Math 341, Jeffrey Adams

Final, May 14, 2011 SOLUTIONS

The average on the final was 170/200 (85%). There were two perfect scores.

(1) [20 points] True or false. Briefly justify your answers. Here R is an open, connected region in $\mathbb{R}^3$, $\mathbf{F}$ is vector field on R, f is a scalar valued function on R, and $\mathbf{F}$ and f are twice differentiable. In each case mark the assertion true only if it is true for all $\mathbf{F}$ and f.

(a) True (easy computation)

(b) False: $\nabla \cdot (\nabla f) = \nabla^2(f) = \sum_1^3 \frac{\partial^2 f}{x_i^2}$

(c) True (easy computation)

(d) False; it is true if R is simply connected by the Main Theorem

(2) [30]

(a) Since $f_x = \frac{x}{x^2+y^2}$, $f = \int \frac{x}{x^2+y^2}\,dx$ Let $u = x^2 + y^2$, $du = 2xdx$ to give $f(x,y) = \frac{1}{2}\ln(x^2+y^2) + C(y)$. Now $f_y = \frac{y}{x^2+y^2} + C'(y)$, and setting this equal to $\frac{y+x^2+y^2}{x^2+y^2}$ gives $C'(y) = \frac{x^2+y^2}{x^2+y^2} = 1$, i.e. $C(y) = y + D$. So

$$f(x,y) = \frac{1}{2}\ln(x^2+y^2) + y + D$$

(b) This is an application of Green's Theorem. The area is $\int_R 1dA$. By Green's Theorem this equals $\int_\gamma \mathbf{F}\cdot\mathbf{dx}$ provided $\nabla\times\mathbf{F} = 1$. Write $\mathbf{F} = (P,Q)$, so we need $Q_x - P_y = 1$. Take $Q(x,y) = \frac{1}{2}x$ and $P(x,y) = -\frac{1}{2}y$. Then $Q_x - P_y = \frac{1}{2} - (-\frac{1}{2}) = 1$. Then

$$\int_\gamma \mathbf{F}\cdot\mathbf{dx} = \int_\gamma Pdx + Qdy = \int_\gamma -\frac{1}{2}ydx + \frac{1}{2}xdy = \frac{1}{2}\int xdy - ydx$$

(3) [30]

(a) This is separable: $y^2dy = -x^2dx$, so $\frac{1}{3}y^3 = -\frac{1}{3}x^3 + c$, or $y^3 = -x^3 + c$, or $y = \sqrt[3]{-x^3+c}$.

(b) Plugging in $x = 1, y = 0$ gives $0 = \sqrt[3]{-1+c}$ or $c = 1$, so $y = \sqrt[3]{1-x^3}$

(4) (a) This is linear nonhomogeneous. Write it as $y' - 3y = e^x + 2$. The homogeneous equation has solution $y = e^{3x}$. Multiply both sides by e^{-3x} to give

$$e^{-3x}y' - 3e^{-3x}y = e^{-3x}(e^x+2)$$

or

$$(e^{-3x}y)' = e^{-3x}(e^x+2)$$

which gives

$$y = e^{3x}\int e^{-3x}(e^x+2)\,dx$$

The integral is

$$\int e^{-2x} + 2e^{-3x}\,dx = -\frac{1}{2}e^{-2x} - \frac{2}{3}e^{-3x}$$

so the solution is

$$e^{3x}(-\frac{1}{2}e^{-2x} - \frac{2}{3}e^{-3x}) = -\frac{1}{2}e^{x} - \frac{2}{3}$$

Therefore the general solution is

$$y = ce^{3x} - \frac{1}{2}e^{x} - \frac{2}{3}$$

(b) The initial condition $x = 0, y = 2$ gives $2 = c - \frac{1}{2} - \frac{2}{3} = c - \frac{7}{6}$, so $c = 2 + \frac{7}{6} = \frac{19}{6}$.
(5) [30] Find the general solution of

$$y'' + 3y' + 2y = 1 + e^{x} + e^{2x} \tag{1}$$

First solve the homogeneous equation. The characteristic equation is $\lambda^2 + 3\lambda + 2 = 0$, i.e. $(\lambda + 1)(\lambda + 2) = 0$, so $\lambda = -1, -2$. The homogenous solutions are e^{-x}, e^{-2x}.

Use variation of parameters. Set $y = ue^{-x}$. Then

$$\begin{aligned} y &= e^{-x} \\ y' &= u'e^{-x} - ue^{-x} \\ y'' &= u''e^{-x} - 2u'e^{-x} + ue^{-x} \end{aligned}$$

and plugging this in gives

$$e^{-x}[(u'' - 2u' + u) + 3(u' - u) + 2u] = 1 + e^{x} + e^{2x}$$

The terms involving u (not u', u'') cancel, to give

$$e^{-x}[u'' - 2u' + 3u'] = 1 + e^{x} + e^{2x}$$

or

$$u'' + u' = e^{x}(1 + e^{x} + e^{2x})$$

Let $v = u'$, so

$$v' + v = e^{x} + e^{2x} + e^{3x}$$

The homogeneous equation is $v' + v = 0$, or $v = e^{-}x$, so multiply both sides by e^{x}:

$$e^{x}v' + e^{x}v = e^{2x} + e^{3x} + e^{4x}$$

or

$$(e^{x}v)' = e^{2x} + e^{3x} + e^{4x}$$

So

$$
\begin{aligned}
v &= e^{-x}\int e^{2x}+e^{3x}+e^{4x}dx \\
&= e^{-x}[\frac{1}{2}e^{2x}+\frac{1}{3}e^{3x}+\frac{1}{4}e^{4x}] \\
&= \frac{1}{2}e^{x}+\frac{1}{3}e^{2x}+\frac{1}{4}e^{3x}]
\end{aligned}
$$

(you can ignore the $+c$. Then

$$u = \int v dx = \frac{1}{2}e^{x}+\frac{1}{6}e^{2x}+\frac{1}{12}e^{3x}$$

and finally

$$y = ue^{-x} = \frac{1}{2}+\frac{1}{6}e^{x}+\frac{1}{12}e^{2x}$$

The general solution is therefore

$$y = c_1e^{-x}+c_2e^{-2x}+\frac{1}{2}+\frac{1}{6}e^{x}+\frac{1}{12}e^{2x}$$

You can also do this by undetermined coefficients: guess $y = a+be^{x}+ce^{2x}$ and solve for a, b, c.

(6) [30] The equilibrium points where where $H' = P' = 0$, i.e. $H = P = 0$, or $(H, P) = (\frac{2}{3}, \frac{1}{2})$.

Write $F = (1-2P)H$, $G = (3H-2)P$. Then $F_H = 1-2P, F_P = -2H$, $G_H = 3P$, $G_P = 3H-2$, so the Jacobian is

$$\begin{pmatrix} 1-2P & -2H \\ 3P & 3H-2. \end{pmatrix}$$

At $(0, 0)$ the Jacobian is

$$\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

which gives an unstable saddle point.

At $(\frac{2}{3}, \frac{1}{2})$ the Jacobian is

$$\begin{pmatrix} 0 & -\frac{4}{3} \\ \frac{3}{2} & 0 \end{pmatrix}$$

The eigenvalues are the roots of $\lambda^2 - (-\frac{4}{3})(\frac{3}{2}) = \lambda^2 + 2$, i.e. $\lambda = \pm 2i$. This is a stable point, with periodic orbits (ellipses).

At $(0,0)$ the nonlinear system is unstable, since there is a positive eigenvalue. At $(\frac{2}{3}, \frac{1}{2})$ you cannot determine stability from the linearization Theorem, since the eigenvalues have 0 real part. (In fact, from the discussion in class, they are stable, with periodic orbits).
Note: Although you can write

$$\begin{pmatrix} H \\ P \end{pmatrix}' = \begin{pmatrix} 1 & -2H \\ 3P & -2 \end{pmatrix} \begin{pmatrix} H \\ P \end{pmatrix}$$

this doesn't help you to compute the Jacobian, which you need to do; the Jacobian is *not* just some sort of derivative of this matrix.
(7) [30] Solve the Euler equation (for $x > 0$):

$$x^2 y'' + 3xy' + y = 0, \quad y(1) = 2,\ y'(1) = 1.$$

Plug in x^r: $x^2 r(r-1)x^r + 3rx^r + x^r = 0$, i.e. $r(r-1) + 3r + 1 = 0$, or $r^2 + 2r + 1 = 0$, or $(r+1)^2 = 0$. This has only one root $r = -1$. So $y = x^{-1}$ is a solution.

The other solution? It is $\ln(x)x^{-1}$. So

$$y = c_1 x^{-1} + c_2 x^{-1}\ln(x)$$

For the initial conditions, $x = 1, y = 2$ gives $c_1 = 2$. Compute

$$y' = -c_1 x^{-2} + c_2[-x^{-2}\ln(x) + \frac{1}{x^2}]$$

and plug in $x = 1$, $y' = 1$ to get $1 = -c_1 + c_2 = -2 + c_2$ so $c_2 = 3$. The solution is

$$\frac{2}{x} + \frac{3\ln(x)}{x}$$