DIRECTIONS:

- Each problem should be done using methods from the chapter specified.
- Problems should be clearly explained and steps should be given; don’t just write down the mathematics without context and definitely don’t write down just the final answers. Use coherent sentences to explain what you are doing. Even if those sentences are brief, they should provide large-scale context to the process.
- All matrices, vectors, calculations, code, etc. which are relevant to each problem or which are used in each problem should be shown inline with the rest of the question. This doesn’t necessarily mean showing the algebraic details, for example if you’re multiplying two matrices you should show the matrices and the result (in context) but you can do the product elsewhere, like in Matlab.
- Reasonably accurate approximations are sufficient unless otherwise specified.
- You may use any resources you wish except for other people. The only person (other than yourself) you can ask questions to is me, and any answers I give will be general in the sense that the answer must apply to everyone in the class and will be shared with the entire class in order to be fair.
- If you have questions, ask me ASAP!
- You will be graded on neatness, organization and correctness.
- Note: Neatness is worth 10 points!
1. Chapter 6 ($\mathbb{R}P^2$ Transformations of Lines):
Suppose the line represented by $\mathbf{L} = [3; 2; -6]$ is rotated by $\theta$ radians counterclockwise about the point $[0; 2]$. Find all $\theta \in [0, 2\pi)$ so that the rotated line passes through the origin.

2. Chapter 6 ($\mathbb{R}P^2$ Transforms of Points and Lines):
Suppose we have two points $\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{R}^2$ and suppose $M$ is a transformation. Consider the equation:

$$(M^{-1})^T (\mathbf{P}_1 \times \mathbf{P}_2) = (M \mathbf{P}_1) \times (M \mathbf{P}_2)$$

(a) Pick three pairs of points and three transformations (rotations, translations, etc.) and test this equation to ensure it is true.

(b) Without proving this equation explain why it makes sense in terms of what is happening projectively.

3. Chapter 7 ($\mathbb{R}P^3$ Transformations of Points and Planes):
For $t < 0$ an object follows a path in $\mathbb{R}^3$ with parametrization

$$\mathbf{r}(t) = \cos(t) \mathbf{i} + 2 \sin(t) \mathbf{j} - 10 \mathbf{k}$$

Hence it is in the plane $z = -10$. Starting at $t = 0$ the plane and the object start rotating about the axis $x = 0, z = -7$ at one rotation every $2\pi$ seconds.

(a) Find the vector representing the plane at time $t < 0$.

(b) Find the vector representing the plane at time $t \geq 0$.

(c) Find the equation of the object at time $t \geq 0$.

(d) Naturally the rotating object should remain on the rotating plane. Show that this is in fact the case.

4. Chapter 7 ($\mathbb{R}P^3$ Perspective Projection):
An object follows a path in $\mathbb{R}^3$ with parametrization

$$\mathbf{r}(t) = \sin(t) \mathbf{i} + 2 \cos(t) \mathbf{j} - t \mathbf{k}$$

(a) Find the image of the path resulting when the position of the object is projected to the $xy$-plane using viewpoint $z = 100$.

(b) Describe the image of the path as $t \to \infty$. What happens both as $t$ gets larger and in the limit?

5. Chapter 7 (Plücker Coordinates):
Given three planes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ which meet at a single point it’s clear that the intersection lines $\mathcal{P}_1 \cap \mathcal{P}_2$ and $\mathcal{P}_1 \cap \mathcal{P}_3$ meet because they meet at that same point.

(a) Show computationally that this is true for the planes $\mathcal{P}_1 : x + y - z = 1$, $\mathcal{P}_2 : 2x - y - 5z = 7$, and $\mathcal{P}_3 : 6x + y + 2z = 2$. In other words find the Plücker coordinates for the line $\mathcal{P}_1 \cap \mathcal{P}_2$ and for the line $\mathcal{P}_1 \cap \mathcal{P}_3$ and then find the intersection point using Plücker methods.

(b) Show in general that this is in fact true. This is a proof but it’s really just simplifying the appropriate calculation. You can assume the outcome, just show that the calculation is valid,
6. Chapter 8 (Rotation - Note: This is essentially most of a problem from the first take-home exam except the method of solution is now with geometric algebra. The solutions to the first take-home exam are posted so you know what you should be getting, just show how the steps work out in geometric algebra.) [10 pts]

Suppose \( \hat{x}(t) \) and \( \hat{y}(t) \) are parametrized orthonormal vectors in \( \mathbb{R}^3 \) and \( \mathcal{P} \) is the plane they span. Moreover assume that an object travels in this plane according to:

\[
r(t) = t\hat{x}(t) + t^2\hat{y}(t) \text{ for } t \geq 0
\]

(a) Suppose at \( t = 0 \) we have:

\[
\hat{x}(0) = \frac{1}{\sqrt{5}}(1\mathbf{e_1} + 2\mathbf{e_2} + 0\mathbf{e_3}) \\
\hat{y}(0) = \frac{1}{\sqrt{6}}(2\mathbf{e_1} - 1\mathbf{e_2} + 1\mathbf{e_3})
\]

If at \( t = 0 \) the plane \( \mathcal{P} \) starts rotating via the right-hand rule about the axis \( \hat{x}(t) \times \hat{y}(t) \) at a rate of 0.1 radians per second, find \( \hat{x}(t) \) and \( \hat{y}(t) \) for all \( t \geq 0 \). These should be exact.

(b) Determine the location in \( \mathbb{R}^3 \) of the object at time \( t \). This should be exact.

7. Chapter 8 (Projection): [10 pts]

Consider the subspace \( B = \text{Span}\{2\mathbf{e_1} + 3\mathbf{e_2}, \mathbf{e_1} - 5\mathbf{e_3}\} \) and the vector \( \mathbf{a} = 10\mathbf{e_1} + 15\mathbf{e_2} + 20\mathbf{e_3} \).

(a) Use geometric algebra to find \( \text{Proj}_B \mathbf{a} \).

(b) Use standard linear algebra methods to do the same thing and check that the results match.

8. Chapter 9 (Bezier Curves): [10 pts]

Given the eight control points:

\[
b_0 = [0; 10], \ b_1 = [3; 13], \ b_2 = [5; 15], \\
b_3 = [8; 13], \ b_4 = [10; 10], \ b_5 = [12; 8], \\
b_6 = [14; 3], \ b_7 = [18; 1]
\]

(a) For the degree 7 Bezier curve \( \mathbf{b}(t) \) apply the de Casteljau algorithm to find \( \mathbf{b}(0.3) \).

(b) List the control points for each of the two degree 7 Bezier curves which result from breaking the above curve at \( t = 0.3 \).

9. Chapter 9 (Bezier Curves): For the first letter of your first name create sets of points which draw the letter using linear and quadratic Bezier curves. Give the sets of points, the Bezier curve parametrization for each set and a picture of the final result (all curves together) You must have at least one quadratic curve. Repeat for the first letter of your last name.