Math 341, Jeffrey Adams
Test I, March 18, 2011 SOLUTIONS
Each problem is worth 20 points

(1)
(a) This is an easy calculation. $\nabla(f) = (f_x, f_y, f_z)$ and $\text{curl}(\nabla f) = \text{curl}(f_x, f_y, f_z) = (f_{zy} - f_{yz}, -f_{zx} + f_{xz}, f_{yx} - f_{xy}) = (0, 0, 0)$.
(b) No. This is the main theorem, and the statement is true if S is simply connected, but not necessarily otherwise. The point is you would like to define $f(\mathbf{x}) = \int_\gamma \mathbf{F} \cdot \mathbf{dx}$ where γ is a path from a fixed point to $\mathbf{x}$. If this is well defined then $\nabla f = \mathbf{F}$. If the region is simply connected then this *is* well defined: any closed path δ is the boundary of a region B, and $\int_\delta \mathbf{F} \cdot \mathbf{dx} = \int_B \text{curl}\mathbf{F} \cdot dS = 0$.

For example take $\mathbf{F} = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0)$ in $\mathbb{R}^3 - (0, 0, 0)$.

(2) Suppose $\alpha > 0$ is a constant, and

$$\mathbf{F}_\alpha(x, y) = (\frac{-y}{(x^2 + y^2)^\alpha}, \frac{x}{(x^2 + y^2)^\alpha})$$

Let $(x(t), y(t)) = (R\cos(t), R\sin(t))$, so $x'(t) = -R\sin(t)$, $y'(t) = R\cos(t)$.
(a) Note that on the circle $(x^2 + y^2)^\alpha = R^{2\alpha}$. The integral is:

$$\int_0^{2\pi} (\frac{-R\sin(t)}{R^{2\alpha}}, \frac{R\cos(t)}{R^{2\alpha}}) \cdot (-R\sin(t), R\cos(t))dt$$
$$= \int_0^{2\pi} \frac{R^2}{R^{2\alpha}}(\sin^2(t) + \cos^2(t))\, dt$$
$$= R^{2-2\alpha} \int_0^{2\pi} dt = 2\pi R^{2-2\alpha}.$$

(b)

$$\begin{aligned}
\text{curl}(\mathbf{F}_\alpha) &= \partial_x(\frac{x}{(x^2 + y^2)^\alpha}) - \partial_y(\frac{-y}{(x^2 + y^2)^\alpha}) \\
&= \frac{(x^2 + y^2)^\alpha - x\alpha(x^2 + y^2)^{\alpha-1}(2x)}{(x^2 + y^2)^{2\alpha}} - \frac{-(x^2 + y^2)^\alpha + y\alpha(x^2 + y^2)^{\alpha-1}(2y)}{(x^2 + y^2)^{2\alpha}} \\
&= \frac{2(x^2 + y^2)^\alpha - 2\alpha(x + y)(x^2 + y^2)^{\alpha-1}}{(x^2 + y^2)^{2\alpha}} \\
&= \frac{(2 - 2\alpha)(x^2 + y^2)^\alpha}{(x^2 + y^2)^{2\alpha}} \\
&= 2\frac{(1 - \alpha)}{(x^2 + y^2)^\alpha}.
\end{aligned}$$

(c) If $R < \sqrt{x_0^2 + y_0^2}$ then the circle doesn't contain the origin. The curl is 0 on the interior of the circle, which is simply connected, so by the main theorem the integral is 0.

If $R > \sqrt{x_0^2 + y_0^2}$ then the circle goes around the origin once, counter-clockwise. By independence of path, we can replace the path with a circle (of any radius) centered at the origin. By part (b) the answer is 2π (note that $\alpha = 1$ gives $2\pi R^{2-\alpha} = 2\pi$, independent of R).

Take $\alpha = 1$. Let γ be the circle, centered at a point (x_0, y_0), with radius $R \neq \sqrt{x_0^2 + y_0^2}$, traced counter-clockwise. What is $\int_\gamma \mathbf{F}_\alpha \cdot \mathbf{dx}$? Your answer will depend on (x_0, y_0) and R. Justify your answer.

(3) The characteristic equation is $r^2 - 2r + 10 = 0$ which has roots $r = (2 \pm \sqrt{4 - 40}/2 = (2 \pm \sqrt{-6})/2 = (2 \pm 6i)/2 = 1 \pm 3i$. Since these are complex, the general real solution is

$$e^x(c_1 \cos(3x) + c_2 \sin(3x))$$

The particular solution is found as follows.

$y(0) = 2$ implies $2 = e^0(c_1 + 0) = c_1$, so $c_1 = 2$. On the other hand

$$y'(x) = e^x(c_1 \cos(3x) + c_2 \sin(3x)) + e^x(-3c_1 \sin(3c) + 3c_2 \cos(3x))$$

and plugging in $y'(0) = 3$ gives

$$3 = e^0(c_1) + e^0(3c_2) = c_1 + 3c_2 = 2 + 3c2$$

so $3c_2 = 1$ or $c_2 = \frac{1}{3}$. The solution is

$$e^x(\frac{1}{3}\cos(3x) + 2\sin(3x))$$

As $x \to \infty$ this oscillates betwee ∞ and $-\infty$.

(4) First solve the homogenous equation, This has characteristic equation $r^2 - 5r + 4 = 0$, or $(r-1)(r-4) = 0$, so the roots are $1, 4$. The solutions are $y_1 = e^x, y_2 = e^{4x}$.

Now use variation of parameters. The Wronskian is $e^x(4e^{4x}) - e^x e^{4x} = 3e^{5x}$. Write $y = u_1 y_1 + u - 2y_2$ where

$$u_1' = -e^{4x}e^x/3e^{5x} = -\frac{1}{3}$$
$$u_2' = e^x e^x/3e^{5x} = \frac{1}{3}e^{3x}$$

Therefore $u_1 = -\frac{1}{3}x$ and $u_2 = \frac{1}{9}e^{-3x}$, and

$$u_1 = -\frac{1}{3}x,\ u_2 = -\frac{1}{9}e^{x}$$

So

$$y_p = -\frac{1}{3}xe^x - \frac{1}{9}e^{-3x}e^{4x} = -\frac{1}{3}xe^x - \frac{1}{9}e^x$$

The general solution is

$$-\frac{1}{3}xe^x - \frac{1}{9}e^x + c_1e^x + c_2e^{4x} = -\frac{1}{3}xe^x + c_1e^x + c_2e^{4x}$$

You can also solve this by setting $y = ue^x$ and solving for u, or by writing $(D-1)(D-4)y = e^x$, let $v = (D-4)y$, and then $(D-1)v = e^x$, so $(D-1)^2v = 0$ etc.

(5) This is separable. Write $\frac{dy}{dx} = -\sin(3x)y^2$, or $y^{-2}dy = -\sin(3x)dx$. So $-y^{-1} = \frac{1}{3}\cos(3x) + c$. Write this as $y^{-1} = -\frac{1}{3}\cos(3x) + c$, or

$$y = \frac{1}{-\frac{1}{3}\cos(3x) + c}$$

Modifying c this is the same as

$$y = \frac{3}{-\cos(3x) + c}$$

The initial condition gives $1 = \frac{3}{-1+c}$ so $c = 4$, and

$$y = \frac{3}{-\cos(3x) + 4}$$

Finally, $y(\pi) = \frac{3}{-\cos(3\pi)+4} = \frac{3}{5}$