1. Chapter 2: Consider the three points $P_0 = (7, 1)$, $P_2 = (2, 6)$ and $Q = (3, 2)$. [15 pts]

   (a) Show that $P_0$ and $P_1$ are the same distance from $Q$.

   **Solution:**
   We have $|QP_0| = \sqrt{17}$ and $|QP_1| = \sqrt{17}$.

   (b) Calculate the angle between the line segments $QP_0$ and $QP_1$. This should be exact and can contain trig functions.

   **Solution:**
   Since $QP_0 = [4; -1]$ and $QP_1 = [-1; 4]$ we have
   $$\theta = \cos^{-1} \left( \frac{QP_0 \cdot QP_1}{|QP_0||QP_1|} \right) = \cos^{-1} \left( -\frac{8}{17} \right)$$

   (c) Find the parametrization $P(t) = (x(t), y(t))$ which parametrizes the arc centered at $Q$ with $P(0) = P_0$ and $P(1) = P_1$. Use counterclockwise rotation about $Q$ to do this, meaning $P(t)$ should be calculated by rotating $P_0$ part way towards $P_1$. This should be exact and can contain trig functions.

   **Solution:** We have:
   $$P(t) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} \cos(\theta t) & -\sin(\theta t) \\ \sin(\theta t) & \cos(\theta t) \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$
   $$= \begin{bmatrix} 3 + 4\cos(\theta t) + \sin(\theta t) \\ 2 + 4\sin(\theta t) - \cos(\theta t) \end{bmatrix}$$

   (d) Calculate the coordinates of $P(0.2)$, $P(0.4)$, $P(0.6)$ and $P(0.8)$.

   **Solution:** We have:
   $$P(0.2) \approx (7.0656, 2.6861)$$
   $$P(0.4) \approx (6.4504, 4.2572)$$
   $$P(0.6) \approx (5.2572, 5.4504)$$
   $$P(0.8) \approx (3.6861, 6.0656)$$
2. Chapter 2: Suppose  $\hat{x}(t)$ and $\hat{y}(t)$ are parametrized orthonormal vectors in $\mathbb{R}^3$ and $P$ is the plane they span. Moreover assume that an object travels in this plane according to:

$$ r(t) = t\hat{x}(t) + t^2\hat{y}(t) \text{ for } t \geq 0 $$

(a) Describe geometrically what path the object is following in the plane $P$.

**Solution:** The curve is a parabola.

(b) Suppose at $t = 0$ we have:

$$ \hat{x}(0) = \frac{1}{\sqrt{5}}(1\, i + 2\, j + 0\, k) $$
$$ \hat{y}(0) = \frac{1}{\sqrt{6}}(2\, i - 1\, j + 1\, k) $$

If at $t = 0$ the plane $P$ starts rotating via the right-hand rule about the axis $\hat{x}(t) \times \hat{y}(t)$ at a rate of 0.1 radians per second, find $\hat{x}(t)$ and $\hat{y}(t)$ for all $t \geq 0$. These should be exact.

**Solution:**

We know $\hat{x}(t)$ rotates towards $\hat{y}(t)$ and so:

$$ \hat{x}(t) = \cos(0.1t)\hat{x}(0) + \sin(0.1t)\hat{y}(0) = \begin{bmatrix} \frac{1}{\sqrt{5}} \cos(0.1t) + \frac{2}{\sqrt{6}} \sin(0.1t) \\ \frac{2}{\sqrt{5}} \cos(0.1t) - \frac{1}{\sqrt{6}} \sin(0.1t) \\ \frac{1}{\sqrt{6}} \sin(0.1t) \end{bmatrix} $$

We know $\hat{y}(t)$ rotates away from $\hat{x}(t)$ and so:

$$ \hat{y}(t) = \cos(-0.1t)\hat{y}(0) + \sin(-0.1t)\hat{x}(0) = \begin{bmatrix} \frac{2}{\sqrt{6}} \cos(0.1t) - \frac{1}{\sqrt{5}} \sin(0.1t) \\ -\frac{1}{\sqrt{5}} \cos(0.1t) - \frac{2}{\sqrt{6}} \sin(0.1t) \\ \frac{1}{\sqrt{6}} \cos(0.1t) \end{bmatrix} $$

(c) Determine the location in $\mathbb{R}^3$ of the object at time $t$. This should be exact.

**Solution:** We have:

$$ r(t) = t\hat{x}(t) + t^2\hat{y}(t) = \begin{bmatrix} \frac{1}{\sqrt{5}} \cos(0.1t) + \frac{2}{\sqrt{6}} \sin(0.1t) \\ \frac{2}{\sqrt{5}} \cos(0.1t) - \frac{1}{\sqrt{6}} \sin(0.1t) \\ \frac{1}{\sqrt{6}} \sin(0.1t) \end{bmatrix} + \begin{bmatrix} \frac{1}{\sqrt{5}} \cos(0.1t) - \frac{1}{\sqrt{6}} \sin(0.1t) \\ -\frac{1}{\sqrt{5}} \cos(0.1t) - \frac{2}{\sqrt{6}} \sin(0.1t) \\ \frac{1}{\sqrt{6}} \cos(0.1t) \end{bmatrix} $$
3. Chapter 3: Calculate each of the following: [10 pts]

(a) The result when $3 + 4i$ is rotated counterclockwise by 1.75 radians about $8 + 9i$. Express your answer as $a + bi$.

Solution:
The result is:
\[ e^{1.75i} (3 + 4i - (8 + 9i)) + (8 + 9i) \approx 13.8112 + 4.9713i \]

(b) The result when the line parametrized by nearest point $-1 + 2i$ is translated by $2 + 3i$. Express your answer as a nearest point.

Solution:
The result is:
\[ (-1 + 2i) + \frac{-1 + 2i}{| -1 + 2i|^2} \text{Re}((-1 - 2i)(2 + 3i)) = -1.8 + 3.6i \]

(c) The result when the line parametrized by angle $\theta = 2$ radians is rotated counterclockwise by 3 radians about $1 - 1i$. Express your answer as a nearest point.

Solution:
We first translate by $-(1 - 1i) = -1 + i$ to get the line parametrized by closest point:
\[ i e^{i\theta} \text{Im}((-1 + i)e^{-2i}) \approx -0.4484 - 0.2052i \]

We then rotate:
\[ e^{3i} (-0.4484 - 0.2052i) \approx 0.4729 + 0.1399i \]

We then translate back:
\[ (0.4729 + 0.1399i) + \frac{(0.4729 + 0.1399i)}{|0.4729 + 0.1399i|} \text{Re}(0.4729 - 0.1399i)(1 - i) \approx 1.1204 + 0.3314i \]
4. Chapter 3: Let \( \mathcal{L} \) be the line represented by \( 3 - 2i \). If we rotate \( \mathcal{L} \) counterclockwise about \( 3 + 1i \) by \( \theta \) radians, what must \( \theta \) be to ensure that the rotated line meets the origin?

**Solution:**

In order to rotate about \( 3 + 1i \) we need to translate the line by \( -3 - 1i \) so first we find that the translated line is:

\[
3 - 2i + \frac{3 - 2i}{|3 - 2i|^2} \text{Re}((3 + 2i)(-3 - 1i)) = \frac{18}{13} - \frac{12}{13}i
\]

Rotation by \( \theta \) then yields:

\[
(
\cos(\theta) + i \sin(\theta)
\) \left( \frac{18}{13} - \frac{12}{13}i \right) = \frac{1}{13} \left[(18 \cos \theta + 12 \sin \theta) + (-12 \cos \theta + 18 \sin \theta)i \right]
\]

If the original line would meet the origin when rotated then this line would meet \( -3 - 1i \).

This can be written in terms of complex numbers but it’s slightly easier in terms of vectors. If we think of the above complex number as a vector \( \mathbf{v} \) then \([-3, -1]\) will lie on the line that this yields precisely when \((\mathbf{v} - [-3, -1]) \perp \mathbf{v}\). This yields the equation:

\[
7 \cos \theta + 9 \sin \theta + 6 = 0
\]

Wolfram Alpha yields:

\[
\theta = 2 \tan^{-1}\left(9 - \sqrt{94}\right) + 2n\pi
\]

\[
\theta = 2 \tan^{-1}\left(9 + \sqrt{94}\right) + 2n\pi
\]
5. Chapter 5: Suppose a cube starts with its eight corners at:

\[ P_1 = (0,0,0), \ P_2 = (1,0,0), \ P_3 = (1,1,0), \ P_4 = (0,1,0), \]
\[ P_5 = (0,0,1), \ P_6 = (1,0,1), \ P_7 = (1,1,1), \ P_8 = (0,1,1), \]

(a) If an axis passes through \((5,4,0)\) with direction \(1i + 1j + 3k\) and if the cube rotates about this axis by 5.3 radians counterclockwise by the right-hand rule, find the rotated locations of each of the corners.

**Solution:**

We put \(p = \cos(5.3/2) + \frac{1i + 1j + 3k}{\sqrt{11}} \sin(5.3/2)\) and \(v_0 = 5i + 4j + 0k\) and then point by point:

\[
\begin{align*}
    p(0i + 0j + 0k - v_0)p^* + v_0 &= -1.1477i + 5.182j - 1.3447k \\
    p(1i + 0j + 0k - v_0)p^* + v_0 &= -0.55285i + 4.4697j - 0.97227k \\
    p(1i + 1j + 0k - v_0)p^* + v_0 &= 0.24048i + 5.0646j - 1.1017k \\
    p(0i + 1j + 0k - v_0)p^* + v_0 &= -0.35441i + 5.7769j - 1.4742k \\
    p(0i + 0j + 1k - v_0)p^* + v_0 &= -1.2771i + 5.5544j - 0.42577k \\
    p(1i + 0j + 1k - v_0)p^* + v_0 &= -0.68225i + 4.8421j - 0.053297k \\
    p(1i + 1j + 1k - v_0)p^* + v_0 &= +0.11108i + 5.437j - 0.1827k \\
    p(0i + 1j + 1k - v_0)p^* + v_0 &= -0.48381i + 6.1493j - 0.55517k
\end{align*}
\]

(b) If a mirror has equation \(2x + 3y - z = 10\), find the reflected locations of each of the corners.

**Solution:**

We put \(\hat{n} = \frac{2i + 3j - 1k}{\sqrt{14}}\) and \(v_0 = 0i + 0j - 10k\) and then point by point:

\[
\begin{align*}
    \hat{n}(0i + 0j + 0k - v_0)\hat{n} + v_0 &= 2.857i + 4.285j - 1.4286k \\
    \hat{n}(1i + 0j + 0k - v_0)\hat{n} + v_0 &= 3.2857i + 3.4286j - 1.1429k \\
    \hat{n}(1i + 1j + 0k - v_0)\hat{n} + v_0 &= 2.4286i + 3.1429j - 0.71429k \\
    \hat{n}(0i + 1j + 0k - v_0)\hat{n} + v_0 &= 2i + 4j - 1k \\
    \hat{n}(0i + 0j + 1k - v_0)\hat{n} + v_0 &= 3.1429i + 4.7143j - 0.57143k \\
    \hat{n}(1i + 0j + 1k - v_0)\hat{n} + v_0 &= 3.5714i + 3.8571j - 0.28571k \\
    \hat{n}(1i + 1j + 1k - v_0)\hat{n} + v_0 &= 2.7143i + 3.5714j + 0.14286k \\
    \hat{n}(0i + 1j + 1k - v_0)\hat{n} + v_0 &= 2.2857i + 4.4286j - 0.14286k
\end{align*}
\]
6. Chapter 5: A mirror passes through the origin but the mirror is rotating so that for $0 \leq t \leq 1$ [10 pts] the unit normal vector moves along the great arc from $0\hat{i} + 0\hat{j} + 1\hat{k}$ to $\frac{1}{\sqrt{3}}(1\hat{i} + 1\hat{j} + 1\hat{k})$.

(a) Use Slerp to find the normal vector $n(t)$ at time $t$.

**Solution:**

We call the two normals $n_s$ and $n_e$ and then: We find $\theta = \cos^{-1}(n_s \cdot n_e) = \cos^{-1}(1/\sqrt{3}) \approx 0.9953$. Then with Slerp:

$$n(t) = \frac{\sin(0.9953(1-t))}{\sin(0.9953)} n_s + \frac{\sin(0.9953t)}{\sin(0.9953)} n_e$$

$$= \begin{bmatrix} 0.7071 \sin(0.9553t) \\ 0.7071 \sin(0.9553t) \\ 0.7071 \sin(0.9553t) - 1.2247 \sin(0.9553t - 0.9553) \end{bmatrix}$$

(b) If the point $(5, 10, 15)$ is reflected in the mirror, find a parametrization $r(t)$ for $0 \leq t \leq 1$ of the reflection.

**Solution:**

We have the following. This is Matlab’s simplification but I suspect it can be simplified more.

$$n(t)(5\hat{i} + 10\hat{j} + 15\hat{k})n(t)$$

$$= \begin{bmatrix} 11.25 \cos(1.9106t) - 3.75 \cos(1.9106t - 1.9106) - 8.6603 \cos(1.9106t - 0.95532) - 2.5 \\ 7.5 \cos(1.9106t) - 7.5 \cos(1.9106t - 1.9106) - 4.3301 \cos(1.9106t - 0.95532) + 2.5 \\ 11.25 \cos(1.9106t - 1.9106) - 25.981 \cos(1.9106t - 0.95532) + 3.75 \cos(1.9106t) \end{bmatrix}$$
7. Chapter 6: A basic rocket shape in $\mathbb{R}^2$ consists of the following points, where $(0,0)$ represents the center of gravity:

$$(0,0), (1,-2), (-1,-2), (1,2), (-1,2), (0,3)$$

Suppose the rocket needs to take a trip around the plane so that its center of gravity travels from:

$$(0,0) \rightarrow (10,50) \rightarrow (30,100) \rightarrow (100,100) \rightarrow (200,100) \rightarrow (100,0) \rightarrow (0,0)$$

Give a series of (matrix) transformations (rotations and translations) which does this. The basic condition of movement is that the rocket must be pointing in the direction it translates (before it translates), meaning each translation should be preceded by the appropriate rotation. Give the matrix of points at each step followed by the transformation, followed by the resulting matrix, and so on.

**Solution:**

I did this in Matlab because sanity. The idea is this: The rocket needs to figure out how to point to its new location. It does this by figuring out the rotation corresponding to undoing the previous rotation (except around its current point) and then rotating toward the new location simply using arctan2 and the $x$ and $y$ differences. It then makes the translation. In the case of the first move there is no original rotation to undo, so that code starts the whole thing.

Here is the code and the results typset. The functions $R$ and $TT$ are rotation and translation.

```matlab
D0 = transpose([0 0 1;1 -2 1;-1 -2 1;1 2 1;-1 2 1;0 3 1])
oldx = 0;
oldy = 0;
newx = 10;
newy = 50;
theta = -atan2(newx,newy);
D1 = TT(10,50) * R(theta) * D0

oldx = newx;
oldy = newy;
unrotate = TT(oldx,oldy)*R(-theta)*TT(-oldx,-oldy);
newx = 30;
newy = 100;
theta = -atan2((newx-oldx),(newy-oldy));
D2 = TT(newx-oldx,newy-oldy) * TT(oldx,oldy)*R(theta)*TT(-oldx,-oldy)*unrotate*D1

oldx = newx;
oldy = newy;
unrotate = TT(oldx,oldy)*R(-theta)*TT(-oldx,-oldy);
newx = 100;
newy = 100;
theta = -atan2((newx-oldx),(newy-oldy));
D3 = TT(newx-oldx,newy-oldy) * TT(oldx,oldy)*R(theta)*TT(-oldx,-oldy)*unrotate*D2

oldx = newx;
```
oldy = newy;
unrotate = TT(oldx,oldy)*R(-theta)*TT(-oldx,-oldy);
newx = 200;
newy = 100;
theta = -atan2((newx-oldx),(newy-oldy));
D4 = TT(newx-oldx,newy-oldy) * TT(oldx,oldy)*R(theta)*TT(-oldx,-oldy)*unrotate*D3

oldx = newx;
oldy = newy;
unrotate = TT(oldx,oldy)*R(-theta)*TT(-oldx,-oldy);
newx = 100;
newy = 0;
theta = -atan2((newx-oldx),(newy-oldy));
D5 = TT(newx-oldx,newy-oldy) * TT(oldx,oldy)*R(theta)*TT(-oldx,-oldy)*unrotate*D4

Rocket at the start:

\[
\begin{bmatrix}
0 & 1.0000 & -1.0000 & 1.0000 & -1.0000 & 0 \\
0 & -2.0000 & -2.0000 & 2.0000 & 2.0000 & 3.0000 \\
1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\
\end{bmatrix}
\]

After rotation at (0, 0) and translation to (10, 50):

\[
\begin{bmatrix}
50.0000 & 47.8427 & 48.2350 & 51.7650 & 52.1573 & 52.9417 \\
1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\
\end{bmatrix}
\]

After rotation at (10, 50) and translation to (30, 100):

\[
\begin{bmatrix}
100.0000 & 97.7717 & 98.5144 & 101.4856 & 102.2283 & 102.7854 \\
1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\
\end{bmatrix}
\]

After rotation at (30, 100) and translation to (100, 500):

\[
\begin{bmatrix}
100.0000 & 98.0000 & 98.0000 & 102.0000 & 102.0000 & 103.0000 \\
100.0000 & 99.0000 & 101.0000 & 99.0000 & 101.0000 & 100.0000 \\
1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\
\end{bmatrix}
\]

After rotation at (100, 100) and translation to (200, 100):

\[
\begin{bmatrix}
200.0000 & 198.0000 & 198.0000 & 202.0000 & 202.0000 & 203.0000 \\
100.0000 & 99.0000 & 101.0000 & 99.0000 & 101.0000 & 100.0000 \\
1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\
\end{bmatrix}
\]

After rotation at (200, 100) and translation to (100, 0):
After rotation at $(100, 0)$ and translation to $(0, 0)$:

\[
\begin{bmatrix}
100.0000 & 100.7071 & 102.1213 & 97.8787 & 99.2929 & 97.8787 \\
0.0000 & 2.1213 & 0.7071 & -0.7071 & -2.1213 & -2.1213 \\
1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000
\end{bmatrix}
\]
8. Chapter 6: Suppose a line segment initially extends from (1, -5) to (3, -5). If the line segment rotates counterclockwise about its left endpoint by 2 revolutions per second starting at $t = 0$, describe the image of the object at time $t$ under the perspective projection with $d = 10$. This answer should be exact.

Hint: The image is also a line segment. Where does it extend from and to?

Solution:

The fixed endpoint of the segment projects to:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -0.1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3/2 \end{bmatrix} \equiv \begin{bmatrix} 2/3 \\ 0 \\ 1 \end{bmatrix}$$

At time $t$ the rotating end projects to:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -0.1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(4\pi t) & -\sin(4\pi t) & 0 \\ \sin(4\pi t) & \cos(4\pi t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \equiv \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 2 \cos(4\pi t) \\ \frac{3}{2} - \frac{1}{5} \sin(4\pi t) \\ \frac{1 + 2 \cos(4\pi t)}{1.5 - 0.2 \sin(4\pi t)} \end{bmatrix} \equiv \begin{bmatrix} 1 \pm 2 \cos(4\pi t) \\ 0 \\ 1 \end{bmatrix}$$

Thus the image is a line segment stretching from $x = 2/3$ to $x = \frac{1 + 2 \cos(4\pi t)}{1.5 - 0.2 \sin(4\pi t)}$. 