1. Write down the matrix product in \( \mathbb{R}P^2 \) which rotates the four points \([1 : 2], [0 : 0], [-5 : 2] \) and \([8 : 10]\) by 4.1 radians about the point (56, 70).

**Solution:**

\[
\begin{bmatrix}
1 & 0 & 56 \\
0 & 1 & 70 \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\cos(4.1) & -\sin(4.1) & 0 \\
\sin(4.1) & \cos(4.1) & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & -56 \\
0 & 1 & -70 \\
0 & 1 & 1
\end{bmatrix}
\]

2. Suppose a transformation of the plane is expressed as \( z \mapsto e^{\pi/6}(z - (2 + 4i)) \) when treating the plane as \( \mathbb{C} \). Rewrite this transformation as a product of matrices in \( \mathbb{R}P^2 \).

**Solution:**

\[
\begin{bmatrix}
cos(\pi/6) & -sin(\pi/6) & 0 \\
sin(\pi/6) & cos(\pi/6) & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & -2 \\
0 & 1 & -4 \\
0 & 0 & 1
\end{bmatrix}
\]

3. Suppose the point and normal pair \((v_0, n_0)\) represents a plane \( \mathcal{P} \) and the point and direction pair \((v_1, d_1)\) represents a line \( \mathcal{L} \).

(a) If \( \mathcal{P} \) is rotated by \( \theta \) radians about \( \mathcal{L} \) according to the right-hand rule applied to \( \mathcal{L} \), write down the algebraic quaternion expressions (point and normal pair) for the resulting plane.

**Solution:**

Let \( p = \cos(\theta/2) + \frac{v_1}{|v_1|} \sin(\theta/2) \) and then:

\[ (v_0, n_0) \mapsto (pv_0 - v_1)p^* + v_1, pn_0p^* \]

(b) If \( \mathcal{L} \) is reflected in \( \mathcal{P} \), write down the algebraic quaternion expressions (point and direction pair) for the resulting line.

**Solution:**

Let \( \hat{n} = \frac{n_0}{|n_0|} \) and then:

\[ (v_1, d_1) \mapsto (\hat{n}(v_0 - v_1))\hat{n} + v_1, \hat{n}d_0\hat{n} \]

4. In \( \mathbb{R}P^2 \) if you follow the ends of the parabola \( x = y^2 \), which point(s) at infinity do they approach?

**Solution:**

For such a point as we go to the right, \( y \rightarrow \pm \infty \) and then:

\[ [x; y; 1] = [y^2; y; 1] \equiv [1; 1/y; 1/y^2] \rightarrow [1; 0; 0] \]
5. Suppose \( L \) is a line in \( \mathbb{C} \) represented by the closest point to the origin \( z_0 \in \mathbb{C} \). Suppose \( L \) is rotated about the point \( c_0 \in \mathbb{C} \). Describe a criteria (with words, pictures, or algebraically) under which the rotation of \( L \) would never pass through the origin, no matter what angle of rotation was used.

**Solution:**

Let \( C \) be the circle of radius \( |c_0| \) centered at \( c_0 \) (meaning the circle which touches the origin). Then no rotation of \( L \) will result in \( L \) hitting the origin iff \( L \) outside this circle. This means that the distance from \( L \) to \( c_0 \) is greater than \( |c_0| \).

6. In \( \mathbb{C} \) show that the composition of two rotations by different angles about different points can be written as a single rotation followed by a translation.

**Solution:**

Two such rotations yield a mapping:

\[
\begin{align*}
\text{Rotation} & : z \mapsto e^{i\theta_2}(e^{i\theta_1}(z - z_1) + z_1 - z_2) + z_2 \\
\rightarrow & e^{i\theta_2}(e^{i\theta_1}z - e^{i\theta_1}z_1 + z_1 - z_2) + z_2 \\
\rightarrow & e^{i(\theta_1+\theta_2)}z - e^{i(\theta_1+\theta_2)}z_1 + e^{i\theta_2}(z_1 - z_2) + z_2 \\
\text{Translation} & : \quad z \mapsto z.
\end{align*}
\]

7. Suppose the location of an object is given by the parametrization \( \mathbf{r}(t) = t \mathbf{i} - t^2 \mathbf{j} \) for \( t \geq 0 \). Consider the perspective projection with \( y = 10 \).

(a) Find the projection of the object on the \( x \)-axis at time \( t \).

**Solution:**

The object is at \([t; -t^2; 1]\) and so:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & -0.1 & 1
\end{bmatrix}
\begin{bmatrix}
t \\
-t^2 \\
1
\end{bmatrix} =
\begin{bmatrix}
t \\
0 \\
1 + 0.1t^2
\end{bmatrix} =
\begin{bmatrix}
t/(1 + 0.1t^2) \\
0 \\
1
\end{bmatrix}
\]

So the object projects to \( x = t/(1 + 0.1t^2) \)

(b) Where does the projection of the object go as \( t \to \infty \)?

**Solution:**

As \( t \to \infty \) we see \( x \to 0 \).

(c) How far to the right on the \( x \)-axis does the object go before reversing course?

**Solution:**

We have:

\[
\frac{d}{dt} \frac{t}{1 + 0.1t^2} = \frac{1(1 + 0.1t^2) - t(0.2t)}{(1 + 0.2t^2)^2}
\]

which equals zero when \( t = \sqrt{10} \) and so the object is at \( x = \sqrt{10}/(1+0.1(\sqrt{10})^2) \).
8. Show algebraically using $\mathbb{H}$ that rotation by $\theta$ followed by rotation by $-\theta$ cancel. [Ch5:10pts]

Solution:

For the axis $\mathbf{u}$ the first rotation has $p_1 = \cos(\theta/2) + \sin(\theta/2)\mathbf{u}$ and the second rotation has $p_2 = \cos(-\theta/2) + \sin(-\theta/2)\mathbf{u} = \cos(\theta/2) - \sin(\theta/2)\mathbf{u} = p_1^*$. The result is then

$$\mathbf{v} \mapsto p_2 p_1 \mathbf{v} p_1^* p_2^* = p_1^* p_1 \mathbf{v} p_1 = \mathbf{v}$$

where the last equality holds since for unit quaternions the conjugate is the inverse.

9. Suppose $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are perpendicular. Write down an expression using simple vector operations (no quaternions) for the result of rotating $\mathbf{v}$ by $\theta$ radians in the direction of $\mathbf{w}$. [Ch2:10pts]

Solution:

The only issue is that $\mathbf{w}$ needs to be the same length as $\mathbf{v}$ in order to use the nicest formula we have. This is easy to fix:

$$\mathbf{v} \mapsto (\cos \theta) \mathbf{v} + (\sin \theta) \frac{\|\mathbf{v}\|}{\|\mathbf{w}\|} \mathbf{w}$$