MATH 431 Exam 2 Solutions

1. Suppose \( L \in \mathbb{RP}^2 \) represents a line in \( E^2 \). Define:

\[
M(\theta) = T(3,4)R(\theta)T(-3,-4)
\]

If you were to solve the following equation for \( \theta \), assuming there is a solution, what would you find?

\[
\left( (M(\theta)^{-1})^T L \right) \times [1; 3; 0] = [0; 0; 0]
\]

**Solution:**

You would find the angle \( \theta \) such that rotating \( L \) by \( \theta \) radians counterclockwise results in the line being on top of the line \([1; 3; 0]\).

Note: Many people said parallel to \([1; 3; 0]\) but if it were parallel it would meet at a point at infinity which looks like a nonzero \([x; y; 0]\). The only way to get \([0; 0; 0]\) is if the two vectors are multiples of one another which is when the lines are exactly the same.

Note: I think I said in class that yielding \([0; 0; 0]\) meant they don’t meet, so that may have led some people to say so. If I did say that, I was wrong. If you said parallel then I only took off 1 point because I felt that on the one hand I couldn’t take too much off because you might have just said what I said, but on the other hand I had to do something because it was technically wrong and some people did get it right. I felt 1 point was just a cursory note.
2. If \( M \) is a transformation in \( \mathbb{RP}^2 \) (rotation, translation, combination, etc.), if \( L \in \mathbb{RP}^2 \) represents a line, and \( x \in E^2 \) is a point, explain how you know that the following is true?

\[
L \cdot x = 0 \iff ((M^{-1})^T L) \cdot (Mx) = 0
\]

**Solution:**

We have \( L \cdot x = 0 \) iff \( L \) is on the line represented by \( L \) which is iff \( Mx \) (the transformed point) is on the line represented by \( (M^{-1})^T L \) (the transformed line) which is iff \( ((M^{-1})^T L) \cdot (Mx) = 0 \).
3. Write down the product of matrices in $\mathbb{R}P^3$ which would rotate by $\pi/6$ about the vector $[1; 0; 1] \in \mathbb{R}^3$ with direction given by the right-hand rule. You do not need to multiply these matrices.

Solution:

There are a few options, one is $R_Y(\pi/4)R_Z(\pi/6)R_Y(-\pi/4)$ which is:

$$
\begin{bmatrix}
\sqrt{2}/2 & 0 & \sqrt{2}/2 & 0 \\
0 & 1 & 0 & 0 \\
-\sqrt{2}/2 & 0 & \sqrt{2}/2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\sqrt{3}/2 & -1/2 & 0 & 0 \\
1/2 & \sqrt{3}/2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\sqrt{2}/2 & 0 & -\sqrt{2}/2 & 0 \\
0 & 1 & 0 & 0 \\
\sqrt{3}/2 & 0 & \sqrt{3}/2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

Note: The other somewhat obvious answer is $R_Y(-\pi/4)R_X(\pi/6)R_Y(\pi/4)$. 
4. If \( p_1, p_2 \in \mathbb{R}^3 \) are points and \([n; n_0]\) represents a plane use Plücker methods to find an expression for the point where the line containing \( p_1 \) and \( p_2 \) meets the plane.

Note: A line \([d; m]\) meets a plane \([n; n_0]\) at \([n \times m - n_0 d; n \cdot d]\).

Solution:

For the line we have \( d = p_1 - p_2 \) and \( v = p_1 \) and so \( m = d \times v = (p_2 - p_1) \times p_1 \) and then they meet the plane at:

\[
[n \times ((p_2 - p_1) \times p_1) - n_0(p_2 - p_1), n \cdot (p_2 - p_1)]
\]

Note: This can be simplified a bit, for example \( d = (p_2 - p_1) \times p_1 = p_2 \times p_1 \), but simplification wasn’t necessary.
5. For $B = 2e_1e_2 - 3e_3e_1$ calculate each of the following: [Ch8: 10 pts]

(a) $B^{-1}$

Solution:
We have:
$$B^{-1} = -\frac{B}{|B|^2} = -\left(\frac{2e_1e_2 - 3e_3e_1}{13}\right)$$

(b) $B^*$

Solution:
We have:
$$B^* = Be_3e_2e_1 = (2e_1e_2 - 3e_3e_1)e_3e_2e_1 = 2e_3 + 3e_2$$

(c) $B^\dagger$

Solution:
We have:
$$B^\dagger = (2e_1e_2 - 3e_3e_1)^\dagger = 2e_2e_1 - 3e_1e_3$$
6. Give an example (with calculations) to show that it is not always the case that for multivectors $A$ and $B$ that $AB = A \cdot B + A \wedge B$.

**Solution:**

Almost anything works. For example if $A = e_1 e_2$ and $B = e_2 e_3$ then:

\[
AB = e_1 e_3 \\
A \cdot B = \langle AB \rangle_{2-2} = \langle e_1 e_3 \rangle_0 = 0 \\
A \wedge B = \langle AB \rangle_{2+2} = \langle e_1 e_3 \rangle_4 = 0
\]
7. Write down geometric algebra expressions for each of the following. You do not need to calculate any geometric products but you do need to calculate everything else, such as wedge products, dot products, inverses, and duals. [Ch8 : 20pts]

(a) The rejection of the vector \( v = 3e_2 \) relative to the plane \( \text{Span}\{e_3, 2e_1 + e_2\} \).

**Solution:**
First note that:
\[
B = e_3 \wedge (2e_1 + e_2) = -1e_2e_3 + 2e_3e_1
\]
Then:
\[
vB = (3e_2)(-1e_2e_3 + 2e_3e_1) = -3e_4 + 6e_1e_2e_3
\]
Then:
\[
\text{Rej}_Bv = (v \wedge B)B^{-1} = (vB)_{2+1}B^{-1} = (6e_1e_2e_3) \left( -\left(\frac{-1e_2e_3 + 2e_3e_1}{6}\right)\right)
\]

(b) The reflection of the vector \( v = 2e_1 - 5e_3 \) in the plane \( \text{Span}\{e_2 + e_3, e_1 + 3e_2\} \).

**Solution:**
First note that:
\[
B = (e_2 + e_3) \wedge (e_1 + 3e_2) = -1e_1e_2 - 3e_2e_3 + 1e_3e_1
\]
Then:
\[
\text{Refl}_Bv = -BvB^{-1} = -(-e_1e_2 - 3e_2e_3 + e_3e_1)(2e_1 - 5e_3) \left( -\left(\frac{-e_1e_2 - 3e_2e_3 + e_3e_1}{11}\right)\right)
\]

(c) The rotation of the vector \( v = e_1 + 3e_2 \) around the axis \( r = e_1 - e_3 \) by \( \theta = \pi/3 \).

**Solution:**
We have:
\[
\hat{r} = \frac{e_1 - e_3}{\sqrt{2}}
\]
And hence:
\[
\hat{r}^* = \left(\frac{e_1 - e_3}{\sqrt{2}}\right)e_3e_2e_1 = \frac{-e_1e_2 - 2e_3}{\sqrt{2}}
\]
Since \( \theta = \pi/3 \) we have \( \theta/2 = \pi/6 \) and so the rotation is:
\[
\exp((\pi/6)(1/\sqrt{2})(-e_1e_2 - e_2e_3))(e_1 + 3e_2)\exp(-(\pi/6)(1/\sqrt{2})(-e_1e_2 - e_2e_3))
\]
8. The following picture is of a quadratic Bezier curve along with the two ending control points $b_0$ and $b_2$.

(Original picture ommited in solutions.)

(a) Use the picture to provide an approximate $b_1$.

**Solution:**

We take the tangents at the ends and see where they meet:

![Diagram showing quadratic Bezier curve with approximate points](image)

The meeting point is approximately [10; 9]. Anything close was acceptable, especially if the picture made it clear.

(b) Use the picture to provide an approximate $b(0.25)$.

**Solution:**

Here are the corresponding lines:

![Diagram showing quadratic Bezier curve with approximate points](image)

The exact point is [5.1875; 4.5625] but anything close was acceptable.
9. Given the three control points: \( b_0 = [10; 10], b_1 = [2; 0], b_2 = [10; 5] \)

(a) Write down the parametrization of the quadratic Bezier curve using these points. You do not need to simplify.

Solution:

We have:

\[
b(t) = (1 - t)^2 [10; 10] + 2t(1 - t) [2; 0] + t^2 [10; 5]
\]

(b) Apply the de Casteljau algorithm to find \( b(0.1) \).

Solution:

We have:

\[
\begin{array}{ccc}
[10; 10] & [2; 0] & [10; 5] \\
[9.2; 9] & [2.8; 0.5] \\
[8.56; 8.15] \\
\end{array}
\]

Thus \( b(0.1) = [8.56; 8.15] \).