MATH 241
HOUR EXAM II November 3, 2000

1. Let \( f(x, y) = \frac{y^2}{x} + xy^2 \).
   
a) If we move in the direction \( \mathbf{u} = (i + j)/\sqrt{2} \) from the point \((2, 1)\), does the function \( f \) increase or decrease?

   b) In what direction should we move from \((2, 1)\) to get the maximum rate of increase? What is the maximum rate of increase?

   c) What is the direction of a tangent vector to the level curve of \( f \) through the point \((2, 1)\)?

2. a) Let \( f(x, y, z) \) be given with

\[
  f_x(1, 0, 2) = 1, \quad f_y(1, 0, 2) = -2, \quad f_z(1, 0, 2) = 3.
\]

Let \( x(t) = t^2, \quad y(t) = \sin(\pi t), \quad z(t) = 2t \). Set \( F(t) = f(x(t), y(t), z(t)) \). What is \( dF(t)/dt \) at \( t = 1 \)?

3. Find the maximum and minimum values of \( f(x, y) = (x - 1)^2 + 2y^2 \) on the set

\[
  R = \{(x, y) : x^2 + y^2 \leq 1\}.
\]

Sketch the set \( R \) and the level curves of \( f \).

4. Write the following iterated integral as a double integral over a set \( R \) in the \( x, y \) plane. Sketch the set \( R \). Then evaluate the integral by changing the order of integration.

\[
  \int_0^1 \int_{y^{1/3}}^1 y \, dx \, dy.
\]

5. Let \( D \) be the solid region which lies inside the sphere \( x^2 + y^2 + z^2 = 4 \), and outside the cylinder \( x^2 + y^2 = 1 \).
   
a) Sketch the intersection of \( D \) with the \( xy \) plane.

   b) Find the volume of \( D \). Use the symmetry of the region to write the integral in polar coordinates. Calculate the integral.