Character Tables for $GL(2), SL(2), PGL(2)$ and $PSL(2)$ over a finite field

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1 Introduction

Let $\mathbb{F} = \mathbb{F}_q$ be the finite field with q elements. We compute the character tables for the groups $GL(2,\mathbb{F}), SL(2,\mathbb{F}), PGL(2,\mathbb{F})$ and $PSL(2,\mathbb{F})$, including the case q even. The results are well known.

A basic references for the representation theory of finite groups and character tables is [4]. For the groups under consideration see [1], [5], [2].

1.1 Notation

Let $GL(2,\mathbb{F})$ be the two—by–two matrices over $\mathbb{F}$ with non–zero determinant. Let $Z = \{diag(x,x) \mid x \in \mathbb{F}^*\}$ be the center of $GL(2,\mathbb{F})$ and

$$\begin{aligned} SL(2,\mathbb{F}) &= \{g \in GL(2,\mathbb{F}) \mid det(g) = 1\} \\ PGL(2,\mathbb{F}) &= GL(2,\mathbb{F})/Z \\ PSL(2,\mathbb{F}) &= SL(2,\mathbb{F})/Z \cap SL(2,\mathbb{F}) \end{aligned}$$

The order of $GL(2,\mathbb{F})$ is $(q+1)q(q-1)^2$. Both $PGL(2,\mathbb{F})$ and $SL(2,\mathbb{F})$ have order $(q+1)q(q-1)$. The order of $PSL(2,\mathbb{F})$ is $(q+1)q(q-1)/2$ if q is odd, and $(q+1)q(q-1)$ if q is even.

If q is odd these groups are all distinct. If q is even then $PGL(2,\mathbb{F}) = PSL(2,\mathbb{F}) = PSL(2,\mathbb{F})$, and if $q = 2$ then also $GL(2,\mathbb{F}) = PGL(2,\mathbb{F})$.

For any finite group G write $\hat{G}$ for the set of equivalence classes of irreducible finite-dimensional complex representations of G. For any finite–dimensional representation π of G write Θ_π for its character.

Let $\mathbb{E} = \mathbb{F}_{q^2}$, the unique quadratic extension of $\mathbb{F}$. If q is odd choose $\Delta \in \mathbb{F}^* - \mathbb{F}^{*2}$ and write $\mathbb{E} = \mathbb{F}(\delta) = \mathbb{F}(\sqrt{\Delta})$. For $z \in \mathbb{E}^*$ let $\overline{z} = z^q$; this is the action of the non–trivial element Galois group of $\mathbb{E}$ over $\mathbb{F}$. The norm map $N : \mathbb{E}^* \to \mathbb{F}^*$ is $N(z) = z\overline{z} = z^{q+1} \in \mathbb{F}$.

For $\chi \in \hat{\mathbb{E}}$ write $\overline{\chi}(z) = \chi(\overline{z})$.

Let $B = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\}$, $T = \{diag(x,y)\} \simeq \mathbb{F}^* \times \mathbb{F}^*$. For $\mu \in \hat{T}$ write $\mu = \mu(\alpha, \beta)$, with $\alpha, \beta \in \widehat{\mathbb{F}^*}$. We also write B, T for the corresponding sugroups of the other groups under consideration. For $\mu \in \hat{T}$ write μ for the one–dimensional representation of B whose restriction to T is equal to μ.

We will write representatives of the conjugacy classes. To say two elements are equal is to say their conjugacy classes are equal.

One can tell whether a finite group G is simple from its character table: G is not simple if and only if there exists a non–trivial element $g \in G$ and a non–trivial representation $\pi \in \hat{G}$ such that $\Theta_\pi(g) = \Theta_\pi(1)$. It follows from the tables that $PSL(2, \mathbb{F}_q)$ is simple if and only if $q \geq 4$.

1.2 Coincidences

For small values of q these groups are isomorphic to some other familiar groups:

1. $PSL(2,\mathbb{F}_2) = PGL(2,\mathbb{F}_2) = SL(2,\mathbb{F}_2) = GL(2,\mathbb{F}_2) \simeq S_3$,

2. $PGL(2,\mathbb{F}_3) \simeq S_4$, $PSL(2,\mathbb{F}_3) \simeq A_4$, $SL(2,\mathbb{F}_3) \simeq$ binary tetrahedral group

3. $PSL(2,\mathbb{F}_4) = PGL(2,\mathbb{F}_4) = SL(2,\mathbb{F}_4) = GL(2,\mathbb{F}_4) \simeq A_5$,

4. $PGL(2,\mathbb{F}_5) \simeq S_5$, $PSL(2,\mathbb{F}_5) \simeq A_5$, $SL(2,\mathbb{F}_5) \simeq$ the binary icosahedral group

5. $PSL(2,9) \simeq A_6$

2 $GL(2,\mathbb{F})$

2.1 Conjugacy Classes:

1. $c_1(x) = \begin{pmatrix} x & \\ & x \end{pmatrix}$ $(x \in \mathbb{F}^*)$,

2. $c_2(x) = \begin{pmatrix} x & 1 \\ & x \end{pmatrix}$ $(x \in \mathbb{F}^*)$,

3. $c_3(x,y) = \begin{pmatrix} x & \\ & y \end{pmatrix}$ $(x \neq y \in \mathbb{F}^*)$; $c_3(x,y) = c_3(y,x)$,

4. $c_4(z) = \begin{pmatrix} x & \Delta y \\ y & x \end{pmatrix}$ $(z = x + \delta y \in \mathbb{E} - \mathbb{F})$; $c_4(z) = c_4(\overline{z})$

Here and elsewhere $c_3(x,y) = c_3(y,x)$ means that the conjugacy classes of these two elements agree.

2.2 Representations:

For $\alpha, \beta \in \widehat{\mathbb{F}^*}$ let $\mu(\alpha,\beta)$ be the corresponding character of $T \simeq \mathbb{F}^* \times \mathbb{F}^*$. For $\mu \in \hat{T}$ let

$$\rho(\mu) = Ind_B^G(\mu)$$

where μ is extended to a one–dimensional representation of B as usual. This is the principal series, of dimension $q+1$. For example see [3], [1] or [2].

Let $\mu^w(\alpha,\beta) = (\beta,\alpha)$. Then $\rho(\mu) = \rho(\mu^w)$, and $\rho(\mu)$ is irreducible if and only if $\mu^w \neq \mu$. For $\alpha \in \widehat{\mathbb{F}^*}$ let $\mu = \mu(\alpha,\alpha)$ and write

$$\rho(\mu) = \overline{\rho}(\alpha) + \rho'(\alpha)$$

where $\dim(\overline{\rho}(\alpha)) = q$ and $\dim(\rho'(\alpha)) = 1$. Then $\rho'(\alpha)(g) = \alpha(det(g))$.

Representations:

1. $\rho(\mu)$ $(\mu^w \neq \mu)$,

2. $\overline{\rho}(\alpha)$ $(\alpha \in \hat{\mathbb{F}}^*)$

3. $\rho'(\alpha)$ $(\alpha \in \hat{\mathbb{F}}^*)$,

4. $\pi(\chi)$ $(\chi \in \hat{\mathbb{E}}, \chi \neq \overline{\chi})$

3 $SL(2,\mathbb{F})$

The order of $SL(2,\mathbb{F})$ is $(q+1)q(q-1)$. If q is even then $SL(2,\mathbb{F}) \simeq PGL(2,\mathbb{F}) \simeq PSL(2,\mathbb{F})$, and these tables contain the character tables for $PGL(2,\mathbb{F})$ and $PSL(2,\mathbb{F})$.

3.1 Notation

Let $\mathbb{E}^1$ be the kernel of the norm map $N : \mathbb{E}^* \to \mathbb{F}^*$. This has order $q+1$.

If q is odd let ζ be the unique non–trivial character of $\mathbb{F}^*$ with $\zeta^2 = 1$. Then

$$\zeta(-1) = (-1)^{\frac{q-1}{2}} = \begin{cases} 1 & q \equiv 1 \mod (4) \\ -1 & q \equiv 3 \mod (4) \end{cases}$$

and $\zeta(-1) = 1$ if and only if $-1 \in \mathbb{F}^{*2}$.

3.2 Conjugacy Classes (q odd):

1. $\pm I$
2. $c_2(\epsilon, \gamma) = \begin{pmatrix} \epsilon & \gamma \\ 0 & \epsilon \end{pmatrix} \quad (\epsilon = \pm 1, \gamma \in \{1, \Delta\})$
3. $c_3(x) = diag(x, x^{-1})$ $(x \neq \pm 1)$, $c_3(x) = c_3(x^{-1})$,
4. $c_4(z) = \begin{pmatrix} x & \Delta y \\ y & x \end{pmatrix}$ $(z = x + \delta y \in \mathbb{E}^1,\ z \neq \pm 1)$, $c_4(z) = c_4(\overline{z})$

3.3 Representations (q odd):

For $\alpha \in \widehat{\mathbb{F}^*}$ let $\rho(\alpha)$ be the restriction of the principal series representation $\rho(\mu(\alpha, 1))$ of $GL(2,\mathbb{F})$ to $SL(2,\mathbb{F})$. Define $\overline{\rho}(\alpha)$ and $\rho'(\alpha)$ similarly. Let $\pi(\chi)$ denote the restriction of the cuspidal representation $\pi(\chi)$ of $GL(2,\mathbb{F})$ to $SL(2,\mathbb{F})$. Let χ_0 be the unique non–trivial quadratic character of $\mathbb{E}^1$. Write

$$\rho(\zeta) = \omega_e^+ + \omega_e^w$$

and

$$\pi(\chi_0) = \omega_o^+ + \omega_o^w$$

Then $\omega_e^{\pm}$ are of dimension $\frac{q+1}{2}$, and $\omega_o^{\pm}$ are of dimension $\frac{q-1}{2}$. These are the four oscillator representations of $SL(2,\mathbb{F})$.

1. $\rho(\alpha)$ $(\alpha \in \widehat{\mathbb{F}^*}, \alpha^2 \neq 1)$,
2. $\overline{\rho}(1)$
3. $\rho'(1)$
4. $\pi(\chi)$ $(\chi \in \widehat{\mathbb{E}^1}, \chi^2 \neq 1)$
5. $\omega_e^{\pm}, \omega_o^{\pm}$

Let $\omega^+ = \omega_e^+ \oplus +\omega_o^+, \omega^- = \omega_e^- \oplus \omega_o^-$. These are the two oscillator representations, realized on $L^2(\mathbb{F})$, and $\omega_e^{\pm}$ (resp. $\omega_o^{\pm}$) consists of the even (resp. odd) functions.

3.4 Conjugacy Classes (q even):

1. I
2. $N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
3. $c_3(x) = diag(x, x^{-1})$, $x \neq 1$, $c_3(x) = c_3(x^{-1})$,
4. $c_4(z) = \begin{pmatrix} x & \Delta y \\ y & x \end{pmatrix}$ $(z = x + \delta y \in \mathbb{E}^1,\ z \neq 1)$, $c_4(z) = c_4(\overline{z})$

3.5 Representations (q even):

1. $\rho(\alpha)$ $(\alpha \neq 1)$
2. $\overline{\rho}(1)$
3. $\rho'(1)$
4. $\pi(\chi)$ $(\chi \in \widehat{\mathbb{E}^1}, \chi \neq 1)$

4 $PGL(2, \mathbb{F})$

Let $G = PGL(2, \mathbb{F}) = GL(2, \mathbb{F})/Z$. We assume q is odd. If q is even $PGL(2, \mathbb{F}) = PSL(2, \mathbb{F}) = SL(2, \mathbb{F})$. See Section 3 for the character table of $SL(2, \mathbb{F})$.

The order of $PGL(2, \mathbb{F}_q)$ is $(q+1)q(q-1)$ (if q is odd).

4.1 Conjugacy Classes

1. I
2. $N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
3. $c_3(x) = diag(x, 1)$ $(x \neq \pm 1)$, $c_3(x) = c_3(x^{-1})$,
4. $c_3(-1) = diag(-1, 1)$
5. $c_4(z)$ $(z \in \mathbb{E}^*/\mathbb{F}^* \simeq \mathbb{E}^1, z \neq \pm\delta)$, $c_4(z) = c_4(\overline{z})$,
6. $c_4(\delta)$

4.2 Representations

Write $\rho(\alpha), \overline{\rho}(\alpha), \rho'(\alpha)$ and $\pi(\chi)$ for the representations of $PGL(2, \mathbb{F})$ obtained from the corresponding representations of $GL(2, \mathbb{F})$ (which factor to $PGL(2, \mathbb{F})$) and passing to the quotient.

1. $\rho(\alpha)$ $(\alpha^2 \neq 1)$, $\rho(\alpha) = \rho(\alpha^{-1})$,
2. $\overline{\rho}(\alpha)$ $(\alpha^2 = 1)$
3. $\rho'(\alpha)$ $(\alpha^2 = 1)$
4. $\pi(\chi)$ $(\chi^2 \neq 1, \chi \neq \overline{\chi})$

5 $PSL(2, \mathbb{F})$

Let $G = PSL(2, \mathbb{F}) = SL(2, \mathbb{F}_q)/Z \cap SL(2, \mathbb{F})$. If q is even $Z \cap SL(2, \mathbb{F}) = I$, and $PSL(2, \mathbb{F}_q) = SL(2, \mathbb{F}_q) = PGL(2, \mathbb{F}_q)$. See Section 3.

We assume q is odd. The order of $PSL(2, \mathbb{F}_q)$ is $(q+1)q(q-1)/2$ (if q is odd).

5.1 Conjugacy Classes

Some notation is as in Section 3.

1. I
2. $c_2(\epsilon, \gamma) = \begin{pmatrix} \epsilon & \gamma \\ 0 & \epsilon \end{pmatrix} \quad (\epsilon = \pm 1, \gamma \in \{1, \Delta\})$
3. $c_3(x)$ $(x \neq \pm 1)$, $c_3(x) = c_3(-x) = c_3(\frac{1}{x}) = c_3(-\frac{1}{x})$
4. $c_4(z)$ $(z \in \mathbb{E}^1, z \neq \pm 1)$, $c_4(z) = c_4(\overline{z}) = c_4(-z) = c_4(-\overline{z})$

5.2 Representations

Some notation is as in Section 3.

1. $\rho(\alpha)$ $(\alpha^2 \neq 1)$, $\rho(\alpha) \simeq \rho(\alpha^{-1})$
2. $\overline{\rho}(1)$
3. $\rho'(1)$
4. $\pi(\chi)$ $(\chi^2 \neq 1, \chi \neq \overline{\chi})$, $\pi(\chi) \simeq \pi(\overline{\chi})$
5. $\omega_e^{\pm}$ if $\zeta(-1) = 1$
6. $\omega_o^{\pm}$ if $\zeta(-1) = -1$

6 Tables

6.1 $GL(2, \mathbb{F})$

Character Table of $GL(2, \mathbb{F}_q)$						
		Number:	$q-1$	$q-1$	$\frac{1}{2}(q-1)(q-2)$	$\frac{1}{2}q(q-1)$
		Size:	1	q^2-1	$q(q+1)$	$q(q-1)$
Rep	Dimension	Number	$c_1(x)$	$c_2(x)$	$c_3(x,y)$	$c_4(z)$
$\rho(\mu)$	$q+1$	$\frac{1}{2}(q-1)(q-2)$	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\mu(g)+\mu^w(g)$	0
$\overline{\rho}(\alpha)$	q	$q-1$	$q\alpha(x^2)$	0	$\alpha(xy)$	$-\alpha(Nz)$
$\rho'(\alpha)$	1	$q-1$	$\alpha(x^2)$	$\alpha(x^2)$	$\alpha(xy)$	$\alpha(Nz)$
$\pi(\chi)$	$q-1$	$\frac{1}{2}q(q-1)$	$(q-1)\chi(x)$	$-\chi(x)$	0	$-\chi(z)-\chi(\overline{z})$

6.2 $SL(2,\mathbb{F})$

Character Table of $SL(2,\mathbb{F})$, q odd							
		Number:	1	1	4	$\frac{q-3}{2}$	$\frac{q-1}{2}$
		Size:	1	1	$\frac{q^2-1}{2}$	$q(q+1)$	$q(q-1)$
Rep	Dimension	Number	I	$-I$	$c_2(\epsilon,\gamma)$	$c_3(x)$	$c_4(z)$
$\rho(\alpha)$	$q+1$	$\frac{q-3}{2}$	$(q+1)$	$(q+1)\alpha(-1)$	$\alpha(\epsilon)$	$\alpha(x)+\alpha(x^{-1})$	0
$\overline{\rho}(1)$	q	1	q	q	0	1	-1
$\rho'(1)$	1	1	1	1	1	1	1
$\pi(\chi)$	$q-1$	$\frac{q-1}{2}$	$q-1$	$(q-1)\chi(-1)$	$-\chi(\epsilon)$	0	$-\chi(z)-\chi(z^{-1})$
$\omega_e^{\pm}$	$\frac{q+1}{2}$	2	$\frac{q+1}{2}$	$\frac{q+1}{2}\zeta(-1)$	$\omega_e^{\pm}(\epsilon,\gamma)$	$\zeta(x)$	0
$\omega_o^{\pm}$	$\frac{q-1}{2}$	2	$\frac{q-1}{2}$	$-\frac{q-1}{2}\zeta(-1)$	$\omega_o^{\pm}(\epsilon,\gamma)$	0	$-\chi_0(z)$
$\omega^{\pm}$	q	2	q	$q\zeta(-1)$	$\kappa_{\pm}(\epsilon,\delta)$	$\zeta(x)$	$-\chi_0(z)$

Notation:

$$\zeta \in \widehat{\mathbb{F}^*/\mathbb{F}^{*2}}, \zeta^2 = 1$$
$$\beta = \zeta(-1)$$
$$\tau = \sqrt{\beta q}$$
$$\omega_e^{\pm}(\epsilon,\gamma) = \frac{1}{2}(\zeta(\epsilon) \pm \zeta(\gamma)\tau)$$
$$\omega_o^{\pm}(\epsilon,\gamma) = \epsilon\frac{1}{2}(-\zeta(\epsilon) \pm \zeta(\gamma)\tau)$$
$$\kappa_{\pm}(\epsilon,\gamma) = \begin{cases} \pm\zeta(\gamma)\tau & \epsilon = 1 \\ \zeta(-1) & \epsilon = -1 \end{cases}$$

Character Table of $SL(2,\mathbb{F})$, q even						
		Number:	1	1	$\frac{q-2}{2}$	$\frac{q}{2}$
		Size:	1	q^2-1	$q(q+1)$	$q(q-1)$
Rep	Dimension	Number	I	N	$c_3(x)$	$c_4(z)$
$\rho(\alpha)$	$q+1$	$\frac{q-2}{2}$	$(q+1)$	1	$\alpha(x)+\alpha(x^{-1})$	0
$\overline{\rho}(1)$	q	1	q	0	1	-1
$\rho'(1)$	1	1	1	1	1	1
$\pi(\chi)$	$q-1$	$\frac{q}{2}$	$q-1$	-1	0	$-\chi(z)-\chi(z^{-1})$

6.3 $PGL(2, \mathbb{F})$

Character Table of $PGL(2, \mathbb{F}_q)$, q odd									
		Number:	1	1	$\frac{q-3}{2}$	1	$\frac{q-1}{2}$	1
		Size:	1	$q^2 - 1$	$q(q+1)$	$\frac{q(q+1)}{2}$	$q(q-1)$	$\frac{q(q-1)}{2}$
Rep	Dimension	Number	1	N	$c_3(x)$	$c_3(-1)$	$c_4(z)$	$c_4(\delta)$
$\rho(\alpha)$	$q+1$	$\frac{q-3}{2}$	$(q+1)$	1	$\alpha(x) + \alpha(x^{-1})$	$2\alpha(-1)$	0	0
$\overline{\rho}(\alpha)$	q	2	q	0	$\alpha(x)$	$\alpha(-1)$	$-\alpha(Nz)$	$\alpha(\Delta)$
$\rho'(\alpha)$	1	2	1	1	$\alpha(x)$	$\alpha(-1)$	$\alpha(Nz)$	$-\alpha(\Delta)$
$\pi(\chi)$	$q-1$	$\frac{q-1}{2}$	$(q-1)$	-1	0	0	$-\chi(z) - \chi(z^{-1})$	$-2\chi(\delta)$

If q is even then $PGL(2, \mathbb{F}) = SL(2, \mathbb{F})$.

6.4 $PSL(2, \mathbb{F})$

Character Table of $PSL(2, \mathbb{F}_q)$, $q \equiv 1 \mod (4)$							
		Number:	1	2	$\frac{q-5}{4}$	1	$\frac{q-1}{4}$
		Size:	1	$(q^2-1)/2$	$q(q+1)$	$\frac{q(q+1)}{2}$	$q(q-1)$
Rep	Dimension	Number	1	$c_2(\gamma)$	$c_3(x)$	$c_3(\sqrt{-1})$	$c_4(z)$
$\rho(\alpha)$	$q+1$	$\frac{q-5}{4}$	$(q+1)$	1	$\alpha(x)+\alpha(x^{-1})$	$2\alpha(\sqrt{-1})$	0
$\overline{\rho}(1)$	q	1	q	0	1	1	-1
$\rho'(1)$	1	1	1	1	1	1	1
$\pi(\chi)$	$q-1$	$\frac{q-1}{4}$	$(q-1)$	-1	0	0	$-\chi(z)-\chi(z^{-1})$
$\omega_e^{\pm}$	$\frac{q+1}{2}$	2	$\frac{q+1}{2}$	$\omega_e^{\pm}(1,\gamma)$	$\zeta(x)$	$\zeta(\sqrt{-1})$	0

Character Table of $PSL(2, \mathbb{F})$, $q \equiv 3 \mod (4)$							
		Number:	1	2	$\frac{q-3}{4}$	$\frac{q-7}{4}$	1
		Size:	1	$(q^2-1)/2$	$q(q+1)$	$q(q-1)$	$\frac{q(q-1)}{2}$
Rep	Dimension	Number	1	$c_2(\gamma)$	$c_3(x)$	$c_4(z)$	$c_4(\delta)$
$\rho(\alpha)$	$q+1$	$\frac{q-3}{4}$	$(q+1)$	1	$\alpha(x)+\alpha(x^{-1})$	0	0
$\overline{\rho}(1)$	q	1	q	0	1	-1	1
$\rho'(1)$	1	1	1	1	1	1	1
$\pi(\chi)$	$q-1$	$\frac{q-3}{4}$	$(q-1)$	-1	0	$-\chi(z)-\chi(z^{-1})$	$-2\chi(\delta)$
$\omega_o^{\pm}$	$\frac{q-1}{2}$	2	$\frac{q-1}{2}$	$\omega_o^{\pm}(1,\gamma)$	0	$-\chi_0(z)$	$-\chi_0(\delta)$

If q is even then $PSL(2, \mathbb{F}) = SL(2, \mathbb{F})$.

7 Proofs

For $GL(2,\mathbb{F})$ see [3], [1], [2].

The character table for $SL(2,\mathbb{F}_q)$ may be found in [5] (q odd) or [6] (q even). (There is a misprint in the table in [5]: the last two columns in the row for $diag(-1,1)$ should each be multiplied by -1.)

Alternatively, once we have $GL(2,\mathbb{F})$ we restrict to $SL(2,\mathbb{F})$; see [2]. For q odd $SL(2,\mathbb{F})Z$ has index 2 in $GL(2,\mathbb{F})$; therefore the the restriction of an irreducible representation π is either irreducible or the direct sum of two irreducible summands of the same dimension. The only hard part is calculating the character of the halves of the oscillator representations $\omega_e^{\pm}, \omega_o^{\pm}$. If q is even $GL(2,\mathbb{F}) = SL(2,\mathbb{F})Z$ and all restrictions are irreducible.

For $PGL(2,\mathbb{F})$ and $PSL(2,\mathbb{F})$ it is merely a question of taking a subset of the corresponding representations of $GL(2,\mathbb{F})$ and $SL(2,\mathbb{F})$. Similar computations give the conjugacy classes and representations.

For example for q odd consider the number of conjugacy classes $diag(x,\frac{1}{x})$ with $x \neq \pm 1$. This is the set $x \neq \pm 1$, modulo $x \to \frac{1}{x}, -x$. If $-1 \notin \mathbb{F}^{*2}$ there are no fixed points of this action, and there are $\frac{q-3}{4}$ such conjugacy classes. If $-1 = i^2$ then $i = -\frac{1}{i}$, so there are $\frac{q-5}{4}+1$ such conjugacy classes. Also note that the Weyl group element is contained in $Stab(c_3(i))$, and the order of this conjugacy classes is $\frac{q(q+1)}{2}$.

Similarly the set of characters α of $\mathbb{F}^*$ which give non–isomorphic irreducible principal series is the set of characters such that $\alpha^2 \neq 1$ and $\alpha(-1) = 1$, modulo $\alpha \to \alpha^{-1}$. There are $\frac{q-1}{2}$ characters of $\mathbb{F}^*/\pm 1$. Suppose $-1 \in \mathbb{F}^{*2}$. The characters of $\mathbb{F}^*/\pm 1$ are $1, \zeta$ and $\frac{q-5}{2}$ others, which consists of $\frac{q-5}{4}$ pairs α, α^{-1}. If $-1 \notin \mathbb{F}^{*2}$ then the $\frac{q-1}{2}$ characters of $\mathbb{F}^*/\pm 1$ consist of the trivial character and $\frac{q-3}{4}$ pairs α, α^{-1}.

Note that $SL(2,\mathbb{F})Z = GL(2,\mathbb{F})_+ := \{g \in GL(2,\mathbb{F}) \,|\, det(g) \in \mathbb{F}^{*2}\}$. If q is even then $\mathbb{F}^{*2} = \mathbb{F}^*$. It follows that $GL(2,\mathbb{F}) = SL(2,\mathbb{F})Z$. Also $PSL(2,\mathbb{F}) = SL(2,\mathbb{F})/\pm I = SL(2,\mathbb{F})$, and $PGL(2,\mathbb{F}) = SL(2,\mathbb{F})Z/Z = SL(2,\mathbb{F})/SL(2,\mathbb{F}) \cap Z = SL(2,\mathbb{F})$.

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