MATH 431 Homework Chapter 2 Solutions

(2.3) Find the result of rotating the point \((-2, -7)\) clockwise by \(7\pi/6\) radians.

Solution: We do:

\[
R(-7\pi/6) \begin{bmatrix} -2 \\ -7 \end{bmatrix} = \begin{bmatrix} \cos(-7\pi/6) & -\sin(-7\pi/6) \\ \sin(-7\pi/6) & \cos(-7\pi/6) \end{bmatrix} \begin{bmatrix} -2 \\ -7 \end{bmatrix} = ...
\]

(2.5) A non-vertical line in space can be given by the slope-intercept form \(y = mx + b\) for \(m, b \in \mathbb{R}\).

(a) Find the image of the point \((0, b)\) under rotation by \(\theta\).

Solution: We do:

\[
R(\theta) \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} -b \sin \theta \\ b \cos \theta \end{bmatrix}
\]

(b) Find the image of the point \((1, m + b)\) under rotation by \(\theta\). Solution: We do:

\[
R(\theta) \begin{bmatrix} 1 \\ m + b \end{bmatrix} = \begin{bmatrix} \cos \theta - (b + m) \sin \theta \\ \sin \theta + (b + m) \cos \theta \end{bmatrix}
\]

(c) Find the slope of the image of the line. This should not contain \(b\). Why does this make sense? For which values of \(\theta\) will this be undefined? Explain geometrically what is happening for such values.

Solution: We just find the slope using the previous two points. The answer is:

\[
\frac{\sin \theta + m \cos \theta}{\cos \theta - m \sin \theta}
\]

It makes sense because the slope of the new line depends only on the rotation, not on the original intercept.

This will be undefined when:

\[
\cos \theta - m \sin \theta = 0 \\
\tan \theta = \frac{1}{m}
\]

What is happening is that the line is rotating to vertical.

(d) Write the equation of the new line in the form \(y = m'x + b'\) where \(m'\) and \(b'\) may depend on \(m, b, \theta\).

Solution: Using point-slope form:

\[
y - b \cos \theta = \frac{\sin \theta + m \cos \theta}{\cos \theta - m \sin \theta} (x + b \sin \theta)
\]

\[
y = \left(\frac{\sin \theta + m \cos \theta}{\cos \theta - m \sin \theta}\right) x + \left(\frac{\sin \theta + m \cos \theta}{\cos \theta - m \sin \theta}\right) b \sin \theta + b \cos \theta
\]

\[
y = \left(\frac{\sin \theta + m \cos \theta}{\cos \theta - m \sin \theta}\right) x + \frac{b}{\cos \theta - m \sin \theta}
\]
(e) Where does the line \( y = 2x + 6 \) get mapped to when \( \theta = \pi/6 \)?

**Solution:** To the line:

\[
y = \left( \frac{\sin \pi/6 + 2 \cos \pi/6}{\cos \pi/6 - 2 \sin \pi/6} \right) x + \frac{6}{\cos \pi/6 - 2 \sin \pi/6}
\]

(f) In the case where the slope is undefined what will the equation of the new line be?

**Solution:** The slope is undefined for a vertical line. This will have equation \( x = \ldots \) and all we need is one point on the line to know what \( x \) is. We saw that the \((0, b)\) goes to a point with \( x = -b \sin \theta \). We also saw that we’ll get undefined when \( \tan \theta = 1/m \) which gives \( \sin \theta = \pm 1/\sqrt{m^2 + 1} \). Thus we have:

\[
x = \pm \frac{b}{\sqrt{m^2 + 1}}
\]

(2.7) It’s fairly clear intuitively that rotation preserves distances (formally it’s an example of an **isometry**) but show computationally that this is true. In other words show that if \( P \) and \( Q \) are points then we have:

\[
dist(R_\theta(P), R_\theta(Q)) = dist(P, Q)
\]

**Solution:** Given points:

\[
P = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}
\]

Observe that:

\[
R_\theta(P) = \begin{bmatrix} x_1 \cos \theta - y_1 \sin \theta \\ y_1 \cos \theta + x_1 \sin \theta \end{bmatrix}
\]

\[
R_\theta(Q) = \begin{bmatrix} x_2 \cos \theta - y_2 \sin \theta \\ y_2 \cos \theta + x_2 \sin \theta \end{bmatrix}
\]

From here it’s just showing that the distance formula applied to the result gives the same thing as the distance formula applied to the original points. Just calculation and trig cancelation.

(2.9) The vectors \( v = 4i + 2j - 3k \) and \( w = 1i + 1j + 2k \) are perpendicular but do not have the same magnitude. Find the vector which results when \( v \) is rotates \( 5\pi/6 \) radians towards \( w \) in the plane they span.

**Solution:** We replace \( w \) by a vector which has the same magnitude as \( v \), which is \( |v| = \sqrt{29} \). Thus we use:

\[
w' = \sqrt{\frac{29}{6}} (1i + 1j + 2k)
\]

and we calculate:

\[
\text{Rot}(v) = (\cos \theta)v + (\sin \theta)w'
\]

\[
= (\cos(5\pi/6))(4i + 2j - 3k) + (\sin(5\pi/6))\sqrt{\frac{29}{6}} (1i + 1j + 2k)
\]

\[
= \ldots
\]
(2.13) Given a plane $P$, a point $p \in P$, and a unit vector $\hat{u}$ anchored at $p$, we can still rotate $P$ about $p$ (and about the axis $\hat{u}$) in a counterclockwise direction as defined by $\hat{u}$ even when $p \neq 0$, although this transformation is not linear. We do so by translating $\mathbb{R}^3$ so that $p$ is at the origin, then rotate, then translate back.

Let $P$ be the plane $x + 2y + 3z = 6$. Let the center and axis be defined by $p = [2, 2, 0]^T$ and $\hat{u}$ be the normalization of $N = 1i + 2j + 3k$. Find the resulting point when $(0, 3, 0)$ is rotated by $\pi/6$ radians in this way.

**Solution:**

The translation will be by $(-2, -2, 0)$ so we translate $(0, 3, 0)$ to $(-2, 1, 0)$, rotate using $\hat{u}$, then translate back.

The result is:

$$\text{Rot}(v) = (\cos \theta)v + (\sin \theta)(\hat{u} \times v)$$

$$= (\cos \pi/6) \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + (\sin \pi/6) \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$= ...$$

(2.15) Use the above formula to calculate the reflection of the point $(5, -3)$ in the vector $v = 2i + 1j$.

**Solution:** The “above formula” is:

$$F_v = R_{2\theta}F_{e_1}$$

We have:

$$R_{2\theta} = \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 \theta - \sin^2 \theta & -2 \sin \theta \cos \theta \\ 2 \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix}$$

Noting that the $\theta$ arises from $v = 2i + 1j$ we have $\sin \theta = 1/\sqrt{5}$ and $\cos \theta = 2/\sqrt{5}$ and so:

$$F_v = \begin{bmatrix} \cos^2 \theta - \sin^2 \theta & -2 \sin \theta \cos \theta \\ 2 \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$$= ...$$
(2.17) The line \( y = mx \) is represented by the vector \( \mathbf{v} = [1, m]^T \) which by basic trigonometry satisfies
\[
1 = \sqrt{1 + m^2} \cos \theta \quad \text{and} \quad m = \sqrt{1 + m^2} \sin \theta
\]
(a picture can help you see this). Use this fact and the above formula for \( F_{\mathbf{v}} \) to show that reflection in the line \( y = mx \) is represented by the matrix:
\[
\frac{1}{m^2 + 1} \begin{bmatrix}
1 - m^2 & 2m \\
2m & m^2 - 1
\end{bmatrix}
\]

**Solution:** We know:
\[
\sin \theta = \frac{m}{\sqrt{m^2 + 1}} \quad \text{and} \quad \cos \theta = \frac{1}{\sqrt{m^2 + 1}}
\]
Thus we have:
\[
R_{2\theta} = \begin{bmatrix}
\cos(2\theta) & -\sin(2\theta) \\
\sin(2\theta) & \cos(2\theta)
\end{bmatrix} = \begin{bmatrix}
\cos^2 \theta - \sin^2 \theta & -2\sin \theta \cos \theta \\
2\sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta
\end{bmatrix} = \begin{bmatrix}
1/(m^2 + 1) - m^2/(m^2 + 1) & -2m/(m^2 + 1) \\
-2m/(m^2 + 1) & 1/(m^2 + 1) - m^2/(m^2 + 1)
\end{bmatrix}
\]
And so:
\[
F_{\mathbf{v}} = \frac{1}{m^2 + 1} \begin{bmatrix}
1 - m^2 & 2m \\
2m & m^2 - 1
\end{bmatrix}
\]

(2.25) Suppose all you have at your disposal is an unknown fixed reflection \( F_{\mathbf{v}} \) and all rotations. Show that you can achieve any reflection by doing a product \( R_{\theta}F_{\mathbf{v}} \) for some \( \theta \).

**Solution:** We know that for our fixed \( \mathbf{v} \) and its associated angle \( \theta_0 \) we have:
\[
F_{\mathbf{v}} = R_{2\theta_0}F_{\mathbf{e}_1}
\]
\[
R_{-2\theta_0}F_{\mathbf{v}} = F_{\mathbf{e}_1}
\]
So then for any \( \mathbf{w} \) we have:
\[
F_{\mathbf{w}} = R_{2\theta}F_{\mathbf{e}_1} = R_{2\theta}R_{-2\theta_0}F_{\mathbf{v}} = R_{2(\theta-\theta_0)}F_{\mathbf{v}}
\]

(2.28) Which point parametrizes the line \( y = 3x + 2 \)?

**Solution:** The closest point to the origin, which is \((-3/5, 1/5)\). This can be calculated a variety of ways.
(2.31) We showed that this perspective projection was not linear by assuming that it was, constructing the associated matrix, then showing that the matrix failed to do what we want. Instead we could take the mapping 
\[
\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} dx/(d - y) \\ 0 \end{bmatrix}
\]
and show that it failed the definition of linearity. Do so.

**Solution:** Observe that if we call this mapping $f$ then:

\[
f \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = f \left( \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 2d/(d + 1) \\ 0 \end{bmatrix}
\]

But:

\[
f \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) + f \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} d/(d + 1) \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (2d + 1)/(d + 1) \\ 0 \end{bmatrix}
\]

These are different.