1. Let \( \bar{x}_0 \in \mathbb{R}^n \) and \( r \in \mathbb{R}^+ \). Prove from the definition of closed that \( S = \{ \bar{x} \in \mathbb{R}^n \mid \|\bar{x} - \bar{x}_0\| < r \} \) is not closed.

2. Let \( A, B \subseteq \mathbb{R} \) be sequentially compact. Prove that \( \{(x, y)\mid x \in A, y \in B\} \subseteq \mathbb{R}^2 \) is sequentially compact.

3. Let \( \mathcal{O} \subseteq \mathbb{R}^n \) be open and \( \bar{x} \in \mathcal{O} \). Suppose \( f : \mathcal{O} \to \mathbb{R} \) is continuously differentiable and \( \nabla f(\bar{x}) \neq 0 \). Prove that the direction of norm 1 at \( \bar{x} \) in which \( f \) increases the fastest is the direction \( \nabla f(\bar{x})/\|\nabla f(\bar{x})\| \).

4. Let \( \mathcal{O} = \{(x, y, z) \in \mathbb{R}^3 \mid xyz > -1\} \) and define \( g : \mathcal{O} \to \mathbb{R} \) by \( g(x, y, z) = \sqrt{1 + xyz} \).

   (a) Define \( \phi : \mathbb{R} \to \mathbb{R} \) by \( \phi(t) = g(3t, 1 - t, t) \). Calculate \( \phi''(0) \) directly.

   (b) Find the Hessian for \( g \) at \((0, 1, 0)\) and use this to calculate \( \phi''(0) \).

5. Let \( a, b \in \mathbb{R} \) be fixed nonzero constants with \( a^2 + b^2 = 1 \). Define \( P : \mathbb{R}^2 \to \mathbb{R} \) to be the projection function to the line \( ax + by = 0 \). Explicitly calculate \( P(x, y) \) for any \((x, y)\). Prove that \( P \) is linear and determine the matrix of the transformation.

6. Define \( f(x, y) = (2xy + y - x + \sin(x + y), x^2 + y - e^x) \).

   (a) Show that the Inverse Function Theorem applies at \((0, 0)\).

   (b) Find \( Df^{-1}(0, -1) \).

   (c) Can we find solutions to \( f(x, y) = (0, -1) \) arbitrarily close to \((0, 0)\)? Justify.

7. Suppose your thermostat has two dials to set, one labeled \( x \) and the other labeled \( y \). The resulting temperature is \( T(x, y) = x^2y + x^3y^3 \). Suppose you set \( x = 3 \) and \( y = 2 \). Use Dini’s Theorem to prove that if \( x \) changes slightly that you can alter \( y \) to maintain the same temperature. Make sure you explain how Dini’s Theorem is applying to the problem.

8. Suppose \( S \) is a bounded countably infinite set of points in \( \mathbb{R}^n \). Prove that an infinite subset may be chosen with Jordan content zero.

9. Define \( I = [0, 1] \times [0, 1] \) and \( f : I \to \mathbb{R} \) by

   \[
   f(x, y) = \begin{cases} 
   0 & \text{for } y \in \mathbb{Q} \\
   2x - 1 & \text{for } y \not\in \mathbb{Q}
   \end{cases}
   \]

   Of the following three, which exist and which do not? Justify informally but clearly.

   (a) \( \int_I f \)

   (b) \( \int_0^1 \int_0^1 f(x, y) \, dx \, dy \)

   (c) \( \int_0^1 \int_0^1 f(x, y) \, dy \, dx \)

10. Prove Green’s Theorem for a generalized rectangle. That is, prove that if \( I \) is a generalized rectangle with boundary \( C \) and \( M, N : I \to \mathbb{R} \) are continuously differentiable then we have \( \int_C M \, dx + N \, dy = \int_I Nx - My \).

11. Let \( C \) consist of the two straight line segments from \((0, 1)\) to \((0, 0)\) to \((1, 0)\). Prove from the definition that \( C \) is rectifiable with length 2.