Final Exam (Saturday, May 16)

Exam is open book, open notes; calculators are allowed for built-in arithmetical operations, not for calculus functions nor any programs you added. Show all work, but cross out any work that you do not want considered for grading. Numerical answers need not be simplified; numbers like $2/\sqrt{5}$ or $\sin^{-1}(1/3)$ in your solutions are best left as is. **However, your answers should not involve complex numbers.** Check your answers when possible; if your answer does not satisfy the equations in the problem statement, and you can't find your error, at least indicate that you know there's an error. There are 6 problems, and each is worth 10 points.

1. Let $y(t)$ be the solution to the initial value problem

$$\frac{dy}{dt} = 2.1 - 2t^2 - 0.1y^2, \qquad y(0) = 1.$$

   (a) Show that the solution exists and that $-2 \le y(t) \le 4$ for $0 \le t \le 1$.

   (b) Use the Euler method with step size $h = 0.5$ to compute an approximation to $y(1)$.

   (c) Determine an upper bound on the error in the approximation in part (b).

2. Consider the initial value problem

$$y'' + y' - 2y = 5 - 6e^{-2t}, \qquad y(0) = -1, \quad y'(0) = 4.$$

   In whichever order you want:

   (a) find the solution;

   (b) find the Laplace Transform of the solution.

3. Consider the initial value problem

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & -5 \\ 1 & 4 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

   (a) Find the solution $\mathbf{x}(t)$.

   (b) Is this solution asymptotically stable, stable (but not asymptotically stable), or unstable?

   (c) Graph the solution in the phase plane (x_1 versus x_2).

4. Consider the system

$$\frac{dx}{dt} = 3y + y^3,$$
$$\frac{dy}{dt} = 1 - e^{bx} - 2y$$

   where b is a real constant. Notice that the origin ($x = y = 0$) is an equilibrium solution.

(a) For which values of b is the origin asymptotically stable?

(b) For which values of b is the origin unstable?

(c) Show that the system has no nontrivial (i.e., non-equilibrium) periodic solutions.

5. Consider the system

$$\begin{aligned}\frac{dx}{dt} &= -xy^3,\\ \frac{dy}{dt} &= -4x^3y.\end{aligned}$$

(a) Show that if $y(0) > 0$, then $y(t) > 0$ for all real t for which the solution exists.

(b) Find the orbits (solution curves) of this system.

(c) Find the orbit with $x(0) = 1$ and $y(0) = 1$. What are the limits of $x(t)$ and $y(t)$ as $t \to \infty$?

6. Let

$$f(x) = \begin{cases} 7 & 0 < x < 3 \\ 5 & 3 \le x < 6. \end{cases}$$

(a) Find the Fourier cosine series for $f(x)$.

(b) Solve the heat equation $u_t = 2u_{xx}$ with boundary conditions $u_x(0,t) = u_x(6,t) = 0$ and initial condition $u(x,0) = f(x)$.