $C$ is rectifiable with length 2.