Chapter 8

Geometric Algebra
### Contents

#### 8 Geometric Algebra

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.1</td>
<td>Introduction and Computation Note</td>
<td>4</td>
</tr>
<tr>
<td>8.2</td>
<td>Scalars, Vectors, Bivectors, Trivectors</td>
<td></td>
</tr>
<tr>
<td>8.2.1</td>
<td>Scalars</td>
<td>4</td>
</tr>
<tr>
<td>8.2.2</td>
<td>Vectors</td>
<td>4</td>
</tr>
<tr>
<td>8.2.3</td>
<td>Bivectors</td>
<td>4</td>
</tr>
<tr>
<td>8.2.4</td>
<td>Trivectors</td>
<td>5</td>
</tr>
<tr>
<td>8.2.5</td>
<td>Multivectors</td>
<td>6</td>
</tr>
<tr>
<td>8.2.6</td>
<td>The Inner Product</td>
<td>6</td>
</tr>
<tr>
<td>8.2.7</td>
<td>The Outer Product $a \wedge b$ Revisited</td>
<td>7</td>
</tr>
<tr>
<td>8.2.8</td>
<td>The Outer Product $a \wedge b \wedge c$ Revisited</td>
<td>10</td>
</tr>
<tr>
<td>8.2.9</td>
<td>The Outer Products $a \wedge (b \wedge c)$ and $(a \wedge b) \wedge c$</td>
<td>11</td>
</tr>
<tr>
<td>8.3</td>
<td>The Geometric Product</td>
<td>11</td>
</tr>
<tr>
<td>8.3.1</td>
<td>The Geometric Product of Vectors</td>
<td>11</td>
</tr>
<tr>
<td>8.3.2</td>
<td>The Geometric Product of Multivectors</td>
<td>13</td>
</tr>
<tr>
<td>8.3.3</td>
<td>Properties of the Geometric Product</td>
<td>14</td>
</tr>
<tr>
<td>8.4</td>
<td>Additional Definitions</td>
<td></td>
</tr>
<tr>
<td>8.4.1</td>
<td>Blades</td>
<td>15</td>
</tr>
<tr>
<td>8.4.2</td>
<td>Norms</td>
<td>17</td>
</tr>
<tr>
<td>8.4.3</td>
<td>Grade Extraction from a Multivector</td>
<td>17</td>
</tr>
<tr>
<td>8.4.4</td>
<td>Reversion</td>
<td>18</td>
</tr>
<tr>
<td>8.4.5</td>
<td>Inversion</td>
<td>18</td>
</tr>
<tr>
<td>8.4.6</td>
<td>Inner and Outer Products of Multivectors</td>
<td>21</td>
</tr>
<tr>
<td>8.4.7</td>
<td>Dual of a Multivector</td>
<td>23</td>
</tr>
<tr>
<td>8.5</td>
<td>Computation in 3D</td>
<td></td>
</tr>
<tr>
<td>8.5.1</td>
<td>Projection and Rejection</td>
<td>25</td>
</tr>
<tr>
<td>8.5.2</td>
<td>Reflection</td>
<td>28</td>
</tr>
<tr>
<td>8.5.3</td>
<td>Rotation</td>
<td>29</td>
</tr>
<tr>
<td>8.6</td>
<td>Generalization of $\mathbb{H}$</td>
<td>33</td>
</tr>
</tbody>
</table>
8.1 Introduction and Computation Note

Some of the calculations in this chapter are fairly intense. I did some of the really bad ones in Python with the clifford package. In those cases I’ve noted this. Some I did by hand for practice.

8.2 Scalars, Vectors, Bivectors, Trivectors

8.2.1 Scalars

A scalar in \( \mathbb{R}^n \) can be thought of as a zero-dimensional entity projecting in zero directions out from the origin. Scalars are said to have grade 0. The set of scalars is denoted and defined by:

\[ \wedge^0 \mathbb{R}^n = \text{span}\{1\} \]

8.2.2 Vectors

A vector in \( \mathbb{R}^n \) can be thought of as a one-dimensional entity projecting out in one direction out from the origin. In \( \mathbb{R}^n \) we have the standard basis:

\[ \{e_1, ..., e_n\} \]

and all vectors may be written as linear combinations of those. Vectors are said to have grade 1. The set of vectors is denoted and defined by:

\[ \wedge^1 \mathbb{R}^n = \text{span}\{a | a \in \mathbb{R}^n\} \]

8.2.3 Bivectors

To create our newest addition we introduce a new way of combining vectors.

Definition 8.2.3.1. Given two vectors \( \mathbf{a} \) and \( \mathbf{b} \) we define the outer product (exterior product, wedge product) \( \mathbf{a} \wedge \mathbf{b} \) as the oriented parallelogram with sides \( \mathbf{a} \) and \( \mathbf{b} \).

By “oriented” the implication is that if we follow \( \mathbf{a} \) first and create a loop around the parallelogram we get an orientation. Thus \( \mathbf{b} \wedge \mathbf{a} \) would have the opposite orientation.

Example 8.1. If \( \mathbf{a} = 5\mathbf{e}_1 + 2\mathbf{e}_2 \) and \( \mathbf{b} = 3\mathbf{e}_1 + 6\mathbf{e}_2 \) then the following two pictures illustrate \( \mathbf{a} \wedge \mathbf{b} \) and \( \mathbf{b} \wedge \mathbf{a} \). Note the orientations given by the circulating arrows.
8.2. SCALARS, VECTORS, BIVECTORS, TRIVECTORS

We adopt the convention that reversing the orientation negates the parallelogram and consequently \( \mathbf{b} \wedge \mathbf{a} = -(\mathbf{a} \wedge \mathbf{b}) \). It follows from this that \( \mathbf{a} \wedge \mathbf{a} = 0 \), which makes sense because no parallelogram (or a degenerate parallelogram) is created.

**Definition 8.2.3.2.** We define a *bivector* to be any linear combination of such outer products (oriented parallelograms). More formally a *bivector* in \( \mathbb{R}^n \) for \( n \geq 2 \) is any element in the set:

\[
\wedge^2 \mathbb{R}^n = \text{span}\{ \mathbf{a} \wedge \mathbf{b} \mid \mathbf{a}, \mathbf{b} \in \mathbb{R}^n \}
\]

Bivectors are said to have *grade* 2.

**Example 8.2.** An example of a bivector in \( \mathbb{R}^3 \) is:

\[
10(2\mathbf{e}_1 \wedge (\mathbf{e}_1 + 2\mathbf{e}_2)) + 7((2\mathbf{e}_2 + 2\mathbf{e}_3) \wedge (4\mathbf{e}_1 - \mathbf{e}_3))
\]

8.2.4 Trivectors

**Definition 8.2.4.1.** Given three vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) we define the outer product (exterior product, wedge product) \( \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \) as the oriented parallelopipid with sides \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \).

By “oriented” the implication is that if we take the orientation of \( \mathbf{a} \wedge \mathbf{b} \) and apply the right hand rule, if \( \mathbf{c} \) agrees we get an orientation. If it disagrees we get the opposite orientation.
Definition 8.2.4.2. We define a *trivector* to be any linear combination of such triple outer products (oriented parallelopipeds). More formally a *trivector* in $\mathbb{R}^n$ for $n \geq 3$ is any element in the set:

$$\wedge^3\mathbb{R}^n = \text{span}\{ \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \mid \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n \}$$

Trivectors are said to have *grade* 3.

8.2.5 Multivectors

We abstract even further to allow sums of various of the former.

Definition 8.2.5.1. A *multivector* is defined as an abstract sum of scalars, vectors, bivectors and (in $\mathbb{R}^3$) trivectors. Denote the set of multivectors in $\mathbb{R}^n$ by $\wedge \mathbb{R}^n$. Thus $\wedge \mathbb{R}^2$ contains abstract sums of scalars, vectors and bivectors, and $\wedge \mathbb{R}^3$ can include trivectors in the sum as well.

Example 8.3. Here are some examples:

(a) $A = 2 + \mathbf{a}$ is a multivector.
(b) $B = 3 + 3\mathbf{e}_1 + (\mathbf{e}_2 \wedge 3\mathbf{e}_3)$ is a multivector.
(c) $C = 2 + \mathbf{a} + 3\mathbf{b} + (\mathbf{a} \wedge \mathbf{b})$ is a multivector where $\mathbf{a}$ and $\mathbf{b}$ are any vectors.
(d) $D = 4\mathbf{a} + \mathbf{b}$ is a multivector where $\mathbf{a}$ and $\mathbf{b}$ are any vectors.

8.2.6 The Inner Product

Definition 8.2.6.1. Given two vectors $\mathbf{a}$ and $\mathbf{b}$ we define the *inner product* (scalar product, dot product):

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| ||\mathbf{b}|| \cos \theta$$

The inner product produces a scalar thereby dropping us from vectors down to scalars.

Exercise 8.1. Suppose $\mathbf{a}$ and $\mathbf{b}$ have lengths 7 and 10 respectively and meet at angle of $\pi/3$. Find their inner product.

Theorem 8.2.6.1. If $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$ then:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$
8.2. SCALARS, VECTORS, BIVECTORS, TRIVECTORS

Proof. From the Law of Cosines we have:

\[ |b - a|^2 = |a|^2 + |b|^2 - 2|a||b|\cos \theta \]
\[ (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2 = a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2a \cdot b \]

The result follows after cancellation. \(\square\)

Practical Calculation 8.1. If you are given \(a\) and \(b\), to calculate \(a \cdot b\) write \(a\) and \(b\) as linear combinations of \(e_i\) and apply the above theorem. The result will be a scalar.

Example 8.4. If \(a = 2e_1 + 3e_2\) and \(b = 5e_1 - 1e_2 + 17e_3\) then:
\[ a \cdot b = (2)(5) + (3)(-1) + (0)(17) = 7 \]

Definition 8.2.6.2. We say that two nonzero vectors \(a\) and \(b\) are perpendicular if the angle between them is \(\pi/2\), which is equivalent to saying that \(a \cdot b = 0\).

8.2.7 The Outer Product \(a \wedge b\) Revisited

For vectors \(a\) and \(b\) we defined the outer product \(a \wedge b\) as the oriented parallelogram with sides \(a\) and \(b\). What we will do next is introduce some axioms which make geometric sense but have possibly unexpected consequences.

Axiom 8.2.7.1. We insist on the following axioms. For each we have noted what these mean geometrically. Consider that they make geometric sense.

(a) \(a \wedge (b \pm c) = (a \wedge b) \pm (a \wedge c)\)
   The meaning of the + version is that the sum of the two parallelograms which meet along the edge \(a\) equals the resulting large parallelogram.

(b) \((b \pm c) \wedge a = (b \wedge a) \pm (c \wedge a)\)
   The meaning is the same as the above.

(c) \(\alpha(a \wedge b) = (\alpha a) \wedge b = a \wedge (\alpha b)\)
   The meaning is that scaling an entire parallelogram by \(\alpha\) is equivalent to scaling just one side by \(\alpha\).

(d) \(b \wedge a = -(a \wedge b)\)
   The meaning is that flipping the order of the sides of the parallelogram reverses the orientation.
   Note that from (d) we get the special case \(a \wedge a = 0\) for any \(a\).

Here is a picture which illustrates the geometric meaning of the addition version of (a) in that the large parallelogram with sides \(a\) and \(b + c\) equals the sum of
the small parallelograms with sides \( \mathbf{a} \) and \( \mathbf{b} \) for the left one and sides \( \mathbf{a} \) and \( \mathbf{c} \) for the right one.

As a consequence of these axioms it turns out that all outer products may be rewritten as linear combinations of various outer products of basis vectors.

**Practical Calculation 8.2.** To calculate the outer product of two vectors use these axioms and organize the result as a linear combination of various outer products of basis vectors.

**Example 8.5.** If \( \mathbf{a} = 5\mathbf{e}_1 + 2\mathbf{e}_2 \) and \( \mathbf{b} = \mathbf{e}_1 + 4\mathbf{e}_2 \) then

\[
\mathbf{a} \land \mathbf{b} = (5\mathbf{e}_1 + 2\mathbf{e}_2) \land (\mathbf{e}_1 + 4\mathbf{e}_2)
\]

\[
= ((5\mathbf{e}_1 + 2\mathbf{e}_2) \land (\mathbf{e}_1)) + ((5\mathbf{e}_1 + 2\mathbf{e}_2) \land (4\mathbf{e}_2))
\]

\[
= 5(\mathbf{e}_1 \land \mathbf{e}_1) + 2(\mathbf{e}_2 \land \mathbf{e}_1) + 20(\mathbf{e}_1 \land \mathbf{e}_2) + 8(\mathbf{e}_2 \land \mathbf{e}_2)
\]

\[
= 5(0) - 2(\mathbf{e}_1 \land \mathbf{e}_2) + 20(\mathbf{e}_1 \land \mathbf{e}_2) + 8(0)
\]

\[
= 18(\mathbf{e}_1 \land \mathbf{e}_2)
\]

Consider that this example states that the oriented parallelogram formed by \( \mathbf{a} \) and \( \mathbf{b} \) equals 18 oriented squares.
8.2. SCALARS, VECTORS, BIVECTORS, TRIVECTORS

Example 8.6. If \( a = 2e_1 - 3e_2 + 4e_3 \) and \( b = e_1 - 7e_2 \) then

\[
\begin{align*}
\mathbf{a} \wedge \mathbf{b} &= (2e_1 - 3e_2 + 4e_3) \wedge (e_1 - 7e_2) \\
&= 2e_1 \wedge e_1 - 14e_1 \wedge e_2 + 3e_2 \wedge e_1 + 12e_2 \wedge e_2 + 4e_3 \wedge e_1 - 28e_2 \wedge e_2 \\
&= 2(0) - 13e_1 \wedge e_2 + 3e_2 \wedge e_1 + 12(0) + 4e_3 \wedge e_1 - 28(0) \\
&= -10e_1 \wedge e_2 + 4e_3 \wedge e_1
\end{align*}
\]

Exercise 8.2. Let \( a = 2e_1 + 3e_2 \) and \( b = 6e_1 - 7e_2 \). Calculate \( \mathbf{a} \wedge \mathbf{b} \) and \( \mathbf{b} \wedge \mathbf{a} \).

Exercise 8.3. Let \( a = 2e_1 + 3e_2 + 8e_3 \) and \( b = -7e_1 - 3e_2 - 2e_3 \). Calculate \( \mathbf{a} \wedge \mathbf{b} \) and \( \mathbf{b} \wedge \mathbf{a} \).

It’s evident as mentioned earlier that every outer product of two vectors generates a linear combination of various \( e_i \wedge e_j \).

However we can be more formal and note:

Theorem 8.2.7.1. In \( \mathbb{R}^2 \) the outer product of any two vectors equals a multiple of \( e_1 \wedge e_2 \). More specifically if \( \mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 \) and \( \mathbf{b} = b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 \) and if \( \theta \) is the angle from \( \mathbf{a} \) to \( \mathbf{b} \) then

\[
\mathbf{a} \wedge \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin \theta (e_1 \wedge e_2)
\]

Since \( |\mathbf{a}||\mathbf{b}| \sin \theta \) is the area of the parallelogram that \( \mathbf{a} \) and \( \mathbf{b} \) create, this theorem basically states that in \( \mathbb{R}^2 \) each parallelogram equals the number of unit squares equal to its area.

Proof. We have:

\[
\mathbf{a} \wedge \mathbf{b} = (a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2) \wedge (b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2)
\]

\[
= a_1b_1(\mathbf{e}_1 \wedge \mathbf{e}_1) + a_1b_2(\mathbf{e}_1 \wedge \mathbf{e}_2) + a_2b_1(\mathbf{e}_2 \wedge \mathbf{e}_1) + a_2b_2(\mathbf{e}_2 \wedge \mathbf{e}_2)
\]

\[
= a_1b_1(0) + +a_1b_2(\mathbf{e}_1 \wedge \mathbf{e}_2) + a_2b_1(-\mathbf{e}_1 \wedge \mathbf{e}_2) + a_2b_2(0)
\]

\[
= (a_1b_2 - a_2b_1)(\mathbf{e}_1 \wedge \mathbf{e}_2)
\]

\[
= \begin{vmatrix}
a_1 & a_2 \\
b_1 & b_2
\end{vmatrix} (\mathbf{e}_1 \wedge \mathbf{e}_2)
\]

\[
= |\mathbf{a}||\mathbf{b}| \sin \theta (e_1 \wedge e_2)
\]

\( \square \)

And:

Theorem 8.2.7.2. In \( \mathbb{R}^3 \) the outer product of any two vectors equals a linear combination of \( e_1 \wedge e_2, e_2 \wedge e_3, \) and \( e_3 \wedge e_1 \). More specifically if \( \mathbf{a} = a_1 \mathbf{e}_1 + \)
\[ a_2 e_2 + a_3 e_3 \text{ and } b = b_1 e_1 + b_2 e_2 + b_3 e_3 \text{ then} \]

\[ a \wedge b = (a_1 b_2 - a_2 b_1)(e_1 \wedge e_2) + (a_2 b_3 - a_3 b_2)(e_2 \wedge e_3) + (a_3 b_1 - a_1 b_3)(e_3 \wedge e_1) \]

\[
\begin{vmatrix}
 a_1 & a_2 \\
 b_1 & b_2 \\
\end{vmatrix}
\begin{vmatrix}
 e_1 & e_2 \\
\end{vmatrix}
+ \begin{vmatrix}
 a_2 & a_3 \\
 b_2 & b_3 \\
\end{vmatrix}
\begin{vmatrix}
 e_2 & e_3 \\
\end{vmatrix}
+ \begin{vmatrix}
 a_3 & a_1 \\
 b_3 & b_1 \\
\end{vmatrix}
\begin{vmatrix}
 e_3 & e_1 \\
\end{vmatrix}
\]

**Proof.** Omitted. \(\square\)

Warning: This looks a bit like a cross product, and is related to a cross product, but isn’t a cross product!

**Example 8.7.** For the vectors:

\[ a = 2e_1 + 3e_2 + 4e_3 \]
\[ b = 7e_1 + 1e_2 + 5e_3 \]

We have:

\[
 a \wedge b = \begin{vmatrix}
 2 & 3 \\
 7 & 1 \\
\end{vmatrix}
\begin{vmatrix}
 e_1 \wedge e_2 \\
\end{vmatrix}
+ \begin{vmatrix}
 3 & 4 \\
 1 & 5 \\
\end{vmatrix}
\begin{vmatrix}
 e_2 \wedge e_3 \\
\end{vmatrix}
+ \begin{vmatrix}
 4 & 2 \\
 5 & 7 \\
\end{vmatrix}
\begin{vmatrix}
 e_3 \wedge e_1 \\
\end{vmatrix}
\]

\[ = -19(e_1 \wedge e_2) + 11(e_2 \wedge e_3) + 18(e_3 \wedge e_1) \]

### 8.2.8 The Outer Product \(a \wedge b \wedge c\) Revisited

For vectors \(a, b\) and \(c\) we defined the outer product \(a \wedge b \wedge c\) as the oriented parallelepiped with sides \(a, b\) and \(c\). What we will do next is introduce some axioms which make geometric sense but have unexpected consequences. These axioms are similar to those for the outer product of two vectors so we’ve been less verbose.

**Axiom 8.2.8.1.** We insist on the following axioms.

(a) The outer product distributes over addition and subtraction.

(b) Multiplication by a scalar is associative.

(c) Interchanging two adjacent terms negates the outer product of three vectors, for example \(a \wedge b \wedge c = -a \wedge c \wedge b\).

Invoking the same axioms above we extend the triple outer product to arbitrary vectors and we get the following in \(\mathbb{R}^3\).

**Theorem 8.2.8.1.** In \(\mathbb{R}^3\) the outer product of any three vectors equals a multiple of \(e_1 \wedge e_2 \wedge e_3\). More formally if \(a = a_1 e_1 + a_2 e_2 + a_3 e_3\) and
8.3. THE GEOMETRIC PRODUCT

\[ b = b_1e_1 + b_2e_2 + b_3e_3 \text{ and } c = c_1e_1 + c_2e_2 + c_3e_3 \text{ then} \]

\[
\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \begin{vmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3 \\
\end{vmatrix} (e_1 \wedge e_2 \wedge e_3)
\]

\textbf{Proof.} Omitted. \hfill \Box

\textbf{Practical Calculation 8.3.} Given vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \), to calculate \( \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \) write \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) as linear combination of \( e_i \) and apply the above axioms. The result will be a linear combination of various \( e_i \wedge e_j \wedge e_k \).

\textbf{Exercise 8.4.} If \( \mathbf{a} = 3e_2 - 2e_2 + 0e_3, \mathbf{b} = 1e_1 + 2e_2 + 4e_3, \) and \( \mathbf{c} = 6e_1 - 1e_2 + 7e_3 \) calculate all possible triple outer products.

Hint: Up to interchange/sign how many are there?

8.2.9 The Outer Products \( \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) \) and \( (\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} \)

Although this looks like a parenthetical hiccup from \( \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \) they do not initially mean the same thing. In fact the expressions:

\[ \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) \text{ and } (\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} \]

have no meaning until we give them meaning. After all, we have no definition of the outer product of a vector like \( \mathbf{a} \) and a parallelogram like \( \mathbf{b} \wedge \mathbf{c} \).

So what we’ll do is insist on equality and associativity and state that:

\[ \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} \]

This actually makes sense. For example given that \( \mathbf{b} \wedge \mathbf{c} \) is an oriented parallelogram we can understand that \( \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) \) is the parallelopipid obtained by extending this parallelogram along the vector \( \mathbf{a} \), which is simply the parallelogram \( \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \).

8.3 The Geometric Product

8.3.1 The Geometric Product of Vectors

\textbf{Definition 8.3.1.1.} Given vectors \( \mathbf{a} \) and \( \mathbf{b} \) we define:

\[ \mathbf{a} \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \]
The result of this is a multivector consisting of a scalar plus a bivector.

Before proceeding the following will be critical:

**Theorem 8.3.1.1.** We have:
(a) For \(i\) we have:
\[
e_i e_i = e_i \cdot e_i + e_i \wedge e_i = 1 + 0 = 1
\]
(b) For \(i \neq j\) we have:
\[
e_i e_j = e_i \cdot e_j + e_i \wedge e_j = 0 + e_i \wedge e_j = e_i \wedge e_j
\] (A 1 × 1 square)
(c) And then it follows that for \(i \neq j\) that:
\[
e_i e_j = e_i \wedge e_j = -e_j \wedge e_i = -e_j e_i
\]
(d) If \(\mathbf{u}\) is any unit vector then:
\[
\mathbf{u} \mathbf{u} = \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \wedge \mathbf{u} = 1 + 0 = 1
\]

**Proof.** The proofs are in the statements. \(\square\)

**Practical Calculation 8.4.** It follows that if \(i \neq j\) then \(e_i \wedge e_j = e_j e_i\) and if \(i = j\) then \(e_i \wedge e_i = 0\). We will always replace outer products of \(e_i\) using these shorthand versions.

More precisely it follows that:

**Theorem 8.3.1.2.** We have:
\[
\wedge^0 \mathbb{R}^2 = \text{span} \{1\}
\]
\[
\wedge^1 \mathbb{R}^2 = \text{span} \{e_1, e_2\}
\]
\[
\wedge^2 \mathbb{R}^2 = \text{span} \{e_1 e_2\}
\]
\[
\wedge \mathbb{R}^2 = \wedge^0 \mathbb{R}^2 \cup \wedge^1 \mathbb{R}^2 \cup \wedge^2 \mathbb{R}^2
\]

And we have:
\[
\wedge^0 \mathbb{R}^3 = \text{span} \{1\}
\]
\[
\wedge^1 \mathbb{R}^3 = \text{span} \{e_1, e_2, e_3\}
\]
\[
\wedge^2 \mathbb{R}^3 = \text{span} \{e_1 e_2, e_2 e_3, e_3 e_1\}
\]
\[
\wedge^3 \mathbb{R}^3 = \text{span} \{e_1 e_2 e_3\}
\]
\[
\wedge \mathbb{R}^3 = \wedge^0 \mathbb{R}^3 \cup \wedge^1 \mathbb{R}^3 \cup \wedge^2 \mathbb{R}^3 \cup \wedge^3 \mathbb{R}^3
\]
Example 8.8. If \( a = 2e_1 - 3e_2 + 4e_3 \) and \( b = e_1 - 7e_2 \) then
\[
a \wedge b = (2e_1 - 3e_2 + 4e_3) \wedge (e_1 - 7e_2) \\
= 2e_1 \wedge e_1 - 14e_1 \wedge e_2 - 3e_2 \wedge e_1 + 21e_2 \wedge e_2 + 4e_3 \wedge e_1 - 28e_3 \wedge e_2 \\
= 2(0) - 14e_1 \wedge e_2 + 3e_1 \wedge e_2 + 21(0) + 4e_3 \wedge e_1 + 28e_2 \wedge e_3 \\
= -11e_1e_2 + 28e_2e_3 + 4e_3e_1
\]

8.3.2 The Geometric Product of Multivectors

We extend the geometric product to \( k \)-vectors and in general to Multivectors by requiring associativity and distributivity over addition and subtraction.

Practical Calculation 8.5. To calculate the geometric product of two \( k \)-vectors or multivectors apply these rules to reduce the problem to geometric products of \( k \)-vectors and then proceed using this requirement. The result will be a multivector.

Here is an example with a 1-vector and a 2-vector:

Example 8.9. Suppose \( A = 2e_1 + e_3 \) and \( B = 3e_1e_2 - 5e_2e_3 \) then:
\[
AB = (2e_1 + e_3)(3e_1e_2 - 5e_2e_3) \\
= 6e_1e_1e_2 - 10e_1e_2e_3 + 3e_3e_1e_2 - 5e_3e_2e_3 \\
= 6e_2 - 10e_1e_2e_3 + 3e_1e_2e_3 + 5e_2 \\
= 11e_2 - 7e_1e_2e_3
\]

Here is an example with two multivectors:

Example 8.10. If \( A = 2 + 3e_1 + e_1e_2 \) and \( B = 4 + e_1 - 2e_2 \) then:
\[
AB = (2 + 3e_1 + e_1e_2)(4 + e_1 - 2e_2) \\
= 2(4 + e_1 - 2e_2) + 3e_1(4 + e_1 - 2e_2) + e_1e_2(4 + e_1 - 2e_2) \\
= 8 + 2e_1 - 4e_2 + 12e_1 + 3e_1e_1 - 6e_1e_2 + 4e_1e_2 + e_1e_2e_1 - 2e_1e_2e_2 \\
= 8 + 2e_1 - 4e_2 + 12e_1 + 3 - 6e_1e_2 + 4e_1e_2e_2 - e_2 - 2e_1 \\
= 8 + 12e_1 - 5e_2 - 2e_1e_2
\]

Here is an example where we have to do a lot more initial work. We are given bivectors \( A \) and \( B \) and we have to rewrite them first:

Example 8.11. Suppose \( A = (2e_1 + 3e_2) \wedge (e_2 - e_3) \) and \( B = 4e_1 \wedge (e_2 + 2e_3) \).
Before proceeding with any calculations we write:

\[ A = 2e_1 \wedge e_2 - 2e_1 \wedge e_3 + 3e_2 \wedge e_2 - 3e_2 \wedge e_3 \]
\[ = 2e_1 e_2 - 2e_1 e_3 + 3(0) - 3e_2 e_3 \]
\[ = 2e_1 e_2 + 2e_3 e_1 - 3e_2 e_3 \]
\[ = 2e_1 e_2 - 3e_2 e_3 + 2e_3 e_1 \]

And we write:

\[ B = 4e_1 \wedge e_2 + 8e_1 \wedge e_3 \]
\[ = 4e_1 e_2 - 8e_3 e_1 \]

Then for example we can do \( AB \) easily:

\[ AB = (2e_1 e_2 - 3e_2 e_3 + 2e_3 e_1)(4e_1 e_2 - 8e_3 e_1) \]
\[ = 2e_1 e_2(4e_1 e_2 - 8e_3 e_1) - 3e_2 e_3(4e_1 e_2 - 8e_3 e_1) + 2e_3 e_1(4e_1 e_2 - 8e_3 e_1) \]
\[ = 8e_1 e_2 e_1 e_2 - 16e_1 e_2 e_3 e_1 - 12e_2 e_3 e_1 e_2 + 24e_2 e_3 e_3 e_1 + 8e_3 e_1 e_1 e_2 - 16e_3 e_1 e_1 e_3 \]
\[ = -8 - 16e_2 e_3 - 12e_3 e_1 - 24e_1 e_2 - 8e_2 e_3 - 16 \]
\[ = -24 - 24e_1 e_2 - 24e_2 e_3 - 12e_3 e_1 \]

**Exercise 8.5.** Let \( A = 4 + e_1 e_3 \) and \( B = 3e_1 + e_2 \). Calculate \( AB \) and \( BA \).

**Exercise 8.6.** Let \( A = 4 - 1e_3 + 2e_1 e_2 + 3e_1 e_3 \) and \( B = 5 + 3e_1 + e_2 e_3 + e_1 e_2 e_3 \). Calculate \( AB \) and \( BA \).

### 8.3.3 Properties of the Geometric Product

We can now in fact rewrite the inner and outer products of vectors in terms of the geometric product:

**Theorem 8.3.3.1.** For vectors \( a \) and \( b \) we have:

(a) \( a \cdot b = \frac{1}{2}(ab + ba) \)

(b) \( a \wedge b = \frac{1}{2}(ab - ba) \)

**Proof.** This follows immediately from:

(a) \( ab = a \cdot b + a \wedge b \)

(b) \( ba = b \cdot a + b \wedge a = a \cdot b - a \wedge b \)

\( \square \)
Theorem 8.3.3.2. Note that when we take the geometric product of multivectors the maximum grade is limited by the ambient space. More specifically in $\mathbb{R}^2$ the maximum result will be dimension 2 and in $\mathbb{R}^3$ it will be dimension 3.

Proof. This is clear in terms of what happens to the geometric products of the building blocks in standard form. 

Example 8.12. Consider:
(a) In $\mathbb{R}^2$ we have $(5\mathbf{e}_1)(7\mathbf{e}_2) = 35\mathbf{e}_1\mathbf{e}_2$.
(b) In $\mathbb{R}^2$ we have $(5\mathbf{e}_2)(7\mathbf{e}_2) = 35$.
(c) In $\mathbb{R}^2$ we have $(5\mathbf{e}_1)(7\mathbf{e}_2\mathbf{e}_1) = -35\mathbf{e}_2$.
(d) In $\mathbb{R}^3$ we have $(2\mathbf{e}_1)(6\mathbf{e}_3\mathbf{e}_2) = -12\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$.
(e) In $\mathbb{R}^3$ we have $(2\mathbf{e}_1\mathbf{e}_3)(6\mathbf{e}_1\mathbf{e}_3\mathbf{e}_2) = 12\mathbf{e}_2$.

8.4 Additional Definitions

8.4.1 Blades

Definition 8.4.1.1. A $k$-blade $B$ is the outer product of $k$ vectors.

Example 8.13. The following are blades:
(a) $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ is a 3-blade.
(b) $\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2$ is a 2-blade.
(c) $\mathbf{a}$ is a 1-blade.
(d) 7 is a 0-blade.

Given $\mathbf{v}_1, ..., \mathbf{v}_k$ with $\mathbf{v}_i \in \mathbb{R}^n$ the $k$-blade $B = \mathbf{v}_1 \wedge ... \wedge \mathbf{v}_k$ can be thought of as representing the subspace: $\text{span} \{\mathbf{v}_1, ..., \mathbf{v}_k\}$.

Example 8.14. In $\mathbb{R}^3$ the blade $\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2$ can be thought of as representing the subspace: $\text{span} \{\mathbf{e}_1, \mathbf{e}_2\}$.

Keep in mind that the blade is not equal to the subspace but rather that it can be thought of as representing one. The reason for packaging the vectors together in an outer product rather than just as a set is that it creates a single entity.

Note that any given subspace is represented by many different blades.
Example 8.15. In $\mathbb{R}^3$ the two blades $B_1 = e_1 \wedge e_2$ and $B_2 = (e_1 + e_2) \wedge (e_1 - e_2)$ represent the same subspace of $\mathbb{R}^3$, that being the set span$\{e_1, e_2\}$.

Theorem 8.4.1.1. In $\mathbb{R}^2$ and $\mathbb{R}^3$ all bivectors are 2-blades.

Take a moment to appreciate what this theorem states, primarily in $\mathbb{R}^3$. It states that a linear combination of outer products of two vectors equals a single outer product of two vectors. This will be extremely useful when we are proving things about bivectors.

Proof. In $\mathbb{R}^2$ observe that all bivectors have the form $\lambda e_1 e_2$ for $\lambda \in \mathbb{R}$ and note that:
\[
\lambda e_1 e_2 = \lambda (e_1 \wedge e_2) = ((\lambda e_1) \wedge e_2)
\]
and we’re done.

In $\mathbb{R}^3$ we know every bivector has the form $\alpha e_1 e_2 + \beta e_2 e_3 + \gamma e_3 e_1$ for $\alpha, \beta, \gamma \in \mathbb{R}$.

If $\beta = \gamma = 0$ then we’re done. Otherwise $\beta^2 + \gamma^2 \neq 0$ and observe that if we assign:
\[
a = \frac{1}{\beta^2 + \gamma^2}(-\alpha \beta e_1 - \alpha \gamma e_2 + (\beta^2 + \gamma^2)e_3) \\
b = \gamma e_1 - \beta e_2 + 0e_3
\]
Then we have:
\[
a \wedge b = \frac{1}{\beta^2 + \gamma^2}(-\alpha \beta e_1 - \alpha \gamma e_2 + (\beta^2 + \gamma^2)e_3) \wedge (\gamma e_1 - \beta e_2 + 0e_3) \\
= \frac{1}{\beta^2 + \gamma^2}[(\alpha \beta^2 + \alpha \gamma^2)e_1 e_2 + (\beta(\beta^2 + \gamma^2))e_2 e_3 + (\gamma(\beta^2 + \gamma^2))e_3 e_1] \\
= \alpha e_1 e_2 + \beta e_2 e_3 + \gamma e_3 e_1
\]

Note that it’s fairly clear from the nature of the proof that there are multiple ways to choose/construct $a$ and $b$.

The fact that the bivector is a 2-blade follows immediately from the definition of a 2-blade.

\[\square\]

In friendlier terms in $\mathbb{R}^3$ bivectors are the same as parallelograms and every linear combination of parallelograms equals a parallelogram.

This is computationally important because it allows us to use the term bivector and the expression $a \wedge b$ interchangeably. We will do this constantly.
8.4. ADDITIONAL DEFINITIONS

Exercise 8.7. Write the bivector $B = 2e_1e_2 + 5e_2e_3$ as a 2-blade.

Exercise 8.8. Write the bivector $B = 2e_1e_2 + 5e_2e_3 - 3e_3e_1$ as a 2-blade.

Note: $\mathbb{R}^3$ is the highest dimension in which this is true. For example in $\mathbb{R}^4$ the bivector $e_1e_2 + e_3e_4$ cannot be written as a blade. This is not obvious.

8.4.2 Norms

Definition 8.4.2.1. The norm of a vector $a$, denoted $|a|$, is simply the length of the vector, hence the square root of the sum of the squares of the coefficients.

Example 8.16. If $a = 2e_1 + 3e_2 + 5e_3$ then

$$|a| = \sqrt{4 + 9 + 25} = \sqrt{38}$$

Definition 8.4.2.2. The norm of a multivector $A$ is defined as the square root of the sum of the squares of the coefficients of the basis vectors making up $A$.

Example 8.17. If $A = 2 + 3e_1 + 4e_2 - 5e_1e_2$ then:

$$|A| = \sqrt{4 + 9 + 16 + 25} = \sqrt{54}$$

If a multivector is not written this way then we must expand it first.

8.4.3 Grade Extraction from a Multivector

Definition 8.4.3.1. For a multivector $A$ we use the notation $\langle A \rangle_k$ to denote the extraction of the $k$-vector from $A$. If $k < 0$ the result will be 0.

Example 8.18. If $A = 2 + 3e_1 - 1e_2 + 5e_1e_2 + 3e_1e_3 + 7e_1e_2e_3$ then:

(a) $\langle A \rangle_0 = 2$

(b) $\langle A \rangle_1 = 3e_1 - 1e_2$

(c) $\langle A \rangle_2 = 5e_1e_2 + 3e_1e_3$

(d) $\langle A \rangle_3 = 7e_1e_2e_3$

(e) $\langle A \rangle_4 = \langle A \rangle_5 = \ldots = 0$

(f) $\langle A \rangle_{-1} = \langle A \rangle_{-2} = \ldots = 0$
8.4.4 Reversion

Computation using multivectors (specifically blades) is something we will be doing. It is made easier by the introduction of a concept called reversion.

Definition 8.4.4.1. We define *reversion*, denoted by \(\dagger\) (sometimes by \(\sim\)), by the following rules:

(a) \(\alpha^\dagger = \alpha\)
(b) \(a^\dagger = a\)
(c) \((AB)^\dagger = B^\dagger A^\dagger\)
(d) \((A + B)^\dagger = A^\dagger + B^\dagger\)

Practical Calculation 8.6. To calculate \(A^\dagger\) use the formulas above.

Example 8.19. For vectors \(a\), \(b\) and \(c\) we have:

\[(ab + c)^\dagger = c + ba\]

Example 8.20. Consider \(B = 2e_1 \wedge (e_2 + 6e_3)\). We rewrite the original \(B\) as \(B = 2e_1e_2 + 12e_1e_3\) and then:

\[B^\dagger = (2e_1e_2)^\dagger + (12e_1e_3)^\dagger\]
\[= 2e_2e_1 + 12e_3e_1\]
\[= -2e_1e_2 - 12e_1e_3\]

Exercise 8.9. Find the reversion of each of the following:

(a) \(abc + b + ac\)
(b) \(5e_1e_2e_3 - 2e_1e_2\)
(c) \(5e_2 \wedge (e_1 + 6e_3)\)

8.4.5 Inversion

Definition 8.4.5.1. We say that a multivector \(A\) is *invertible* if there is some multivector \(B\) with \(AB = BA = 1\). We write \(A^{-1}\) in place of \(B\). In this case the multivectors are the *inverse* of one another.

Finding inverses of nonzero vectors is easy:

Theorem 8.4.5.1. Every nonzero vector is invertible and its inverse is \(v/|v|^2\).
Proof. Suppose \( \mathbf{v} \neq 0 \). Then observe that:

\[
\mathbf{vv} = \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2
\]

\[
\mathbf{v} \left( \frac{\mathbf{v}}{|\mathbf{v}|^2} \right) = 1
\]

From here we can see the inverse of every nonzero vector.

For multivectors which are not vectors it’s not so simple but a theorem will help us with the situations we encounter:

**Theorem 8.4.5.2.** For a geometric product \( \mathbf{A} = \mathbf{a}_1...\mathbf{a}_k \) we have:

\[
\mathbf{A}^{-1} = \frac{\mathbf{A}^\dagger}{\mathbf{AA}^\dagger}
\]

Proof. Observe that:

\[
\mathbf{A} \left( \frac{\mathbf{A}^\dagger}{\mathbf{AA}^\dagger} \right) = \mathbf{A} \left( \frac{\mathbf{A}^\dagger}{\mathbf{a}_1...\mathbf{a}_k\mathbf{a}_k...\mathbf{a}_1} \right)
\]

\[
= \mathbf{A} \left( \frac{\mathbf{A}^\dagger}{|\mathbf{a}_1|^2...|\mathbf{a}_k|^2} \right)
\]

\[
= \frac{\mathbf{A}\mathbf{A}^\dagger}{|\mathbf{a}_1|^2...|\mathbf{a}_k|^2}
\]

\[
= \frac{|\mathbf{a}_1|^2...|\mathbf{a}_k|^2}{|\mathbf{a}_1|^2...|\mathbf{a}_k|^2}
\]

\[
= 1
\]

Note that this is an odd proof. The point is that \( \mathbf{AA}^\dagger \) is a scalar which allows us to do the manipulation.

A similar argument shows that this inverse works on the left as well.

**Corollary 8.4.5.1.** For a bivector \( \mathbf{B} \) we have:

\[
\mathbf{B}^{-1} = -\frac{\mathbf{B}}{|\mathbf{B}|^2}
\]

Proof. Suppose \( \mathbf{B} = \mathbf{b}_1 \wedge \mathbf{b}_2 \). Then \( \mathbf{B} = |\mathbf{B}|\mathbf{b}_1^1 \mathbf{b}_2^2 \) where \( \mathbf{b}_1^1 \) and \( \mathbf{b}_2^2 \) form an
orthonormal basis for the subspace spanned by $b_1$ and $b_2$. Then:

$$B^{-1} = (b_1 \wedge b_2)^{-1}$$

$$= (|B|b_1'b_2')^{-1}$$

$$= \frac{(|B|b_1'b_2')^\dagger}{(|B|b_1'b_2')(|B|b_1'b_2')^\dagger}$$

$$= \frac{1}{|B|}{b_2'b_1'}b_1b_2b_2b_1$$

$$= \frac{1}{|B|}{-b_1'b_2'}$$

$$= \frac{-B}{|B|}$$

$$= -\frac{B}{|B|^2}$$

If these two proofs are frustratingly long here is another proof of the corollary which is far simpler:

Proof. Write $B = \alpha e_1 e_2 + \beta e_2 e_3 + \gamma e_3 e_1$ then calculate:

$$BB = \ldots \text{Calculate...} = -\alpha^2 - \beta^2 - \gamma^2 = -|B|^2$$

The result follows.

Practical Calculation 8.7. To calculate the inverse use either formula as appropriate. In the case of a bivector definitely use the second!

Example 8.21. Consider $B = 2e_1 \wedge (e_2 + 6e_3)$. We rewrite $B = 2e_1 e_2 + 12e_1 e_3$ and then we have:

$$B^{-1} = -\frac{B}{|B|^2}$$

$$= -\frac{2e_1 e_2 + 12e_1 e_3}{|2e_1 e_2 + 12e_1 e_3|}$$

$$= -\frac{2e_1 e_2 - 12e_1 e_3}{(2)^2 + (12)^2} = -\frac{2e_1 e_2 - 12e_1 e_3}{148}$$

Exercise 8.10. Find the inverse of each of the following multivectors:

(a) $5e_1 e_2$

(b) $(2e_1 + 3e_2) \wedge (-5e_1 + 4e_3)$
8.4.6 Inner and Outer Products of Multivectors

We defined the geometric product of multivectors by extending the geometric product of vectors which itself was based on the definition of the inner and outer products of vectors.

What we will do now is use the geometric product of multivectors to define the inner and outer products of multivectors.

**Definition 8.4.6.1.** If $A$ is a $j$-multivector and $B$ is a $k$-multivector then:

$$A \cdot B = \langle AB \rangle_{k-j}$$

$$A \wedge B = \langle AB \rangle_{k+j}$$

It’s critical to note that the dot product of two multivectors does not necessarily yield a constant. Rather it is a grade-lowering operation. Similarly the outer product is a grade-raising operation.

**Practical Calculation 8.8.** To calculate either of these use the definition.

**Example 8.22.** Consider $a = 7e_1 - 5e_3$ and $B = 2e_1e_2 + 12e_1e_3$. Since $a$ is a $j = 1$-vector and $B$ is a $k = 2$ vector it follows that $a \cdot B$ is a $2-1 = 1$-vector and $a \wedge B$ is a $2+1 = 3$-vector. We have:

$$aB = (7e_1 - 5e_3)(2e_1e_2 + 12e_1e_3)$$

$$= 14e_1e_1e_2 + 84e_1e_1e_3 - 10e_3e_1e_2 - 60e_3e_1e_3$$

$$= 14e_2 + 84e_3 - 10e_1e_2e_3 + 60e_1$$

$$= 60e_1 + 14e_2 + 84e_3 - 10e_1e_2e_3$$

Hence we have:

$$a \cdot B = \langle 60e_1 + 14e_2 + 84e_3 - 10e_1e_2e_3 \rangle_1 = 60e_1 + 14e_2 + 84e_3$$

and

$$a \wedge B = \langle 60e_1 + 14e_2 + 84e_3 - 10e_1e_2e_3 \rangle_3 = -10e_1e_2e_3$$

**Exercise 8.11.** Given $a = 2e_1 + 5e_2$ and $B = -4e_3e_1 + e_1e_2$, find $aB$, $Ba$, $a \cdot B$, $a \wedge B$, $B \cdot a$, and $B \wedge a$.

First note that we must ensure that this is compatible with these products for vectors as given initially.
Theorem 8.4.6.1. If \( \mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 \) and \( \mathbf{b} = b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3 \) then:

\[
\langle \mathbf{a} \mathbf{b} \rangle_0 = a_1 b_1 + a_2 b_2 + a_3 b_3
\]
\[
\langle \mathbf{a} \mathbf{b} \rangle_2 = (a_1 b_2 - a_2 b_1) (\mathbf{e}_1 \mathbf{e}_2) + (a_2 b_3 - a_3 b_2) (\mathbf{e}_2 \mathbf{e}_3) + (a_3 b_1 - a_1 b_3) (\mathbf{e}_3 \mathbf{e}_1)
\]

Proof. Observe that:

\[
\mathbf{a} \mathbf{b} = (a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3) (b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3)
\]
\[
= a_1 b_1 + a_1 b_2 \mathbf{e}_1 \mathbf{e}_2 - a_1 b_3 \mathbf{e}_3 \mathbf{e}_1
\]
\[- a_2 b_1 \mathbf{e}_1 \mathbf{e}_2 + a_2 b_2 + a_2 b_3 \mathbf{e}_2 \mathbf{e}_3
\]
\[+ a_3 b_1 \mathbf{e}_3 \mathbf{e}_1 - a_3 b_2 \mathbf{e}_2 \mathbf{e}_3 + a_3 b_3
\]
\[= a_1 b_1 + a_1 b_2 + a_2 b_2 + a_3 b_3
\]
\[+ (a_1 b_2 - a_2 b_1) \mathbf{e}_1 \mathbf{e}_2 + (a_2 b_3 - a_3 b_2) \mathbf{e}_2 \mathbf{e}_3 + (a_3 b_1 - a_1 b_3) \mathbf{e}_3 \mathbf{e}_1
\]

And the results follow immediately. \( \square \)

There is one special case of this which will be useful as we proceed. Notice that the following is different (the signs) from the case of two vectors.

Theorem 8.4.6.2. If \( \mathbf{a} \in \wedge^1 \mathbb{R}^3 \) and \( \mathbf{B} \in \wedge^2 \mathbb{R}^3 \) then:

\[
\mathbf{a} \cdot \mathbf{B} = \frac{1}{2} (\mathbf{aB} - \mathbf{Ba})
\]
\[
\mathbf{a} \wedge \mathbf{B} = \frac{1}{2} (\mathbf{aB} + \mathbf{Ba})
\]

Proof. We know that \( \mathbf{B} = \alpha \mathbf{e}_1 \mathbf{e}_2 + \beta \mathbf{e}_2 \mathbf{e}_3 + \gamma \mathbf{e}_3 \mathbf{e}_1 \) for \( \alpha, \beta, \gamma \in \mathbb{R} \). Observe that for \( \mathbf{e}_1 \) we have:

\[
\mathbf{e}_1 \mathbf{B} = \mathbf{e}_1 (\alpha \mathbf{e}_1 \mathbf{e}_2 + \beta \mathbf{e}_2 \mathbf{e}_3 + \gamma \mathbf{e}_3 \mathbf{e}_1) = \alpha \mathbf{e}_2 + \beta \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 - \gamma \mathbf{e}_3
\]
\[
\mathbf{B} \mathbf{e}_1 = (\alpha \mathbf{e}_1 \mathbf{e}_2 + \beta \mathbf{e}_2 \mathbf{e}_3 + \gamma \mathbf{e}_3 \mathbf{e}_1) \mathbf{e}_1 = -\alpha \mathbf{e}_2 + \beta \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 + \gamma \mathbf{e}_3
\]

Then note:

\[
\frac{1}{2} (\mathbf{e}_1 \mathbf{B} - \mathbf{B} \mathbf{e}_1) = \alpha \mathbf{e}_1 - \gamma \mathbf{e}_3 = \langle \mathbf{e}_1 \mathbf{B} \rangle_1 = \mathbf{e}_1 \cdot \mathbf{B}
\]

and:

\[
\frac{1}{2} (\mathbf{e}_1 \mathbf{B} + \mathbf{B} \mathbf{e}_1) = \beta \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 = \langle \mathbf{e}_1 \mathbf{B} \rangle_3 = \mathbf{e}_1 \wedge \mathbf{B}
\]

Similar results hold for \( \mathbf{e}_2 \) and \( \mathbf{e}_3 \), work omitted. Then for a general vector \( \mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 \) we have:

\[
\mathbf{aB} = (a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3) \mathbf{B} = a_1 \mathbf{e}_1 \mathbf{B} + a_2 \mathbf{e}_2 \mathbf{B} + a_3 \mathbf{e}_3 \mathbf{B}
\]
and then:

\[ a \cdot B = \langle aB \rangle_1 \]

\[ = \langle a_1 e_1 B + a_2 e_2 B + a_3 e_3 B \rangle_1 \]

\[ = a_1 \langle e_1 B \rangle_1 + a_2 \langle e_2 B \rangle_1 + a_3 \langle e_3 B \rangle_1 \]

\[ = a_1 \frac{1}{2}(e_1 B - B e_1) + a_2 \frac{1}{2}(e_2 B - B e_2) + a_3 \frac{1}{2}(e_3 B - B e_3) \]

\[ = \frac{1}{2}((a_1 e_1 + a_2 e_2 + a_3 e_3)B - B(a_1 e_1 + a_2 e_2 + a_3 e_3)) \]

\[ = \frac{1}{2}(aB - B a) \]

And similarly for \( a \wedge B \).

**Corollary 8.4.6.1.** It follows immediately (by adding the two) that for a vector \( a \) and a bivector \( B \) that we have:

\[ aB = a \cdot B + a \wedge B \]

**Exercise 8.12.** Show the similar results for \( e_2 \) and \( e_3 \).

### 8.4.7 Dual of a Multivector

**Definition 8.4.7.1.** Given a multivector \( A \) we define the dual of \( A \), denoted \( A^* \), as:

\[ A^* = AI^{-1} \]

where \( I = e_1 e_2 \) so \( I^{-1} = e_2 e_1 \) in \( \mathbb{R}^2 \) and \( I = e_1 e_2 e_3 \) so \( I^{-1} = e_3 e_2 e_1 \) in \( \mathbb{R}^3 \).

**Example 8.23.** For example in \( \mathbb{R}^2 \) if \( A = a_1 e_1 + a_2 e_2 \) then

\[ A^* = (a_1 e_1 + a_2 e_2)e_2 e_1 = a_2 e_1 - a_1 e_2 \]

**Exercise 8.13.** Find the duals of the following multivectors.

(a) \( 2e_3 e_1 + 8e_1 e_2 \)

(b) \( 3 + e_1 + e_2 + e_1 e_2 - e_2 e_3 + 3e_1 e_2 e_3 \)

(c) \( (3e_1 + 2e_3) \wedge (-e_2 + 6e_3) \)

It’s tempting to believe that the dual of a dual yields the original multivector but this is in fact not the case. Rather there are some sign issues that arise. We will only need this case:

**Theorem 8.4.7.1.** In \( \mathbb{R}^3 \) for any multivector \( A \) we have \( (A^*)^* = -A \).
Proof. Observe that for a given $A$ we simply write down the calculation and switch and cancel the various $e_i$:

$$(A^*)^* = (Ae_3e_2e_1)e_3e_2e_1 = (A(-e_1e_2e_3))e_3e_2e_1 = -A$$

The primary use of the dual is in referring to perpendicular subspaces. We have the following:

**Theorem 8.4.7.2.** If $B$ is a blade representing a subspace $U$ of $\mathbb{R}^n$ then $B^*$ represents the perpendicular complement $U^\perp$ of the subspace.

Proof. Omit.

Rather than giving the proof of this in general, we’ll give two specific cases which will be useful to us.

Here’s the case for a bivector.

**Theorem 8.4.7.3.** In $\mathbb{R}^3$ the dual of a bivector $(a \wedge b)^*$ represents the subspace perpendicular to the plane spanned by $a$ and $b$. More explicitly we have:

$$(a \wedge b)^* = a \times b$$

Proof. This is simply calculation. We have:

$$(a \wedge b)^* = \begin{bmatrix}
  a_1 & a_2 & (e_1 \wedge e_2) + a_2 & a_3 & (e_2 \wedge e_3) + a_3 & a_1 & (e_3 \wedge e_1)
\end{bmatrix}^*$$

$$= \begin{bmatrix}
  a_1 & a_2 & e_1e_2 + a_2 & a_3 & e_2e_3 + a_3 & a_1 & e_3e_1
\end{bmatrix}
= \begin{bmatrix}
  a_1 & a_2 & e_3e_2e_1 + a_2 & a_3 & e_2e_3e_1e_1 + a_3 & a_1 & e_3e_1e_3e_1
\end{bmatrix}
= \begin{bmatrix}
  a_1 & a_2 & e_3 + a_2 & a_3 & e_1 + a_3 & a_1 & e_2
\end{bmatrix}
= a \times b$$

The following corollary will be critical when we discuss rotations.

**Corollary 8.4.7.1.** We have:

$$(a \times b)^* = ((a \wedge b)^*)^* = -(a \wedge b)$$

Proof. Immediate.
Here’s the case for a vector.

**Lemma 8.4.7.1.** In $\mathbb{R}^3$ if $a \wedge b = c \wedge d$ then both represent the same subspace.

**Proof.** Since $a \wedge b = c \wedge d$ we have

$$a \times b = (a \wedge b)^* = (c \wedge d)^* = c \times d$$

The result follows.

**Theorem 8.4.7.4.** In $\mathbb{R}^3$ the dual of a vector $v^*$ represents the perpendicular subspace $v^\perp$. In other when $v^*$ is written in the form $a \wedge b$ then all vectors in $\text{span}\{a, b\}$ are perpendicular to $v$.

**Proof.** First observe that if $v = v_1e_1 + v_2e_2 + v_3e_3$ then:

$$v^* = (v_1e_1 + v_2e_2 + v_3e_3)e_3e_2e_1$$

$$= -v_3e_1e_2 - v_1e_2e_3 - v_2e_3e_1$$

Now then we can rewrite this in the form $a \wedge b$ using a previous theorem and using:

$$a = \frac{1}{v_1^2 + v_2^2} (-v_1v_3e_1 - v_2v_3e_2 + (v_1^2 + v_2^2)e_3)$$

$$b = -v_2e_1 + v_1e_2 + 0e_3$$

Since $a \cdot v = b \cdot v = 0$ (just straight calculation), both $a$ and $b$ are perpendicular to $v$ and thus so is every linear combination of $a$ and $b$.

Our specific decomposition of $v^*$ doesn’t matter since by the lemma above any other $c \wedge d$ represents the same subspace. Thus the proof is complete.

The following will be useful in the section on projection and rejection later.

**Theorem 8.4.7.5.** If $B$ is a bivector then $BB^* = B^*B$.

**Proof.** This is just brute force.

8.5 Computation in 3D

8.5.1 Projection and Rejection

The formula for the projection of $a$ onto $b$ is familiar from calculus:

$$\text{Proj}_b a = \frac{a \cdot b}{b \cdot b}$$
In addition to that (and not so familiar) is the rejection, which is the part of \( \mathbf{a} \) which is perpendicular to \( \mathbf{b} \). If this is ever needed in calculus we typically do:

\[
\text{Rej}_b \mathbf{a} = \mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b}
\]

This could be done more elegantly by taking the subspace of \( \mathbb{R}^3 \) which is perpendicular to \( \mathbf{b} \) and projecting to that, but calculus doesn’t make this computationally easy. Geometric algebra does, however.

**Lemma 8.5.1.1.** Let \( \mathbf{a} \) be a vector and \( B \) be a 2-blade. Let \( \text{Proj}_B \mathbf{a} \) be the projection of \( \mathbf{a} \) onto the subspace represented by \( B \) and let \( \text{Rej}_B \mathbf{a} \) be the rejection of \( \mathbf{a} \) relative to the subspace represented by \( B \). Then:

\[
(\text{Proj}_B \mathbf{a}) \wedge B = 0 \\
(\text{Rej}_B \mathbf{a}) \cdot B = 0
\]

**Proof.** For the first, note that the projection is in the subspace represented by \( B \) and thus \( (\text{Proj}_B \mathbf{a}) \wedge B \) creates a parallelopipid with volume 0, hence equals 0.

For the second, note that since \( \text{Rej}_B \mathbf{a} \) is perpendicular to \( B \) it represents the dual \( (\text{Rej}_B \mathbf{a})^* \) and thus \( \text{Rej}_B \mathbf{a} = \alpha B^* \) for some \( \alpha \in \mathbb{R} \). Then observe that:

\[
(\text{Rej}_B \mathbf{a}) B = \alpha B^* B = \alpha BB^* = B\alpha B^* = B(\text{Rej}_B \mathbf{a})
\]

From here we get:

\[
(\text{Rej}_B \mathbf{a}) \cdot B = \frac{1}{2} ((\text{Rej}_B \mathbf{a}) B - B(\text{Rej}_B \mathbf{a})) = 0
\]

\( \square \)

**Theorem 8.5.1.1.** Let \( \mathbf{a} \) be a vector and \( B \) be a 2-blade then the projection and rejection of \( \mathbf{a} \) in relation to the subspace represented by \( B \) are:

\[
\text{Proj}_B \mathbf{a} = (\mathbf{a} \cdot B) B^{-1} = \langle \mathbf{a} B \rangle_1 B^{-1} \\
\text{Rej}_B \mathbf{a} = (\mathbf{a} \wedge B) B^{-1} = \langle \mathbf{a} B \rangle_3 B^{-1}
\]

**Proof.** We have:

\[
(\text{Proj}_B \mathbf{a}) B = (\text{Proj}_B \mathbf{a}) \cdot B + (\text{Proj}_B \mathbf{a}) \wedge B = (\text{Proj}_B \mathbf{a}) \cdot B + 0 = (\text{Proj}_B \mathbf{a}) \cdot B + (\text{Rej}_B \mathbf{a}) \cdot B = (\text{Proj}_B \mathbf{a} + \text{Rej}_B \mathbf{a}) \cdot B = \mathbf{a} \cdot B
\]

\[
\text{Proj}_B \mathbf{a} = (\mathbf{a} \cdot B) B^{-1}
\]
The second equation is similar.

**Exercise 8.14.** Prove the second equation.

Note that for a vector \( \mathbf{b} \) we saw that \( \mathbf{b}^{-1} = \mathbf{b}/|\mathbf{b}|^2 \) and so this formula becomes the traditional formula in that case.

**Practical Calculation 8.9.** To use this formula just dig in and do the calculations.

**Example 8.24.** Suppose we wish to project the vector \( \mathbf{a} = 7\mathbf{e}_1 - 5\mathbf{e}_3 \) onto the subspace represented by the blade \( B = 2\mathbf{e}_1 \wedge (\mathbf{e}_2 + 6\mathbf{e}_3) \). The result is:

\[
\text{Proj}_B \mathbf{a} = (\mathbf{a} \cdot B)B^{-1}
\]

We’ll work this out bit by bit. Note:

\[
B = 2\mathbf{e}_1 \wedge (\mathbf{e}_2 + 6\mathbf{e}_3) = 2\mathbf{e}_1 \mathbf{e}_2 + 12\mathbf{e}_1 \mathbf{e}_3
\]

First, noting that \( \mathbf{a} \) is a 1-vector and \( B \) is a 2-blade, the inner product extracts the \( 2 - 1 = 1 \)-grade.

\[
\mathbf{a} \cdot B = (7\mathbf{e}_1 - 5\mathbf{e}_3) \cdot (2\mathbf{e}_1 \mathbf{e}_2 + 12\mathbf{e}_1 \mathbf{e}_3)
\]

\[
= \langle (7\mathbf{e}_1 - 5\mathbf{e}_3)(2\mathbf{e}_1 \mathbf{e}_2 + 12\mathbf{e}_1 \mathbf{e}_3) \rangle_1
\]

\[
= \langle 14\mathbf{e}_1 \mathbf{e}_2 + 84\mathbf{e}_1 \mathbf{e}_3 - 10\mathbf{e}_3 \mathbf{e}_1 \mathbf{e}_2 - 60\mathbf{e}_3 \mathbf{e}_1 \mathbf{e}_3 \rangle_1
\]

\[
= \langle 14\mathbf{e}_2 + 84\mathbf{e}_3 - 10\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 + 60\mathbf{e}_1 \rangle_1
\]

\[
= 60\mathbf{e}_1 + 14\mathbf{e}_2 + 84\mathbf{e}_3
\]

Second:

\[
B^{-1} = \frac{B}{|B|^2}
\]

\[
= \frac{2\mathbf{e}_1 \mathbf{e}_2 + 12\mathbf{e}_1 \mathbf{e}_3}{|2\mathbf{e}_1 \mathbf{e}_2 + 12\mathbf{e}_1 \mathbf{e}_3|^2}
\]

\[
= \frac{-2\mathbf{e}_1 \mathbf{e}_2 - 12\mathbf{e}_1 \mathbf{e}_3}{148}
\]
Then:

\[
\text{Proj}_B a = (60e_1 + 14e_2 + 84e_3) \left( \frac{-2e_1e_2 - 12e_1e_3}{148} \right)
\]

\[
= \frac{1}{148} (60e_1 + 14e_2 + 84e_3)(-2e_1e_2 - 12e_1e_3)
\]

\[
= \frac{1}{148} (-120e_1e_1e_2 - 720e_1e_1e_3 - 28e_2e_1e_2 - 168e_2e_1e_3 - 168e_3e_1e_2 - 1008e_3e_1e_3)
\]

\[
= \frac{1}{148} (1036e_1 - 120e_2 - 720e_3)
\]

\[
= \frac{1}{37} (259e_1 - 30e_2 - 180e_3)
\]

It's worth noting that this can be done fairly quickly in \( \mathbb{R}^3 \). Notice how the vectors in the following correspond to the data given above. The subspace \( B \) has orthogonal basis \( \{[2; 0; 0], [0; 1; 6]\} \) and so the projection of \( [7; 0; -5] \) onto \( B \) can be found by adding the projections onto the basis vectors. We have:

\[
\text{Proj}_{[2;0;0]} [7; 0; -5] = [7; 0; 0]
\]

\[
\text{Proj}_{[0;1;6]} [7; 0; -5] = -\frac{30}{37} [0; 1; 6]
\]

Thus the overall projection is:

\[
[7; 0; 0] - \frac{30}{37} [0; 1; 6] = \frac{1}{37} [259; -30; -180]
\]

You might wonder what is gained in the geometric algebra way. The answer is that it doesn’t care if we have an orthogonal basis and it’s formulaic in all cases.

**Example 8.25.** Note that for the previous example to do the rejection we’ve done most of the work already. Instead of \( a \cdot B \) we find \( a \wedge B \) but this just involves extracting the grade 3 component \( \langle aB \rangle_3 \) and multiplying by \( B^{-1} \):

\[
(-10e_1e_2e_3) \left( \frac{-2e_1e_2 - 12e_1e_3}{148} \right) = \frac{30}{37} e_2 - \frac{5}{37} e_3
\]

Note that the projection and the rejection add up to \( a \), which makes sense.

**Exercise 8.15.** Find the projection and rejection of \( a = 2e_1 - 3e_2 + 5e_3 \) relative to the subspace span \( \{4e_1 + 1e_2 - 1e_3, 2e_1 + 2e_3\} \).

### 8.5.2 Reflection

**Theorem 8.5.2.1.** Given a vector \( \mathbf{v} \) and a 2-blade \( B \) the result of reflecting \( \mathbf{v} \) in the subspace represented by \( B \) is:

\[ \mathbf{v} \mapsto -B\mathbf{v}B^{-1} \]
8.5. COMPUTATION IN 3D

Proof. To see that this does the required job, we decompose \( \mathbf{v} \) into two parts:

\[
\mathbf{v} = (\mathbf{v} \cdot B)B^{-1} + (\mathbf{v} \wedge B)B^{-1}
\]

A reflection should negate the second part and leave the first part alone. Thus:

\[
\text{Refl}_B \mathbf{v} = (\mathbf{v} \cdot B)B^{-1} - (\mathbf{v} \wedge B)B^{-1}
\]

\[
= (\mathbf{v} \cdot B - \mathbf{v} \wedge B)B^{-1}
\]

\[
= \left( \frac{1}{2}(\mathbf{v}B - B\mathbf{v}) - \frac{1}{2}(\mathbf{v}B + B\mathbf{v}) \right)B^{-1}
\]

\[
= (-B\mathbf{v})B^{-1}
\]

\[
= -B\mathbf{v}B^{-1}
\]

\[\square\]

Example 8.26. Suppose we wish to reflect the vector \( \mathbf{v} = 2\mathbf{e}_1 + 5\mathbf{e}_2 + 7\mathbf{e}_3 \) in the subspace spanned by \( \mathbf{e}_1 - \mathbf{e}_2 \) and \( \mathbf{e}_1 + 3\mathbf{e}_3 \). We set:

\[
B = (\mathbf{e}_1 - \mathbf{e}_2) \wedge (\mathbf{e}_1 + 3\mathbf{e}_3) = \mathbf{e}_1\mathbf{e}_2 - 3\mathbf{e}_2\mathbf{e}_3 - 3\mathbf{e}_3\mathbf{e}_1
\]

We note:

\[
B^{-1} = \frac{B}{|B|^2} = \frac{\mathbf{e}_1\mathbf{e}_2 - 3\mathbf{e}_2\mathbf{e}_3 - 3\mathbf{e}_3\mathbf{e}_1}{|\mathbf{e}_1\mathbf{e}_2 - 3\mathbf{e}_2\mathbf{e}_3 - 3\mathbf{e}_3\mathbf{e}_1|^2} = \frac{-\mathbf{e}_1\mathbf{e}_2 + 3\mathbf{e}_2\mathbf{e}_3 + 3\mathbf{e}_3\mathbf{e}_1}{19}
\]

Then we calculate:

\[
\text{Refl}_B \mathbf{v} = -B\mathbf{v}B^{-1}
\]

\[
= -(\mathbf{e}_1\mathbf{e}_2 - 3\mathbf{e}_2\mathbf{e}_3 - 3\mathbf{e}_3\mathbf{e}_1)(2\mathbf{e}_1 + 5\mathbf{e}_2 + 7\mathbf{e}_3)\left(\frac{-\mathbf{e}_1\mathbf{e}_2 + 3\mathbf{e}_2\mathbf{e}_3 + 3\mathbf{e}_3\mathbf{e}_1}{19}\right)
\]

\[= \ldots \text{Python}\ldots \]

\[= -\frac{1}{19}(-46\mathbf{e}_1 + 11\mathbf{e}_2 + 161\mathbf{e}_3)\]

\[= \frac{46}{19}\mathbf{e}_1 - \frac{11}{19}\mathbf{e}_2 + \frac{161}{19}\mathbf{e}_3
\]

Exercise 8.16. Find the result of reflecting the vector \(-4\mathbf{e}_1 + 2\mathbf{e}_2 - 7\mathbf{e}_3\) in the subspace spanned by \(3\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\) and \(\mathbf{e}_1 - 2\mathbf{e}_2\).

8.5.3 Rotation

In order to see how rotations are constructed recall from previous chapters that if we take two planes which intersect in a line and if we reflect first in one and
then in the other, the result is a rotation around the intersecting lines. Not just that, but any rotation around the axis may be constructed using two such reflections.

So now suppose we’re given a unit axis $\hat{r}$ that we wish to rotate around and an angle $\theta$ according to the right hand rule.

Let $\hat{m}$ and $\hat{n}$ be any two unit vectors perpendicular to $\hat{r}$ and with an angle of $\theta/2$ between them as measured from $\hat{m}$ to $\hat{n}$ according to the right hand rule with respect to $\hat{r}$.

Noting that $\hat{m} \perp \hat{r}$ and $\hat{n} \perp \hat{r}$, consider now the two planes:

$$\hat{m} \hat{r} = \hat{m} \wedge \hat{r} + \hat{m} \cdot \hat{r} = \hat{m} \wedge \hat{r}$$

and

$$\hat{n} \hat{r} = \hat{n} \wedge \hat{r} + \hat{n} \cdot \hat{r} = \hat{n} \wedge \hat{r}$$

It’s clear that reflection in $\hat{m} \hat{r}$ followed by reflection in $\hat{n} \hat{r}$ performs the desired rotation. Since these are both bivectors we can use the previous section to calculate this.

Before the calculation, note the following:

$$(\hat{m} \hat{r})^{-1} = \hat{r} \hat{m}$$

and

$$(\hat{n} \hat{r})^{-1} = \hat{r} \hat{n}$$

and

$$\pm \hat{m} \hat{r} = \pm \hat{m} \wedge \hat{r} = \mp \hat{r} \wedge \hat{m} = \mp \hat{r} \hat{m}$$

Now we calculate:

$$a \mapsto -(\hat{n} \hat{r})(-(\hat{m} \hat{r})a(\hat{m} \hat{r})^{-1})(\hat{n} \hat{r})^{-1}$$

$$\mapsto (\hat{n} \hat{r})(\hat{m} \hat{r})a(\hat{r} \hat{m})(\hat{r} \hat{n})$$

$$\mapsto (\hat{n} \hat{r})(-\hat{r} \hat{m})a(-\hat{m} \hat{r})(\hat{r} \hat{n})$$

$$\mapsto (\hat{n} \hat{r})(\hat{r} \hat{m})a(\hat{m} \hat{r})(\hat{r} \hat{n})$$

$$\mapsto \hat{n} \hat{m} \hat{n} \hat{m}$$

$$\mapsto (\hat{n} \hat{m})a(\hat{n} \hat{m})^\dagger$$

Now then, $\hat{n} \hat{m}$ can be thought of as the plane of rotation, more specifically the plane of rotation and its geometric product construction (which includes the
angle between them). In other words we can think of rotating a plane (and all parallel planes) by assigning that plane using a geometric product of two vectors which span the plane and meet at an angle of $\theta/2$.

Moreover if we choose any unit bivector $\hat{B}$ which spans the same plane as $\hat{m}$ and $\hat{n}$ and has the same orientation as $\hat{m} \wedge \hat{n}$ and then observe that since $|\hat{m}| = |\hat{n}| = 1$ we have:

\[
\hat{m}\hat{n} = \hat{m} \cdot \hat{m} + \hat{m} \wedge \hat{n} \\
= |\hat{m}| |\hat{n}| \cos(\theta/2) + |\hat{m}| |\hat{n}| \sin(\theta/2) \hat{B} \\
= \cos(\theta/2) + (\sin(\theta/2)) \hat{B} \\
= \exp((\theta/2) \hat{B})
\]

and:

\[
\hat{n}\hat{m} = \hat{n} \cdot \hat{m} + \hat{n} \wedge \hat{m} \\
= |\hat{n}| |\hat{m}| \cos(-\theta/2) + |\hat{n}| |\hat{m}| \sin(-\theta/2) \hat{B} \\
= \cos(\theta/2) - (\sin(\theta/2)) \hat{B} \\
= \exp(-\theta/2) \hat{B}
\]

Thus we can write our rotation as:

\[
a \mapsto \exp(-\theta/2) \hat{B} a \exp(\theta/2) \hat{B}
\]

However lastly and most carefully we may in fact want to calculate the rotation via the right hand rule given an axis $\hat{r}$ and an angle $\theta$.

It’s tempting to simply set $\hat{B} = \hat{r}^*$ but this does not quite work. The reason is that although the subspace represented by $\hat{B}$ is the correct plane of rotation, it has the incorrect orientation.

The reason for this is as follows. In the above argument $\hat{m}$ and $\hat{n}$ were chosen so that the direction of $\hat{m} \times \hat{n}$ is that of $\hat{r}$. It follows that the orientation of $(\hat{m} \times \hat{n})^*$ matches that of $\hat{r}^*$ since they only vary by length. However since $(\hat{m} \times \hat{n})^* = -(\hat{m} \wedge \hat{n})$ it follows that the orientation of $(\hat{m} \times \hat{n})^*$ is opposite that of $\hat{m} \wedge \hat{n}$ and consequently the orientation of $\hat{r}^*$ is opposite that of $\hat{m} \wedge \hat{n}$.

The solution is to set $\hat{B} = -\hat{r}^*$ and then the rotation becomes:

\[
a \mapsto \exp((\theta/2) \hat{r}^*) a \exp(-\theta/2) \hat{r}^*)
\]

We close with the previous results packaged into a theorem:
Theorem 8.5.3.1. Given a unit vector \( \hat{r} \) designating an axis of rotation and an angle \( \theta \), rotation by \( \theta \) radians about \( \hat{r} \) in accordance to the right-hand rule may be achieved by assigning two unit vectors \( \hat{m} \) and \( \hat{n} \) such that the angle between them is \( \theta/2 \) and then using the mapping:

\[
a \mapsto (\hat{m}\hat{n})a(\hat{m}\hat{n})^\dagger
\]

Alternately if \( \hat{B} \) is any unit bivector spanning the same plane as \( \hat{m} \) and \( \hat{n} \) with the same orientation as \( \hat{m} \wedge \hat{n} \) then we may use the mapping:

\[
a \mapsto \exp(-\theta/2 \hat{B}) a \exp((\theta/2) \hat{B})
\]

Lastly we can simply use the mapping:

\[
a \mapsto \exp((\theta/2)\hat{r}^*) a \exp(-(\theta/2)\hat{r}^*)
\]

Example 8.27. Suppose we wish to rotate the plane by \( \theta = 2.4 \) radians about the axis \( \hat{r} = \frac{1}{\sqrt{14}}(2\hat{e}_1 + 3\hat{e}_2 - \hat{e}_3) \). We find:

\[
\hat{r}^* = \frac{1}{\sqrt{14}}(2\hat{e}_1 + 3\hat{e}_2 - \hat{e}_3)e_3e_2e_1
\]

\[
= \frac{1}{\sqrt{14}}(-2\hat{e}_2\hat{e}_3 - 3\hat{e}_3\hat{e}_1 + e_1e_2)
\]

\[
= \frac{1}{\sqrt{14}}(e_1e_2 - 2\hat{e}_2\hat{e}_3 - 3\hat{e}_3e_1)
\]

The mapping is then given by:

\[
a \mapsto \exp((\theta/2)\hat{r}^*) a \exp(-(\theta/2)\hat{r}^*)
\]

For \( \theta = 2.4 \) and \( \hat{r}^* \) above.

To rotate the point \( a = -2\hat{e}_1 + 2\hat{e}_2 + 7\hat{e}_3 \) we calculate:

\[
a \mapsto \exp((\theta/2)\hat{r}^*) (-2\hat{e}_1 + 2\hat{e}_2 + 7\hat{e}_3) \exp(-(\theta/2)\hat{r}^*)
\]

\[
\mapsto \text{...Python...}
\]

\[
\mapsto 4.3859\hat{e}_1 - 5.5026\hat{e}_2 - 2.7360\hat{e}_3
\]

Exercise 8.17. What is the mapping which rotates around the axis \( \hat{r} = \hat{e}_1 + \hat{e}_3 \) by \( \theta = 4 \) radians? Make sure you normalize \( \hat{r} \) first. Then find the result of rotating the point \( a = 2\hat{e}_1 + 5\hat{e}_2 - 3\hat{e}_3 \).
8.6 Generalization of $\mathbb{H}$

Consider the following correspondance:

\[
\begin{align*}
e_3e_2 & \leftrightarrow i \\
e_1e_3 & \leftrightarrow j \\
e_2e_1 & \leftrightarrow k
\end{align*}
\]

Then multivectors of the form:

\[
\alpha + \beta e_3e_2 + \delta e_1e_3 + \gamma e_2e_1
\]

correspond to quaternions of the form:

\[
\alpha + \beta \hat{i} + \delta \hat{j} + \gamma \hat{k}
\]

This correspondance is not meaningless. For example from it we get the following correspondances:

\[
\begin{align*}
(e_3e_2)(e_1e_3) &= e_2e_1 \leftrightarrow i \hat{j} = \hat{k} \\
(e_1e_3)(e_2e_1) &= e_3e_2 \leftrightarrow j \hat{k} = i \\
(e_2e_1)(e_3e_2) &= e_1e_3 \leftrightarrow \hat{k} \hat{i} = j
\end{align*}
\]

It follows that multivectors of this form form a closed structure with the geometric product acting in a corresponding manner to the quaternion product.

For example the geometric product:

\[
(7+2e_3e_2+5e_1e_3+8e_2e_1)(2-7e_3e_2+2e_1e_3-6e_2e_1) = 66-91e_3e_2-20e_1e_3+13e_2e_1
\]

corresponds to the quaternion product:

\[
(7 + 2i + 5j + 8\hat{k})(2 - 7i + 2j - 6\hat{k}) = 66 - 91i - 20j + 13\hat{k}
\]

What’s happening here is that the quaternions appear structurally as a subset of the geometric algebra.

We can see this parallel even more clearly when we look at a computation such as rotation.

In the previous example we rotated the point $-2e_1 + 2e_2 + 7e_3$ by $\theta = 2.4$ radians about the axis $\hat{r} = \frac{1}{14}(2e_1 + 3e_2 - e_3)$.

Here is that calculation reorganized and approximated so that we can easily trace the values.
We have: \[ \hat{r} = 0.2857e_1 + 0.4286e_2 - 0.1429e_3 \]
and then: \[ \hat{r}^* = 0.2857e_3e_2 + 0.4286e_1e_3 - 0.1429e_2e_1 \]

We have:
\[ \exp((2/2)\hat{r}^*) = 0.36 + 0.50e_2e_3 + 0.75e_1e_3 - 0.25e_2e_1 \]
and:
\[ \exp((-2/2)\hat{r}^*) = 0.36 - 0.50e_2e_3 - 0.75e_1e_3 + 0.25e_2e_1 \]

and then the product is
\[ \exp((2/2)\hat{r}^*)a\exp((-2/2)\hat{r}^*) \]
\[ = (0.36+0.50e_2e_3+0.75e_1e_3-0.25e_2e_1)(-2e_1+2e_2+7e_3(0.36-0.50e_2e_3-0.75e_1e_3+0.25e_2e_1)) \]
\[ = 4.39e_1 - 5.50e_2 - 2.74e_3 \]

If we approached this as a quaternion problem we would assign:
\[ p = \cos(2.4/2) + \sin(2.4/2)\hat{r} = 0.36 + 0.50i + 0.75j - 0.25k \]
Then:
\[ p^* = \cos(2.4/2) + \sin(2.4/2)\hat{r} = 0.36 - 0.50i - 0.75j + 0.25k \]

Then the product is:
\[ pap^* = (0.36+0.50i+0.75j-0.25k)(-2i+2j+7k)(0.36-0.50i-0.75j+0.25k) \]
\[ = 4.39i - 5.50j - 2.74k \]
Index

$k$-blade, 18
bivector, 5
dot product, 6
dual, 23
exterior product, 4, 5
grade, 4, 6
inner product, 6
inverse, 18
invertible, 18
multivector, 6
norm of a multivector, 17
norm of a vector, 17
outer product, 4, 5
perpendicular, 7
reversion, 18
scalar product, 6
trivector, 6
wedge product, 4, 5