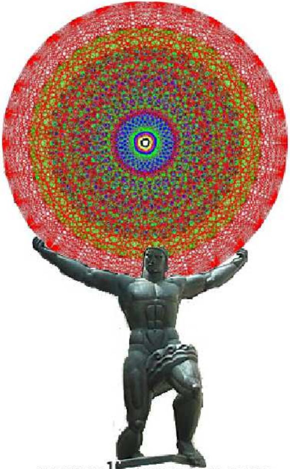
Computing Global Characters



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Roughly: fix H,

$$\theta_\pi(g) = \frac{\sum a(\pi, w)e^{w\lambda}(g)}{\Delta(g)}$$

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Not known... use character theory to get some information (see www.liegroups.org/tables/unipotent)

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$(A, B) \sim (g^t A, B g^{-1})$ for $g \in GL(m, \mathbb{Z})$

Example: Here is complete information about representations of $SL(2,\mathbb{R})$, including their characters.

```
block: block
0(0,1):  0  [i1]  1   (2,*)    0  e
1(1,1):  0  [i1]  0   (2,*)    0  e
2(2,0):  1  [r1]  2   (0,1)    1  1

block: klbasis
0: 0: 1
1: 1: 1
2: 0: 1
   1: 1
   2: 1

5 nonzero polynomials, and 0 zero polynomials,
at 5 Bruhat-comparable pairs.
```

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Other approaches (Schmid, Goresky-Kottwitz-MacPherson, Zuckerman,…)

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Theorem:

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$$(H,\Lambda) \to \begin{cases} I(H,\Lambda) & \text{standard (induced) module} \\ \pi(H,\Lambda) & \text{irreducible Langlands quotient} \end{cases}$$

Fix $H, \Delta^+,$

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(drop Δ^+)

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Proposition: Formula for $\theta_{I(H,\Lambda)}$ on $H(\mathbb{R})$:

$$\theta_{I(H,\Lambda)}(h) = \frac{\sum_{W_{\mathbb{R}}} \mathrm{sgn}(w)(w\Lambda)(h)}{D(h)} \quad (h \in H(\mathbb{R})_+)$$

$W_{\mathbb{R}} = W(G(\mathbb{R}), H(\mathbb{R})) \subset W(G, H)$

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Corollary: $\Gamma \in \widehat{H(\mathbb{R})}_\rho$:

$$a(I(H,\Lambda),\Gamma) = \begin{cases} \pm 1 & \Gamma = w\Lambda \\ 0 & \text{otherwise} \end{cases}$$

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If

$$(*) \qquad \mathrm{Re}\langle d\Lambda, \alpha^\vee\rangle \geq 0 \quad \text{for all } \alpha \in \Delta^+$$

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Theorem: Fix (H,Λ) satisfying (*):

$$a(I(H',\Lambda'),\Lambda) = \begin{cases} \pm 1 & (H,\Lambda)\sim(H',\Lambda') \\ 0 & \text{otherwise} \end{cases}$$

$\{\pi(H,\Lambda)\}$ and $\{I(H,\Lambda)\}$ are both bases of the Grothendieck group

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$$\pi = \sum \mathrm{M}(I,\pi)I \text{ (character formula)}$$

This is precisely what is computed by the Kazhdan-Lustig-Vogan polynomials (the `klbasis` command)

Corollary: Assuming (*),

$$a(\pi, \Lambda) = \pm \mathrm{M}(I(H, \Lambda), \pi)$$

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Conclusion: KLV-polynomials $\Rightarrow$ explicit formulas for all $a(\pi,\Lambda)$

Example: $Sp(4,\mathbb{R})$

```
 0( 0,6):  0  [i1,i1]   1   2   ( 4, *)  ( 5, *)   0  e
 1( 1,6):  0  [i1,i1]   0   3   ( 4, *)  ( 6, *)   0  e
 2( 2,6):  0  [ic,i1]   2   0   ( *, *)  ( 5, *)   0  e
 3( 3,6):  0  [ic,i1]   3   1   ( *, *)  ( 6, *)   0  e
 4( 4,5):  1  [r1,C+]   4   9   ( 0, 1)  ( *, *)   1  1
 5( 5,4):  1  [C+,r1]   7   5   ( *, *)  ( 0, 2)   2  2
 6( 6,4):  1  [C+,r1]   8   6   ( *, *)  ( 1, 3)   2  2
 7( 7,3):  2  [C-,i1]   5   8   ( *, *)  (10, *)   2  1,2,1
 8( 8,3):  2  [C-,i1]   6   7   ( *, *)  (10, *)   2  1,2,1
 9( 9,2):  2  [i2,C-]   9   4   (10,11)  ( *, *)   1  2,1,2
10(10,0):  3  [r2,r1]  11  10   ( 9, *)  ( 7, 8)   3  2,1,2,1
11(10,1):  3  [r2,rn]  10  11   ( 9, *)  ( *, *)   3  2,1,2,1
```

```
0:  0: 1
1:  1: 1
2:  2: 1
3:  3: 1
4:  0: 1
    1: 1
    4: 1
5:  0: 1
    2: 1
    5: 1
6:  1: 1
    3: 1
    6: 1
7:  0: 1
    1: 1
    2: 1
    4: 1
    5: 1
    7: 1
8:  0: 1
    1: 1
    3: 1
    4: 1
    6: 1
    8: 1

 9:  0: 1
     1: 1
     2: 1
     3: 1
     4: 1
     5: 1
     6: 1
     9: 1

10:  0: 1
     1: 1
     2: 1
     3: 1
     4: 1
     5: 1
     6: 1
     7: 1
     8: 1
     9: 1
    10: 1

11: 2: q
    3: q
    9: 1
   11: 1
```

	$\mathbb{R}^* \times \mathbb{R}^*$: Irreducible Modules							
π	(2,1)	(1, 2)	$(2, -1)$	$(-1, 2)$	$(1, -2)$	$(-2, 1)$	$(-1, -2)$	$(-2, -1)$
$\pi(0)$						1, 1	1, 0	0, 1
$\pi(1)$						1, 1	1, 0	0, 1
$\pi(2)$							1, 0	$-1, 0$
$\pi(3)$							1, 0	$-1, 0$
$\pi(4)$				1,1		-1,-1	-1,1	1,-1
$\pi(5)$					1,0	-1,0	-1,0	1,0
$\pi(6)$					1,0	-1,0	-1,0	1,0
$\pi(7)$			1,0	-1,0	-1,0	1,0	1,0	-1,0
$\pi(8)$			1,0	-1,0	-1,0	1,0	1,0	-1,0
$\pi(9)$		1,1		-1,-1	-1,1	2,0	1,-1	-2,0
$\pi(10)$	1,0	-1,0	-1,0	1,0	1,0	-1,0	-1,0	1,0
$\pi(11)$	0,1	0,-1	0,1	1,0	0,-1	-1,0	-1,0	1,0