Characters of Non-Linear Groups

Notes available at www.math.umd.edu/ ˜jda/preprints.

General setup:

$\mathbb{F}$: a local field of characteristic 0

$\mathbb{G}$: an algebraic group over $\mathbb{F}$, often simply connected, $G = \mathbb{G}(\mathbb{F})$,

$\tilde{G}$ a central extension of G:

$$1 \longrightarrow A \longrightarrow \tilde{G} \xrightarrow{p} G \longrightarrow 1$$

$(|A| < \infty)$.

When $\mathbb{G}$ is simply connected $\tilde{G}$ is a non-linear group.

Relevant semisimple orbits

Definition: An irreducible representation π of $\tilde{G}$ is *genuine* if the central character χ_π is an injection when restricted to A.

Fix a Cartan subgroup T of G, let $\tilde{T} = p^{-1}(T)$, $Z(\tilde{T})$ the center of $\tilde{T}$.

Definition: If $g \in \tilde{T}$ is regular,

$$g \text{ is } \textit{relevant} \text{ if } g \in Z(\tilde{T})$$

The next Proposition originates in [Flicker]:

Proposition: Suppose π is an irreducible genuine representation, with character Θ_π. If g is not relevant then

$$\Theta_\pi(g) = 0$$

Proof: Suppose g is not relevant. Then

$$g \underset{\tilde{T}}{\sim} g'$$

for some $g' \neq g$. Projecting both sides to G gives $p(g) \underset{T}{\sim} p(g') \Rightarrow p(g) = p(g') \Rightarrow g' = ag$ for some $a \in A, a \neq 1$. Therefore

$$g \sim g' = ga$$

Since Θ_π is conjugation invariant this gives

$$\Theta_\pi(g) = \Theta_\pi(ga) = \chi_\pi(a)\Theta_\pi(g).$$

Since π is genuine, $\chi_\pi(a) \neq 1$ proving the result.

Remark: Rebecca Herb noticed an error in one version of Harish-Chandra's character formula for the discrete series. In [Acta 1965] formula (pg. 304) is correct, but in (Lemma 57) the terms are rearranged, and written:

$$\Theta_\pi(g) = \frac{1}{\Delta(g)} \sum_{t\in w\backslash W_G} \epsilon(t)\lambda^t(h_1) \sum_{s\in W(A^+)} \epsilon(s)c_\lambda(s:t:A^+)exp(st(\lambda)^y(H_2))$$

The first sum is not well defined for $g \notin Z(T)$. However by the preceding discussion this may be corrected as follows:

$$\Theta_\pi(g) = \begin{cases} \frac{1}{\Delta(g)} \sum_{t\in w\backslash W_G} \epsilon(t)\lambda^t(h_1) & \\ \qquad \sum_{s\in W(A^+)} \epsilon(s)c_\lambda(s:t:A^+)exp(st(\lambda)^y(H_2)) & g\in Z(T) \\ 0 & g\notin Z(T) \end{cases}$$

Remark: A similar phenomenon holds for $\tilde{G}$ a *linear* cover of G. Recall g is regular if $Cent_G(g)^0$ is a torus, and strongly regular if $Cent_G(g)$ is a torus. Suppose $g\in\tilde{G}$ and $p(g)$ is regular, but not strongly regular. Then $\Theta_\pi(g) = 0$ for any genuine representation of $\tilde{G}$.

Example: $G = SO(2n), \tilde{G} = Spin(2n)$. Suppose $g \in SO(2n)$ is regular. Then g is not strongly regular if and only both ± 1 are eigenvalues of g.

$$|\Theta_{Spin_+} \pm \Theta_{Spin_-}(g)| = |det(1\pm p(g))|^{\frac{1}{2}}$$

It follows that $\Theta_{Spin_\pm}(g) = 0$ if and only if $p(g)$ is not strongly regular.

__**Spin-Oscillator Duality**__

Let $G = Sp(2n,\mathbb{R}), G' = SO(n+1,n)$. For g, g' semisimple elements, write $g \leftrightarrow g'$ if g, g' have the same eigenvalues (This is the stabilized orbit correspondence.)

Let $osc = osc_+ \oplus osc_-$ be the oscillator representation of $\widetilde{Sp}(2n,\mathbb{R})$. Then (Howe)

$$|\Theta_{osc}(g)| = |det(1-p(g))|^{-\frac{1}{2}}$$

Proposition: The correspondence $g \to g'$ may be lifted uniquely to a correspondence $\tilde{g} \to \tilde{g}'$ between $\widetilde{Sp}(2n,\mathbb{R})$ and $Spin(n+1,n)$ with the following property:

$$\Theta_{osc_+ - osc_-}(\tilde{g})\Theta_{Spin}(\tilde{g}') = 1$$

Remark: In absolute value this follows from

$$\begin{aligned} |\Theta_{Spin}(g)| &= |det(1+p(g))|^{\frac{1}{2}} \\ |\Theta_{osc_+\pm osc_-}(g)| &= |det(1\mp p(g))|^{-\frac{1}{2}} \end{aligned}$$

Flicker/Kazhdan/Patterson Lifting

Let $G = GL(n, \mathbb{F})$, $\tilde{G}$ an N-fold central extension.

Flicker (for $GL(2)$) and Kazhdan/Patterson (for $GL(n)$) defined an operation t_*, taking a virtual character of G to a genuine virtual character of $\tilde{G}$. If σ is irreducible and tempered, then $t_*(\sigma)$ is zero or $\pm$ an irreducible tempered representation. Over $\mathbb{C}$ (Tadic) and $\mathbb{R}$ (Adams-Huang), for σ irreducible unitary, $t_*(\sigma)$ is zero, or $\pm$ an irreducible unitary representation.

Inversion

Flicker/Kazhdan/Patterson lifting completely describe the irreducible characters of covers of $GL(n)$. What about $\widetilde{SL(n)}$?

Consider a non-trivial $N-fold$ cover $\widetilde{SL(n)}$ of $SL(n)$, and *assume N divides n*, for example the 2-fold cover of $SL(2)$. This extends to a cover $\widetilde{GL(n)}$ of $GL(n)$ ($c = 0$ in Kazhdan/Patterson's notation).

Let π be an irreducible genuine representation of $\widetilde{GL(n)}$, and write $\sum_i \pi_i$ for the restriction of π to $\widetilde{SL(n)}$.

Let

$$GL(n)_+ = SL(n)D = \{g \in GL(n) \,|\, det(g) \in \mathbb{F}^{*n}\}$$

The key point is:

$$Z(\widetilde{GL(n)}_+)/Z(\widetilde{GL(n)}) \simeq \mathbb{F}^*/\mathbb{F}^{*N}$$

Write χ_i for the central character of π_i considered as a representation of $\widetilde{GL(n)}_+$. Choose representatives x_i of $\mathbb{F}^*/\mathbb{F}^{*N}$, and $z_i \in Z(\widetilde{GL(n)}_+), p(z_i) = x_i I$.

Proposition:

$$\theta_{\pi_i}(g) = \frac{1}{m}\sum_j \chi_i(z_j)^{-1}\theta_\pi(z_j g)$$

Remark: This is very different from the case of linear groups, for which the character of a representation of $SL(n)$ is not obtained directly from the corresponding characters of $GL(n)$. In fact for $SL(2)$ this is the first example of endoscopy, in which an endoscopic group T (an elliptic torus) plays a role.

Support of Distributions

It is interesting to consider the restriction to $\widetilde{SL}(n)$ of invariant distributions of $\widetilde{SL(n)}$. By lifting a virtual character of $GL(n)$ which vanishes on $SL(n)$, we obtain a virtual character of $SL(n)$ with small support. To obtain such a virtual character of $GL(n)$, simply take $\pi - \pi \otimes \beta$ for some character β.

For simplicity we consider the case $GL(2)$, $N = 2$. (The general situation is similar, up to complications arising from the center of $\widetilde{GL}(n)$). Suppose σ is a virtual representation of $GL(2)$, and let $\pi = t_*(\sigma)$. By [Flicker],

$$\Theta_\pi(g) = \sum_{\substack{h \\ h^2 = p(g)}} \Phi(h)\Theta_\sigma(h)$$

Here $\Phi(h)$ is a (transfer) factor which need not concern us here. Note that if $p(g) = h^2$ is elliptic, the sum is over $\pm h$.

Now let σ be an irreducible representation of $GL(2)$, with $t_*(\sigma) \neq 0$. Fix a character β_0 of $\mathbb{F}^*$ with $\beta_0(-1) = -1$, and let $\beta(g) = \beta_0(det(g))$. Let $\pi = t_*(\sigma - \sigma \otimes \beta)$, restricted to $\widetilde{SL}(2)$.

Suppose g is elliptic. By the preceding formula, $\Theta_\pi(g) = 0$ if

(1) $p(g) = h^2$ with $h \in SL(2)$,

(2) $p(g) \neq h^2$ for any $h \in GL(2)$

Therefore: the support of Θ_π, restricted to the regular elliptic set in $\widetilde{SL}(2)$, is

$$\{g \,|\, p(g) = h^2, det(h) = -1\}$$

This is a fairly small but non-empty set. Suppose T is an elliptic torus in $GL(2)$, isomorphic to the multiplicative group $\mathbb{E}^*$ of a quadratic extension $\mathbb{E} = \mathbb{F}(\sqrt{\Delta})$. The determinant of an element $z \in \mathbb{E}^*$ is the norm $N(z)$, the torus in $SL(2)$ is the norm one elements, and the set we are considering is

$$\{z^2 \,|\, N(z) = -1\}$$

This is non-empty if and only if $(-1, \Delta)_{\mathbb{F}} = 1$. For example if -1 is not a square in $\mathbb{F}$ and the residual characteristic is odd, this holds if and only if $\Delta = -1$. In this case Θ_π is supported on:

$$\{g \in T - T^2 \,|\, T \text{ unramified}\}.$$

Note that $T - T^2 = -T^2$ is a coset of index 2 in T.

Finally we note that π may be zero. By considering the hyperbolic set, we see $\pi \neq 0$ if $\Theta_\sigma(g) \neq 0$ for some $g = \begin{pmatrix} x & \\ & -\frac{1}{x} \end{pmatrix}$.

For example, taking σ to be the trivial representation gives pi is the sum of the oscillator representations of $\widetilde{SL}(2)$ (there are $|\mathbb{F}^*/\mathbb{F}^{*2}|$ such representations).