Duality for Nonlinear Simply Laced Groups

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1 Introduction

The motivation for the current paper is to extend some of the formalism of the Langlands classification to certain nonlinear (that is, nonalgebraic) double covers of the real points of a reductive algebraic group defined over $\mathbb{R}$. Such nonalgebraic groups have long been known to play an interesting role in the theory of automorphic forms of algebraic groups.

Suppose $\mathbb{G}$ is a connected reductive algebraic group defined over a local field $\mathbb{F}$. For simplicity assume $G = \mathbb{G}(F)$ is split. The local Langlands conjecture, among other things, implies that (L-packets of) irreducible admissible representations of G are parametrized by (admissible) homomorphisms from $W_{\mathbb{F}}$ to the complex "dual" group $\mathbb{G}^\vee(\mathbb{C})$ of $\mathbb{G}$. Here $W_{\mathbb{F}}$ is the Weil group of $\mathbb{F}$. This is known to hold in the case $\mathbb{F} = \mathbb{R}$ [L] when $G = \mathrm{GL}(n)$ over any local field [LRS], [HT], [He] and in some other limited cases.

One of the remarkable features of the Langlands conjecture is the appearance of the dual group $\mathbb{G}^\vee$. An important first step in extending Langlands' formalism to nonlinear groups is to find the correct analog of $\mathbb{G}^\vee$. There is no obvious natural candidate for it.

We consult the refinement of the local Langlands conjecture given in [V6]. The viewpoint adopted there, originating in work of Zelevinsky, Kazhdan, Lusztig and others, is that the right dual object is a complex algebraic variety X equipped with an action of $\mathbb{G}^\vee(\mathbb{C})$. The $\mathbb{G}^\vee(\mathbb{C})$ orbits on X should capture detailed information about the representation theory of G.

For nonarchimedian fields X is precisely the space of admissible Weil group homomorphisms alluded to above. For archimedian fields that space must be modified as in [ABV]. Something remarkable happens in this case: $\mathbb{G}^\vee(\mathbb{C})$-orbits on X also parametrize representation theory of real forms of the dual group $\mathbb{G}^\vee$. The local Langlands conjecture of [V6], cf. [ABV, Theorem 10.4], then becomes a duality (sometimes referred to as "Vogan duality") between representations of real forms of $\mathbb{G}$ and those of $\mathbb{G}^\vee$. In this form the duality reduces to [V4, Theorem 13.13].

The point of this discussion is that it suggests the possibility of extending the duality theory of [V4] to nonlinear groups as a means to extending the Langlands formalism to them. The first successes in this direction were [RT1]–[RT3] which treated all simple groups of type A as well as the metaplectic double cover of the symplectic group. (The arguments in these papers were often case-by-case.) The purpose of the present paper is to give a unified duality theory for all nonlinear double covers of simply laced real reductive groups.

To make these ideas more precise we briefly describe the main result of [V4]. Let $\mathcal{B}$ be a block of representations of G with regular infinitesimal character; this is a finite set of irreducible representations. (For terminology related to blocks and infinitesimal characters, see the discussions at the beginning of Section 4 and preceding Theorem 7.5 respectively.) Write $\mathcal{M}$ for the $\mathbb{Z}$-module spanned by the elements of $\mathcal{B}$ viewed as a submodule of the Grothendieck group. Then $\mathcal{M}$ has two distinguished bases: the irreducible representations in $\mathcal{B}$, and a corresponding basis of standard modules. We may thus consider the matrix relating these two bases. (The Kazhdan-Lusztig-Vogan algorithm of [V3] provides a means to compute this matrix.) We say a block $\mathcal{B}'$ of representations is *dual* to $\mathcal{B}$ if there is a bijection between $\mathcal{B}$ and $\mathcal{B}'$ for which, roughly speaking, the two corresponding change of basis matrices for $\mathcal{B}$ and $\mathcal{B}'$ are inverse-transposes of each other. See Section 4 for precise definitions.

Suppose the infinitesimal character λ of $\mathcal{B}$ is integral. According to Vogan [V4, Theorem 13.13] there is a real form $G^\vee$ of $\mathbb{G}^\vee$ and a block $\mathcal{B}^\vee$ for $G^\vee$, so that $\mathcal{B}$ is dual to $\mathcal{B}^\vee$. If λ is not integral the same result holds with $\mathbb{G}^\vee$ replaced by a subgroup of $\mathbb{G}^\vee$. (The case of singular infinitesimal character introduces some subtleties; see Remark 13.14.)

All of the ingredients entering the statement of [V4, Theorem 13.13] still make sense for nonlinear groups. (The computability of the change basis matrix in the nonlinear setting is established in [RT3].) Thus we seek to establish a similar statement in the nonlinear case.

The main result is:

Theorem 1.1 (cf. Theorem 13.13) *Suppose G is a real reductive linear* *group; see Section 2 for precise assumptions. We assume the root system of G is simply laced. Suppose $\widetilde{G}$ is an admissible two-fold cover of G (Definition 3.4). Let $\mathcal{B}$ be a block of genuine representations of $\widetilde{G}$, with regular infinitesimal character. Then there is a real reductive linear group G', an admissible two-fold cover $\widetilde{G'}$ of G', and a block of genuine representations of $\widetilde{G'}$ such that $\mathcal{B}'$ is dual to $\mathcal{B}$.*

By construction the $\mathrm{Lie}_{\mathbb{C}}(G') \subset \mathrm{Lie}_{\mathbb{C}}(G)$, with equality if the infinitesimal character of $\mathcal{B}$ is half-integral (Definition 9.4).

Often, but not always, $\widetilde{G'}$ is a nonlinear group. (Roughly speaking it is nonlinear whenever G' admits such a cover.) An important difference between Theorem 1.1 and [V4] is that $\mathrm{Lie}(G')$ is a real form of (a subalgebra of) $\mathrm{Lie}_{\mathbb{C}}(G)$, rather than its dual. While this is not a meaningful distinction in the simply laced case, this distinction does of course appear in the general case. For example if $\widetilde{G}$ is the two-fold cover $\widetilde{Sp}(2n,\mathbb{R})$ of $Sp(2n,\mathbb{R})$, then the dual group is $\widetilde{Sp}(2n,\mathbb{R})$, rather than a cover of $SO(2n+1)$ [RT1].

It is worth noting that Theorem 13.13 together with [RT2] give a duality theory for all simple, simply connected groups except the odd Spin groups and those of type F_4. (The case of G_2 is covered by this paper; see the end of Section 2). However, as is also true in [V4], treatment of simple groups is inadequate to handle general reductive groups. In particular Theorem 13.13 cannot be reduced to the case of simple groups (although it comes much closer to being true than in the linear case). We plan to return to the non-simply laced case in a subsequent paper.

There is a close relationship between duality theorems for nonlinear groups and character theory. For instance [A2] and [R1] give "lifting of characters" relating characters of $\widetilde{\mathrm{Sp}}(2n)$ and $SO(2n+1)$. Similarly [KP] relates characters of the two-fold cover $\widetilde{GL}(n,\mathbb{R})$ to those of $GL(n,\mathbb{R})$. The papers [RT2]–[RT3] give the relationship between duality theorems like Theorem 1.1 and these character theories, and also suggest a framework for the general situation. Meanwhile the manuscript [AH] generalizes the character theory of [A2] and [KP] to all simply laced groups. The character theory of a nonlinear two-fold cover of such a group is related to that of a linear group. We return to the relationship of this character theory and duality, as predicted in [RT3], in a subsequent paper.

It is worth pointing out that the genuine representation theory of a simply laced nonlinear group $\widetilde{G}$ is in many ways much simpler than that of a linear group. For example $\widetilde{G}$ has at most one genuine discrete series representation with given infinitesimal and central characters. The same result holds for principal series of a split group (and can be considered the dual of the preceding assertion). See Remark 7.7. Furthermore, while disconnectedness is a major complication in the proofs in [V4], it plays no role for simply laced nonlinear groups (Proposition 9.14).

Finally we point out that it appears difficult to formulate a good general duality theory for covers other than the two-fold covers of real groups. It is perhaps reassuring that the existence of higher covers is a somewhat anomalous feature of the real case. For instance, over a nonarchimedian local field $\mathbb{F}$, the fundamental group of the $\mathbb{F}$-points of a connected, simple, simply connected group $\mathbb{G}$ is a subgroup of the roots of unity of $\mathbb{F}$. By contrast, if $\mathbb{F} = \mathbb{R}$ and the corresponding fundamental group has more than two elements then it is in fact infinite.

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2 Some notation and structure theory

By a *real reductive linear group* we mean a group in the category defined in [V1, Section 0.1]. Thus $\mathfrak{g} = \mathrm{Lie}_{\mathbb{C}}(G)$ is a reductive Lie algebra, G is a real group in Harish-Chandra's class [KV, Definition 4.29], has a faithful finite-dimensional representation, and has abelian Cartan subgroups. Recall that a Cartan subgroup of G is the centralizer of a Cartan subalgebra of $\mathfrak{g}$. Examples of real reductive linear groups include the real points of connected complex reductive groups, and any subgroup of finite index of such a group.

We will sometimes work in the greater generality of a *real reductive group in Harish-Chandra's class*, i.e. $\mathfrak{g}$ is reductive and G is a real group in Harish-Chandra's class. This allows for Cartan subgroups to be nonabelian. Examples include any two-fold cover of a real reductive linear group (see Section 3).

We will always assume we have chosen a Cartan involution θ of G and let $K = G^{\theta}$, a maximal compact subgroup of G. Fix a θ-stable Cartan subgroup H of G. (Here "Cartan subgroup" means the centralizer in G of a Cartan subalgebra of $\mathfrak{g}$.) Write $H = TA$ with $T = H \cap K$ and A connected and simply connected. We denote the Lie algebras of $G, H, K \ldots$

by $\mathfrak{g}_0, \mathfrak{h}_0, \mathfrak{k}_0$, and their complexifications by $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}$. The Cartan involution acts on the roots $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$. We use r, i, cx, c, n to denote the real, imaginary, complex, compact, and noncompact roots, respectively (as in [V1], for example). We say G is simply laced if all roots in each simple factor of Δ have the same length. We only assume G is simply laced when necessary. We declare all the roots to be long in this case.

We denote the center of a group G by $Z(G)$, and the identity component by G^0.

All of the results of this paper apply to G_2, due to the fact that all real roots in type G_2 are metaplectic (Definition 3.2). Consequently we adopt the convention that the root system of type G_2 is simply laced.

3 Nonlinear Groups

We introduce some notation and basic results for two-fold covers. The definitions and statements may be modified in the obvious way for higher covers, but we have no occasion to do so. Throughout this section G is a real reductive linear group (cf. Section 2).

Definition 3.1 We say a real Lie group $\widetilde{G}$ is a two-fold cover of G if $\widetilde{G}$ is a central extension of G by $\mathbb{Z}/2\mathbb{Z}$. Thus there is an exact sequence of Lie groups

$$1 \to A \to \widetilde{G} \to G \to 1$$

with $A \simeq \mathbb{Z}/2\mathbb{Z}$ central in $\widetilde{G}$. We say $\widetilde{G}$ is nonlinear if it is not a linear group, i.e. $\widetilde{G}$ does not admit a faithful finite-dimensional representation. If H is a subgroup of G, we let $\widetilde{H}$ denote the inverse image of H in G. A *genuine* representation of $\widetilde{G}$ is one which does not factor to G.

Suppose H is a θ-stable Cartan subgroup of G, and α is a real or imaginary root. Corresponding to α is the subalgebra $\mathfrak{m}_\alpha$ of $\mathfrak{g}$ generated by the root vectors $X_{\pm\alpha}$. The corresponding subalgebra of $\mathfrak{g}_0$ is isomorphic to $\mathfrak{su}(2)$ if α is imaginary and compact, or $\mathfrak{sl}(2, \mathbb{R})$ otherwise. Let M_α be the corresponding analytic subgroup of G. The following definition is fundamental for the study of nonlinear groups and appears in many places.

Definition 3.2 Fix a two-fold cover $\widetilde{G}$ of G. Fix a θ-stable Cartan subgroup H of G and a real or noncompact imaginary root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$. We say that α

is *metaplectic* if $\widetilde{M_\alpha}$ is a nonlinear group, i.e. is the nontrivial two-fold cover of $\mathrm{SL}(2,\mathbb{R})$

For the next result see [M, Section 2.3] or [A1, Section 1]. (The third statement below follows easily from the first two.)

Lemma 3.3 *1. Fix a two-fold cover $\widetilde{G}$ of linear real reductive group G and assume $\mathfrak{g}_0$ is simple. Suppose H is a Cartan subgroup of G and $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is a long real or long noncompact imaginary root. Then $\widetilde{G}$ is nonlinear if and only if α is metaplectic.*

2. Suppose further that G is the real points of a connected, simply connected, simple algebraic group defined over $\mathbb{R}$. Then G admits a nonlinear two-fold cover (Definition 3.1) if and only if there is a θ-stable Cartan subgroup H and a long real root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$. The same conclusion holds with "non-compact imaginary" in place of "real". This cover is unique up to isomorphism.

3. Suppose G is as in (2), and also assume it is simply laced. Let H be any Cartan subgroup of G. Then G admits a nonlinear two-fold cover (Definition 3.1) if and only if there is a real or noncompact imaginary root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$.

Definition 3.4 We say a two-fold cover $\widetilde{G}$ of a linear real reductive group G is admissible if for every θ-stable Cartan subgroup H every long real or long noncompact imaginary root in $\Delta(\mathfrak{g}, \mathfrak{h})$ is metaplectic.

Lemma 3.5 *Let G be a real linear reductive Lie group, and retain the notation of Definition 3.1. Let $G_1, \ldots, G_k$ denote the (connected) analytic subgroups of G corresponding to the simple factors of the Lie algebra of G. A two-fold cover $\widetilde{G}$ of G is admissible if and only if $\widetilde{G}_i$ is nonlinear for each i such that G_i admits a nonlinear cover.*

This is immediate from Lemma 3.3(1) and the definitions. Roughly speaking, the result says admissible covers are as nonlinear as possible.

Proposition 3.6 *Assume G is the real points of a connected, simply connected, semisimple algebraic group $\mathbb{G}$ defined over $\mathbb{R}$. Then G admits an admissible two-fold cover, which is unique up to isomorphism.*

Proof. By [PR, Theorem 2.6(2)] (for example) we have that the complex points $\mathbb{G}(\mathbb{C})$ factors as $\prod_{i=1}^n G_i(\mathbb{C})$ with each $G_i(\mathbb{C})$ simple, and $G \simeq \prod_{i=1}^n G_i$ accordingly. Let $\overline{G} = \prod \widetilde{G}_i$ where $\widetilde{G}_i$ is the unique nontrivial two-fold cover of G_i, if it exists, or the trivial cover otherwise. (Here we are using Lemma 3.3.) Then $\overline{G}$ has a natural quotient which is an admissible cover of G.

Conversely if $\widetilde{G}$ is an admissible cover of G then $G = \prod \widetilde{G_i}$ (the product taken in $\widetilde{G}$; the terms $\widetilde{G}_i$ necessarily commute). Then $\widetilde{G}$ is a quotient of $\overline{G}$, and is isomorphic to the cover constructed in the previous paragraph. □

If G is reductive there may be a obstruction to the existence of an admissible cover, although this is rare. The following result along these lines will be sufficient for our purposes.

Proposition 3.7 *Suppose $\mathfrak{g}_0$ is a real reductive Lie algebra. There exists a real reductive linear group G with* $\mathrm{Lie}(G) = \mathfrak{g}_0$ *such that there exists an admissible two-fold cover of G.*

Proof. Let $G(\mathbb{C}) = T(\mathbb{C}) \times G_d(\mathbb{C})$, where $T(\mathbb{C})$ is a torus, $G_d(\mathbb{C})$ is simply connected and semisimple, and $\mathrm{Lie}(G(\mathbb{C})) = \mathfrak{g}_0 \otimes \mathbb{C}$. Let $G = T \times G_d$ be the real form of $G(\mathbb{C})$ corresponding to $\mathfrak{g}_0$. By Proposition 3.6 G_d has an admissible cover $\widetilde{G_d}$; take $\widetilde{G} = T \times \widetilde{G_d}$. □

Example 3.8

1. Every cover of an abelian Lie group is admissible.

2. The group $GL(n,\mathbb{R})$ $(n \geq 2)$ has two admissible covers, up to isomorphism, all with the same non-trivial restriction to $SL(n,\mathbb{R})$. These correspond to the two two-fold covers of $O(n)$, sometimes denotes $Pin^{\pm}$.

3. The group $U(p,q)$ $(pq > 0)$ has three nonisomorphic admissible covers, up to isomorphism, corresponding to the three nontrivial two-fold covers of $K = U(p) \times U(q)$, all having the same nontrivial restriction to $SU(p,q)$.

4. Suppose each analytic subgroup of G corresponding to a simple factor of $\mathfrak{g}_0$ admits no nonlinear cover. (For example, suppose each such analytic subgroup is compact, complex, or $\mathrm{Spin}(n,1)$ for $n \geq 3$.) Then every two-fold cover of G is admissible.

4 Definition of Duality

We recall some definitions and notation from [V1] and [V4]. In this section we let G be any real reductive group in Harish-Chandra's class (cf. Section 2).

Recall that block equivalence is the relation on irreducible modules generated by $\pi \sim \eta$ if $\mathrm{Ext}^*(\pi, \eta)$ is nonzero. See [V1, Section 9.1]. (That reference assumes that all Cartans subgroups are abelian, but the main results for block equivalence do not depend in an essential way on this assumption.) All elements in a block have the same infinitesimal character, which we refer to as the infinitesimal character of the block, and therefore each block is a finite set.

Fix a block $\mathcal{B}$ with regular infinitesimal character. Theorem 7.5 below provides a parameter set $\mathcal{S}$ and for each $\gamma \in \mathcal{S}$ a "standard" representation $\pi(\gamma)$, with a unique irreducible quotient $\pi(\gamma)$. The map from $\mathcal{S}$ to $\mathcal{B}$ given by $\gamma \to \overline{\pi}(\gamma)$ is a bijection.

Given a Harish-Chandra module X, we let $[X]$ denote it image in the Grothendieck group of all Harish-Chandra modules. Consider the $\mathbb{Z}$-module $\mathbb{Z}[\mathcal{B}]$ with basis $\{[\overline{\pi}(\gamma)] \mid \gamma \in \mathcal{S}\}$, viewed as a subgroup of the Grothendieck group. Then $\mathbb{Z}[\mathcal{B}]$ is also spanned by the standard modules $\{[\pi(\gamma)] \mid \gamma \in \mathcal{S}\}$. We may thus consider the change of basis matrices in the Grothendieck group:

$$\begin{aligned} [\pi(\delta)] &= \sum_{\gamma \in \mathcal{S}} m(\delta, \gamma)[\overline{\pi}(\gamma)]; \\ [\overline{\pi}(\delta)] &= \sum_{\gamma \in \mathcal{S}} M(\delta, \gamma)[\pi(\gamma)]. \end{aligned} \tag{4.1}$$

For general groups in Harish-Chandra's class there is no known effective algorithm for computing these matrices. For the linear groups considered in Section 2 they are computable by Vogan's algorithm [V3]. For any two-fold cover of such a group they are computable by the main results of [RT3, Part I].

Definition 4.2 Suppose G and G' are real reductive groups in Harish-Chandra's class, with blocks $\mathcal{B}$ and $\mathcal{B}'$ having regular infinitesimal character. Write $\mathcal{S}$ and $\mathcal{S}'$ for the sets parametrizing $\mathcal{B}$ and $\mathcal{B}'$.

Suppose there exists a bijection

$$\begin{aligned} \phi \ : \ \mathcal{S} &\longrightarrow \mathcal{S}' \\ \delta &\longrightarrow \delta' \end{aligned}$$

and a parity function

$$\epsilon \ : \ \mathcal{S} \times \mathcal{S}' \longrightarrow \{\pm 1\}$$

satisfying

$$\epsilon(\gamma, \delta)\epsilon(\delta, \eta) = \epsilon(\gamma, \eta) \qquad \epsilon(\gamma, \gamma) = +1.$$

Then we say that $\mathcal{B}$ is dual to $\mathcal{B}'$ (with respect to ϕ) if for all $\delta, \gamma \in \mathcal{B}$,

(4.3)(a) $\quad m(\delta, \gamma) = \epsilon(\delta, \gamma) M(\gamma', \delta');$ or equivalently,

(4.3)(b) $\quad M(\delta, \gamma) = \epsilon(\delta, \gamma) m(\gamma', \delta').$

We say $\mathcal{B}'$ is *dual* to $\mathcal{B}$ if it is dual to $\mathcal{B}$ with respect to some bijection ϕ.

In cases in which the matrices $M(\gamma, \delta)$ are computable, the computation depends in an essential way on a length function $l : \mathcal{S} \to \mathbb{N}$. The correct notion is the extended integral length of [RT3]. (When G is linear, this reduces to the notion of integral length defined in [V1].) Then the parity function appearing in Definition 4.2 may be taken to be

$$\epsilon(\gamma, \delta) = (-1)^{l(\gamma)+l(\delta)}.$$

5 Cartan Subgroups

In this section G is a real reductive linear group (Section 2) and let $\widetilde{G}$ be a two-fold cover of G. Since $\widetilde{G}$ is a covering group, a Cartan subgroup of $\widetilde{G}$ is the inverse image of a Cartan subgroup H of G. We have an exact sequence $1 \to A \to \widetilde{H} \to H \to 1$. It is often the case that $\widetilde{H}$ is not abelian. For later use we note that

$$\widetilde{H^0} \subset Z(\widetilde{H}) \tag{5.1}$$

since the map $(g, h) \to ghg^{-1}h^{-1}$ is a continuous map from $\widetilde{H} \times \widetilde{H}$ to ± 1. In particular $|\widetilde{H}/Z(\widetilde{H})|$ is finite.

The irreducible genuine representations of $\widetilde{H}$ are parametrized by genuine characters of $Z(\widetilde{H})$ according to the next lemma.

Lemma 5.2 *Retain the setting above. Write* $\Pi(Z(\widetilde{H}))$ *and* $\Pi(\widetilde{H})$ *for the irreducible genuine representations of* $Z(\widetilde{H})$ *and* $\widetilde{H}$*, respectively, and let* $n = |\widetilde{H}/Z(\widetilde{H})|^{\frac{1}{2}}$*. For every* $\chi \in \Pi(Z(\widetilde{H}))$ *there is a unique* $\pi = \pi(\chi) \in \Pi(\widetilde{H})$ *for which* $\pi|_{Z(\widetilde{H})}$ *is a multiple of* χ*. The map* $\chi \to \pi(\chi)$ *is a bijection* $\Pi(Z(\widetilde{H})) \to \Pi(\widetilde{H})$*. The dimension of* $\pi(\chi)$ *is* n*, and* $Ind_{\widetilde{Z(H)}}^{\widetilde{H}}(\chi) = n\pi$*.*

The proof is elementary; see [ABPTV, Proposition 2.2], for instance. The lemma shows that an irreducible representation of $\widetilde{H}$ is determined by a character of $Z(\widetilde{H})$. The fact that this is much smaller than $\widetilde{H}$ makes the representation theory of $\widetilde{G}$ in many ways much simpler than that of G.

Proposition 5.3 ([AH]) *Suppose* G *is simply laced,* $\widetilde{G}$ *is an admissible two-fold cover of* G*, and* H *is a Cartan subgroup of* G*. Then*

$$Z(\widetilde{H}) = Z(\widetilde{G})\widetilde{H^0}. \tag{5.4}$$

In particular a genuine character of $Z(\widetilde{H})$ *is determined by its restriction to* $Z(\widetilde{G})$ *and its differential.*

6 Passing from $\mathfrak{h}$ to $\mathfrak{h}^*$

Fix a complex reductive Lie algebra $\mathfrak{g}$ and let $\mathfrak{h}$ be a Cartan subalgebra. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$, and let $\Delta^\vee \subset \mathfrak{h}$ be the set of coroots [Bou, Definition 1, Chapter VI]. Let $(\cdot, \cdot)$ be a nondegenerate Weyl group invariant bilinear form on $\mathfrak{h}$ satisfying $(\alpha^\vee, \alpha^\vee) = 2$ for all long roots α. Such a form exists and is uniquely determined on the span of the coroots.

Write $\langle \cdot, \cdot \rangle$ for the natural pairing $\mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$. The form $(\cdot, \cdot)$ induces an isomorphism

$$\Phi : \mathfrak{h}^* \simeq \mathfrak{h}$$

defined by

$$(\Phi(\lambda), X) = \langle \lambda, X \rangle \quad (\lambda \in \mathfrak{h}^*, X \in \mathfrak{h}). \tag{6.1}$$

We have

(6.2)(a)
$$\Phi(\alpha) = \frac{2}{(\alpha^\vee, \alpha^\vee)}\alpha^\vee \quad (\alpha \in \Delta)$$

and

$$\text{(6.2)(b)} \qquad \langle \alpha, \Phi(\lambda)\rangle = \frac{2}{(\alpha^\vee, \alpha^\vee)}\langle \lambda, \alpha^\vee\rangle.$$

If $\mathfrak{g}$ is simply laced, these formulas become

$$\text{(6.2)(c)} \qquad \Phi(\alpha) = \alpha^\vee \quad (\alpha \in \Delta)$$

and

$$\text{(6.2)(d)} \qquad \langle \alpha, \Phi(\lambda)\rangle = \langle \lambda, \alpha^\vee\rangle \quad (\alpha \in \Delta, \lambda \in \mathfrak{h}^*).$$

7 Regular Characters

In this section, we recall the definition of regular character of $\widetilde{G}$. See, for example, [V5, Section 2].

Definition 7.1 Suppose G is a real reductive group in Harish-Chandra's class. A regular character of G is a triple

$$\text{(7.2)} \qquad \gamma = (H, \Gamma, \lambda)$$

consisting of a θ-stable Cartan subgroup H, an irreducible representation Γ of H, and $\lambda \in \mathfrak{h}^*$, satisfying the following conditions. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$. The first condition is

$$\text{(7.3)(a)} \qquad \langle \lambda, \alpha^\vee\rangle \in \mathbb{R}^\times \text{ for all } \alpha \in \Delta_i.$$

Let

$$\text{(7.3)(b)} \qquad \rho_i(\lambda) = \frac{1}{2}\sum_{\substack{\alpha\in\Delta_i \\ \langle\lambda,\alpha^\vee\rangle>0}} \alpha, \qquad \rho_{i,c}(\lambda) = \frac{1}{2}\sum_{\substack{\alpha\in\Delta_{i,c} \\ \langle\lambda,\alpha^\vee\rangle>0}} \alpha.$$

The second condition is

$$\text{(7.3)(c)} \qquad d\Gamma = \lambda + \rho_i(\lambda) - 2\rho_{i,c}(\lambda).$$

To be precise, if H is not abelian, the left hand means $d\Gamma_0$ where $\Gamma|_{H^0}$ is a multiple of Γ_0 (cf. Lemma 5.2). We will further assume that

$$\text{(7.4)} \qquad \langle \lambda, \alpha^\vee\rangle \neq 0 \text{ for all } \alpha \in \Delta.$$

If H is given we write $\gamma = (\Gamma, \lambda)$ and say γ is a regular character of H.

Let M be the centralizer of A in G. The conditions on γ imply that there exists a unique discrete series representation of M, say π_M, with Harish-Chandra parameter λ whose lowest $M \cap K$-type has Γ as a highest weight. Define a parabolic subgroup MN by requiring that the real part of λ restricted to $\mathfrak{a}_\circ$ be (strictly) positive on the roots of $\mathfrak{a}$ in $\mathfrak{n}$. Then set

$$\pi(\gamma) = \mathrm{ind}_{MN}^{G}(\pi_M \otimes \mathbb{1}).$$

By the choice of N and (7.4), $\pi(\gamma)$ has a unique irreducible quotient which we denote $\overline{\pi}(\gamma)$.

The maximal compact subgroup K acts in the obvious way on the set of regular characters, and the equivalence class of $\overline{\pi}(\gamma)$ only depends on the K orbit of γ. We write $\gamma \sim \gamma'$ if γ is K-conjugate to γ'.

Fix a codimension one ideal I in the center of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. We call I an infinitesimal character for $\mathfrak{g}$, and (as usual) say that a $U(\mathfrak{g})$ module has infinitesimal character I if it is annihilated by I. Once we specify a Cartan subalgebra $\mathfrak{h}$, I determines a $W(\mathfrak{h},\mathfrak{g})$-orbit $W \cdot \lambda$ in $\mathfrak{h}^*$ (via the Harish-Chandra homomorphism). We say that I is a regular infinitesimal character if λ is regular. (This condition is independent of the choice of $\mathfrak{h}$.) If $\gamma = (H, \Gamma, \lambda)$ is a regular character, we say that γ has infinitesimal character I if λ is in the W-orbit specified by I. Let $\mathcal{S}_I \subset \mathcal{S}$ denote the set of K-orbits of regular characters $\gamma = (H, \Gamma, \lambda)$ with infinitesimal character I.

In this setting the Langlands classification takes the following form. See [V5, Section 2], for example.

Theorem 7.5 *Suppose G is a real reductive group in Harish-Chandra's class. Fix a codimension ideal I in the center of $U(\mathfrak{g})$ giving a regular infinitesimal character for $\mathfrak{g}$. The map $\gamma \to \overline{\pi}(\gamma)$ is a bijection from $\mathcal{S}_I/K$ to the set of equivalence classes of irreducible admissible representations of G with infinitesimal character I.*

We conclude with an easy consequence of Proposition 5.3 that we shall have occasion to use later.

Corollary 7.6 *Let $\widetilde{G}$ be an admissible two-fold cover of a simply-laced real linear reductive group G. Suppose $\gamma = (H, \Gamma, \lambda)$ is a genuine character of $\widetilde{G}$. Then γ is determined by λ and the restriction of Γ to $Z(\widetilde{G})$.*

Remark 7.7 Fix a genuine central character and an infinitesimal character for $\widetilde{G}$. If G has a compact Cartan subgroup the corollary implies that $\widetilde{G}$ has at most one discrete series representation with the given infinitesimal character and central character. If G is split there is precisely one minimal principal series representation with the given infinitesimal character and central characters. This is one example of how the genuine representation theory of $\widetilde{G}$ is simpler than the representation theory of G. See the examples in Section 14.

8 Characters of Cartan subgroups

We need some properties of characters of Cartan subgroups. Throughout this section G is a *simply-laced* real reductive linear group (cf. Section 2), and $\widetilde{G}$ is an admissible two-fold cover of G (Definition 3.4). Fix a θ-stable Cartan subgroup H of G. Write $\exp : \mathfrak{g}_0 \to G$ for the exponential map, and $\widetilde{\exp}$ for the exponential map $\mathfrak{g}_0 \to \widetilde{G}$.

For α an imaginary or complex root in $\Delta(\mathfrak{g}, \mathfrak{h})$ let

$$(8.1) \qquad Z_\alpha = \begin{cases} 2\pi i \alpha^\vee & \alpha \text{ imaginary} \\ 2\pi i(\alpha^\vee + \theta(\alpha^\vee)) & \alpha \text{ complex.} \end{cases}$$

Lemma 8.2 *Recall that G is simply-laced. The kernel of $\widetilde{\exp}$ restricted to $\mathbb{R}Z_\alpha$ is equal to the integer multiples of $c_\alpha Z_\alpha$ where*

$$(8.3) \qquad c_\alpha = \begin{cases} \text{either } 1 \text{ or } \frac{1}{2} & \text{if } \alpha \text{ is compact imaginary} \\ 2 & \text{if } \alpha \text{ is noncompact imaginary} \end{cases}$$

and

$$(8.4) \qquad c_\alpha = \begin{cases} \text{either } 1 \text{ or } \frac{1}{2} & \text{if } \alpha \text{ is complex and } \langle \alpha, \theta(\alpha^\vee) \rangle = 0 \\ 2 & \text{if } \alpha \text{ is complex and } \langle \alpha, \theta(\alpha^\vee) \rangle \neq 0 \end{cases} .$$

If G is the real points of a simply connected complex group then $c_\alpha \neq \frac{1}{2}$.

Proof. First assume α is imaginary. Recall the group M_α introduced in Section 3 (just before Definition 3.2). The proposition reduces to the case of $G = M_\alpha$, and M_α is locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$ or $\mathrm{SU}(2)$. If α is compact then M_α and the identity component of $\widetilde{M_\alpha}$ are either isomorphic to $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$. Thus $c_\alpha = 1$ (in the first case) and $\frac{1}{2}$ (in the second) and the proposition holds in this case. If α is noncompact then M_α is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$ and $\widetilde{M}_\alpha$ is the nonlinear cover $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ by the admissibility assumption on the cover $\widetilde{G}$. (Here we are using G is simply laced and applying the results of Lemma 3.3 and Proposition 3.6.) If H is the compact Cartan subgroup of $\mathrm{SL}(2,\mathbb{R})$ then $\widetilde{H} \to H$ is a nontrivial cover; it follows that $c_\alpha = 2$ in this case.

If G is the real points of a simply connected complex group and α is imaginary, then $M_\alpha \simeq \mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SU}(2)$ and the final assertion follows for imaginary roots.

Now suppose α is complex. If $\langle \alpha, \theta(\alpha^\vee)\rangle = -1$ then $\beta = \alpha + \theta(\alpha)$ is an imaginary root, and $\beta^\vee = \alpha^\vee + \theta(\alpha^\vee)$. Thus $c_\alpha = 2$ by the previous case.

If $\langle \alpha, \theta(\alpha^\vee)\rangle = 1$, then $\beta = \alpha - \theta(\alpha)$ is a real root. By taking a Cayley transform by β we obtain a new Cartan subgroup with an imaginary noncompact root, and again reduce to the previous case. We omit further details.

Finally suppose $\langle \alpha, \theta(\alpha^\vee)\rangle = 0$. Let $\mathfrak{l}_\alpha$ be the subalgebra of $\mathfrak{g}$ generated by root vectors $X_{\pm\alpha}, X_{\pm\theta(\alpha)}$. Let L_α be the subgroup of G with Lie algebra $\mathfrak{l}_\alpha \cap \mathfrak{g}_0$. The assumption $\langle \alpha, \theta(\alpha^\vee)\rangle = 0$ implies L_α is locally isomorphic to $SL(2,\mathbb{C})$. Therefore L_α has no nonlinear cover, so the identity component of $\widetilde{L_\alpha}$ is isomorphic to $SL(2,\mathbb{C})$ or $PSL(2,\mathbb{C})$. Then $c_\alpha = \frac{1}{2}$ in the first case and 2 in the second. The final assertion for complex roots also follows easily. □

The group L_α in the final paragraph of the proof may be defined for any complex root. An examination of rank two root systems shows that L_α is locally isomorphic to $SL(2,\mathbb{C})$, $SU(2,1)$ or $SL(3,\mathbb{C})$, if $\langle \alpha, \theta(\alpha^\vee)\rangle = 0, 1$ or -1, respectively.

Corollary 8.5 *Suppose Λ is a genuine character of $Z(\widetilde{H})$. Let $\lambda \in \mathfrak{h}^*$ be its complexified differential. Then*

(8.6)(a)
$$\langle \lambda, \alpha^\vee\rangle \in \begin{cases} \mathbb{Z} & \alpha \text{ is compact imaginary} \\ \mathbb{Z} + \frac{1}{2} & \alpha \text{ is noncompact imaginary} \end{cases}$$

and

(8.6)(b)
$$\langle \lambda, \alpha^\vee + \theta(\alpha^\vee)\rangle \in \begin{cases} \mathbb{Z} & \text{if } \alpha \text{ is complex and } \langle \alpha, \theta(\alpha^\vee)\rangle = 0 \\ \mathbb{Z} + \frac{1}{2} & \text{if } \alpha \text{ is complex and } \langle \alpha, \theta(\alpha^\vee)\rangle \neq 0 \end{cases}.$$

Proof. Suppose α is imaginary and set $t_\alpha = \widetilde{\exp}(Z_\alpha)$. By Lemma 8.2, $t_\alpha = 1$ if α is compact and t_α is the nontrivial inverse image of 1 otherwise. Since Λ is a genuine character we conclude

$$\text{(8.7)(a)} \qquad \Lambda(t_\alpha) = \begin{cases} 1 & \alpha \text{ compact} \\ -1 & \alpha \text{ noncompact.} \end{cases}$$

On the other hand

$$\text{(8.7)(b)} \qquad \begin{aligned} \Lambda(t_\alpha) &= e^{\langle \lambda, 2\pi i \alpha^\vee \rangle} \\ &= (-1)^{\langle 2\lambda, \alpha^\vee \rangle} \end{aligned}$$

(Note that Λ^2 factors to a character of H, with differential 2λ; therefore $\langle 2\lambda, \alpha^\vee \rangle \in \mathbb{Z}$, so $(-1)^{\langle 2\lambda, \alpha^\vee \rangle}$ is well-defined.) This gives (8.6)(a), and (8.6)(b) is similar. □

Corollary 8.8 *Retain the setting of Corollary 8.5. Suppose $\gamma = (H, \Gamma, \lambda)$ is a genuine regular character for $\widetilde{G}$ (Definition 7.1). Then λ satisfies conditions* (8.6).

Proof. By (7.3)(c), $d\Gamma = \lambda + \rho_i(\lambda) - 2\rho_{i,c}(\lambda)$, and by Corollary 8.5 $d\Gamma$ satisfies conditions (8.6). Since $2\rho_{i,c}(\lambda)$ is a sum of roots $\langle 2\rho_{i,c}(\lambda), \alpha^\vee \rangle \in \mathbb{Z}$ for all α. The result then follows from the fact that if α is imaginary then $\langle \rho_i(\lambda), \alpha^\vee \rangle \in \mathbb{Z}$, and if α is complex then $\langle \rho_i(\lambda), \alpha^\vee \pm \theta(\alpha^\vee) \rangle \in \mathbb{Z}$. □

We need a partial converse to this result, i.e. a condition giving the existence of a genuine regular character $\gamma = (H, \Gamma, \lambda)$ with given λ.

Proposition 8.9 *Suppose G is the real points of a connected, semisimple, simply connected algebraic group $\mathbb{G}$ defined over $\mathbb{R}$. Assume, as usual, that $\widetilde{G}$ is an admissible two-fold cover of G (Definition 3.4). Fix a θ-stable Cartan subgroup H and and $\lambda_0 \in \mathfrak{h}^*$. Then there exists a genuine character of $Z(\widetilde{H})$ with differential λ_0 if and only if λ_0 satisfies the conditions of* (8.6).

Proof. Write $H = TA$ as usual. It is enough to show there is a character of $\widetilde{T^0}$ with differential $\lambda_0|_{\mathfrak{t}_0}$.

Choose a set of positive roots of $\Delta(\mathfrak{g}, \mathfrak{h})$ so that if $\alpha > 0$ is complex then $\theta(\alpha) > 0$. Let S be the set of simple roots which are either imaginary or complex. Then $\{Z_\alpha \,|\, \alpha \in S\}$ is a basis of $\mathfrak{t}_0$, with Z_α as defined in (8.1).

For $\alpha \in S$ let $T_\alpha = \exp(\mathbb{R}Z_\alpha)$, so $T^0 \simeq \prod_S T_\alpha$. Define a character Λ_α of $\widetilde{T_\alpha}$ by

$$\Lambda_\alpha(\widetilde{\exp}(X)) = e^{\lambda(Z_\alpha)} = \begin{cases} e^{\langle\lambda,\alpha^\vee\rangle} & \alpha \text{ imaginary} \\ e^{\langle\lambda,\alpha^\vee+\theta(\alpha^\vee)\rangle} & \alpha \text{ complex.} \end{cases}$$

By Proposition 8.5, Λ_α is a well defined genuine character of $\widetilde{T_\alpha}$. Let $\overline{T^0} = \prod_S \widetilde{T_\alpha}$ and let let $\overline{\Lambda} = \prod_S \Lambda_\alpha$, a character of $\overline{T^0}$. There is a surjective map $\overline{T^0} \to \widetilde{T^0}$ and it follows easily that $\overline{\Lambda}$ factors to a character of $\widetilde{T^0}$. □

Corollary 8.10 *Retain the setting of Proposition 8.9 but instead assume that $\mathbb{G}$ is reductive (rather than semisimple). There is a genuine character Λ of $Z(\widetilde{H})$ satisfying*

$$\langle d\Lambda, \alpha^\vee\rangle = \langle\lambda_0, \alpha^\vee\rangle \quad \textit{for all } \alpha \in \Delta \tag{8.11}$$

if and only if λ_0 satisfies the conditions of (8.6).

Proof. The only if statement is clear from the previous proposition. For the other direction write $\mathbb{G} = \mathbb{T}\mathbb{G}_d$ where $\mathbb{T}$ is a central torus, defined over $\mathbb{R}$, with real points T. Apply the proposition to construct a character Λ_d of $\widetilde{H^0_d}$, where $H_d = H \cap \mathbb{G}_d(\mathbb{C})$. Choose a genuine character τ of $\widetilde{T^0}$ with the same restriction to the finite group $\widetilde{T^0 \cap H_d}$ as Λ_d. Then $\tau\Lambda_d$ is a genuine character of $\widetilde{T^0}\widetilde{H^0_d} = \widetilde{H^0}$ satisfying (8.11). Extend this arbitrarily to $Z(\widetilde{H})$. □

Corollary 8.12 *In the setting of Corollary 8.10 there is a genuine regular character $\gamma = (H, \Gamma, \lambda)$ of $\widetilde{G}$ satisfying*

$$\langle\lambda, \alpha^\vee\rangle = \langle\lambda_0, \alpha^\vee\rangle \quad \textit{for all } \alpha \in \Delta$$

if and only if λ_0 satisfies the conditions of (8.6).

9 Bigradings

We state versions of some of the results of [V4] in our setting. We follow [RT3] to a large extent. Throughout this section let G be a real reductive linear group and $\widetilde{G}$ an admissible two-fold cover of G (Section 2 and Definition 3.4).

Fix a θ-stable Cartan subgroup H of G. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ and set

$$\Delta(\lambda) = \{\alpha \in \Delta \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\}, \tag{9.1}$$

the set of integral roots defined by λ. We will also use the *half-integral* roots: in the notation of Section 6,

$$\Delta_{\frac{1}{2}}(\lambda) = \{\alpha \in \Delta \mid \langle \lambda, \Gamma(\alpha) \rangle \in \frac{1}{2}\mathbb{Z}\}. \tag{9.2}$$

This is a root system. If G is simply laced this is the same as

$$\Delta_{\frac{1}{2}}(\lambda) = \{\alpha \in \Delta \mid \langle \lambda, \alpha^\vee \rangle \in \frac{1}{2}\mathbb{Z}\} = \Delta(2\lambda). \tag{9.3}$$

Let $W_{\frac{1}{2}}(\lambda) = W(\Delta_{\frac{1}{2}}(\lambda))$.

Definition 9.4 We say λ is half-integral if $\Delta_{\frac{1}{2}}(\lambda) = \Delta$.

Recall Δ^r is the set of real roots, i.e. those roots for which $\theta(\alpha) = -\alpha$. Let

$$\begin{aligned} \Delta^r(\lambda) &= \Delta^r \cap \Delta(\lambda) \\ &= \{\alpha \in \Delta(\lambda) \mid, \theta(\alpha) = -\alpha\} \\ \Delta^r_{\frac{1}{2}}(\lambda) &= \Delta^r_{\frac{1}{2}} \cap \Delta(\lambda) \\ &= \{\alpha \in \Delta_{\frac{1}{2}}(\lambda) \mid, \theta(\alpha) = -\alpha\} \end{aligned} \tag{9.5}$$

Note that Δ^r depends on $\theta|_{\mathfrak{h}}$, and $\Delta^r(\lambda)$ also depends on λ. To indicate this dependence we may write $\Delta^r(\theta|_{\mathfrak{h}})$, $\Delta^r(\lambda, \theta|_{\mathfrak{h}})$ and $\Delta^r_{\frac{1}{2}}(\lambda, \theta|_{\mathfrak{h}})$. We may also write $\theta|_\Delta$ for the restriction of the transpose of θ to Δ. Define $\Delta^i(\lambda) = \Delta^i(\lambda, \theta|_{\mathfrak{h}})$ and and $\Delta^i_{\frac{1}{2}}(\lambda) = \Delta^i_{\frac{1}{2}}(\lambda, \theta|_{\mathfrak{h}})$ similarly.

Now fix a genuine regular character $\gamma = (H, \Gamma, \lambda)$ (Definition 7.1). For $\alpha \in \Delta^r_{\frac{1}{2}}(\lambda, \theta|_{\mathfrak{h}})$ let $m_\alpha \in \widetilde{G}$ be defined as in [RT3, Section 5]. Thus m_α is an inverse image of the corresponding element of G of [V1, 4.3.6]. Recall Definition 3.2. Then it is follows easily that

$$\alpha \text{ is metaplectic if and only if } m_\alpha \text{ has order 4.} \tag{9.6}$$

We say α satisfies the parity condition with respect to γ if

$$\text{the eigenvalues of } \Gamma(m_\alpha) \text{ are of the form } -\epsilon_\alpha e^{\pm \pi i \langle \lambda, \alpha^\vee \rangle}; \tag{9.7}$$

here $\epsilon_\alpha = \pm 1$ as in [V1, Definition 8.3.11]. Note that if $\alpha \in \Delta(\lambda)$ then $e^{\pm\pi i\langle\lambda,\alpha^\vee\rangle} = (-1)^{\langle\lambda,\alpha^\vee\rangle}$, and this definition agrees with [V1, Definition 8.3.11].

Let

(9.8)(a) $\Delta^{r,+}_{\frac{1}{2}}(\gamma) = \{\alpha \in \Delta^r_{\frac{1}{2}}(\lambda) \,|\, \alpha \text{ does not satisfy the parity condition}\}$

(9.8)(b) $\Delta^{r,-}_{\frac{1}{2}}(\gamma) = \{\alpha \in \Delta^r_{\frac{1}{2}}(\lambda) \,|\, \alpha \text{ satisfies the parity condition}\}$

and

(9.8)(c) $$\Delta^{i,+}_{\frac{1}{2}}(\gamma) = \{\alpha \in \Delta^i_{\frac{1}{2}}(\lambda) \,|\, \alpha \text{ is compact}\}$$

(9.8)(d) $$\Delta^{i,-}_{\frac{1}{2}}(\gamma) = \{\alpha \in \Delta^i_{\frac{1}{2}}(\lambda) \,|\, \alpha \text{ is noncompact}\}.$$

Finally let

(9.8)(e) $$\begin{aligned} \Delta^{i,\pm}(\gamma) &= \Delta^{i,\pm}_{\frac{1}{2}} \cap \Delta(\lambda) \\ \Delta^{r,\pm}(\gamma) &= \Delta^{r,\pm}_{\frac{1}{2}} \cap \Delta(\lambda). \end{aligned}$$

These are the usual integral imaginary compact and noncompact roots, and the integral real roots which satisfy (or do not satisfy) the parity condition.

It is a remarkable fact that $\Delta^{i,\pm}_{\frac{1}{2}}(\gamma)$ and $\Delta^{r,\pm}_{\frac{1}{2}}(\gamma)$ only depend on H and λ, as the next Proposition shows. This is very different from the linear case.

Proposition 9.9 (Corollaries 6.9 and 6.10 of [RT3]) *Let $\widetilde{G}$ be an admissible cover of a simply laced, real reductive linear group G. Let $\gamma = (H, \Gamma, \lambda)$ be a genuine regular character for $\widetilde{G}$. Then*

(9.10)(a) $$\Delta^{r,+}_{\frac{1}{2}}(\gamma) = \Delta^r(\lambda)$$

(9.10)(b) $$\Delta^{i,+}_{\frac{1}{2}}(\gamma) = \Delta^i(\lambda)$$

Proof. Suppose α is a real root. According to Definition 3.4, α is metaplectic, so by (9.6) m_α has order 4. Therefore, since Γ is genuine, $\Gamma(m_\alpha)$ has eigenvalues $\pm i$. On the other hand $-\epsilon_\alpha e^{\pi i\langle\lambda,\alpha^\vee\rangle} = \pm 1$ if $\alpha \in \Delta(\lambda)$, or $\pm i$ if $\alpha \in \Delta_{\frac{1}{2}}(\lambda) \setminus \Delta(\lambda)$. Therefore the parity condition (9.7) fails if and only if $\alpha \in \Delta(\lambda)$, proving (9.10)(a). Statement (9.10)(b) follows from (8.6)(a). □

We recall some definitions from [V4, Section 3]. Suppose Δ is a root system. A *grading* of Δ is a map $\epsilon : \Delta \to \pm 1$, such that $\epsilon(\alpha) = \epsilon(-\alpha)$

and $\epsilon(\alpha+\beta)=\epsilon(\alpha)\epsilon(\beta)$ whenever α,β and $\alpha+\beta\in\Delta$. A *cograding* of Δ is a map $\delta^\vee:\Delta\to\pm1$ such that the dual map $\delta:\Delta^\vee\to\pm1$ (defined by $\delta(\alpha^\vee)=\delta^\vee(\alpha)$) is a grading of the dual root system $\Delta^\vee$. We identify a grading ϵ with its kernel $\epsilon^{-1}(1)$, and similarly for a cograding.

A *bigrading* is a triple $g=(\theta,\epsilon,\delta^\vee)$ where θ is an involution of Δ, ϵ is a grading of $\Delta^i=\Delta^\theta$, and $\delta^\vee$ is a cograding of $\Delta^r=\Delta^{-\theta}$. (This is a *weak bigrading* of [V4, Definition 3.22].) Alternatively we write $g=(\theta,\Delta^{i,+},\Delta^{r,+})$ where $\Delta^{i,+}\subset\Delta^i$ is the kernel of ϵ and $\Delta^{r,+}\subset\Delta^r$ is the kernel of $\delta^\vee$. The *dual bigrading* of g is the bigrading $g^\vee=(-\theta^\vee,\delta,\epsilon^\vee)$ of $\Delta^\vee$, where $\theta^\vee(\alpha^\vee)=\theta(\alpha)$.

It is easy to see that $\Delta^{i,+}_{\frac12}(\gamma)$ is a grading of $\Delta^i_{\frac12}(\lambda)$. By (9.10)(a) $\Delta^{r,+}_{\frac12}(\gamma)$ is a grading of $\Delta^r_{\frac12}(\lambda)$. If Δ is simply laced, then $\Delta^{r,+}_{\frac12}(\lambda)$ is also a cograding of $\Delta^r_{\frac12}(\lambda)$.

Definition 9.11 The bigrading of $\Delta_{\frac12}(\lambda)$ defined by a genuine regular character γ is

$$g_{\frac12}(\gamma)=(\theta|_\Delta,\Delta^{i,+}_{\frac12}(\gamma),\Delta^{r,+}_{\frac12}(\gamma)).$$

Note that with the obvious notation $g(\gamma)=g_{\frac12}(\gamma)\cap\Delta(\lambda)$ is a bigrading of $\Delta(\lambda)$.

The point of this definition is that is contains the information necessary to prove a duality theorem for simply laced admissible double covers. To the reader familiar with [V4] (where much more detailed strong bigradings are needed), this it is perhaps surprising that we can get away with so little. We offer a few comments explaining this.

Let G be a real reductive linear group. Suppose $\gamma=(H,\Gamma,\lambda)$ is a regular character of G (Definition 7.1). Write $H=TA$ as usual, and let M be the centralizer of A in G. Section 4 of [V4] associates a strong bigrading of of the integral root system $\Delta(\lambda)=\Delta(\mathfrak{g},\mathfrak{h})(\lambda)$ to γ. One component of the strong bigrading is a group $W_1^i(\gamma)$ subject to the following containment relations,

(9.12)(a) $$W(\Delta^{i,+}(\lambda))\subset W_1^i(\gamma)\subset \mathrm{Norm}_{W(\Delta^i(\lambda))}(\Delta^{i,+}(\lambda)).$$

The outer terms in (9.12)(a) depend only on Lie algebra data, while $W_1^i(\gamma)$ depends (roughly speaking) on how disconnected G is: [V4, Proposition 4.11] defines $W_1^i(\gamma)$ to be $W(\Delta^i(\lambda))\cap W(G,H)$. The dual component of a strong bigrading is a group $W_1^r(\gamma)$ subject to the containment relations

(9.12)(b) $$W_0^r(\lambda)\subset W_1^r(\gamma)\subset \mathrm{Norm}_{W(\Delta^r(\lambda))}(W_0^r(\lambda)),$$

where $W_0^r(\lambda)$ is the Weyl group of the zeroth piece of the grading corresponding to the dual of the cograding defined by the parity condition.

Now let $\gamma = (H, \Gamma, \lambda)$ be a genuine regular character of an admissible cover $\widetilde{G}$ of G. Assume further that G is simply laced. Then Proposition 9.9 implies that the outer two terms in (9.12)(a)-(9.12)(b) are the same, and hence the middle term is fixed in each case:

$$W_1^i(\gamma) = W(\Delta^i(\lambda)) \qquad W_1^r(\gamma) = W(\Delta^r(\lambda)). \tag{9.13}$$

Thus the data of the strong bigrading of the integral roots defined in [V4, Proposition 4.11] is contained in the data of Definition 9.11. (Of course the latter contains extra information too.)

We might consider cutting from here to the end of this section.

Incidentally, this discussion suggests that the character theory of nonlinear simply laced groups, in the sense of Section 4, should depend only on the identity component of $\widetilde{G}$. We give a version of this principle, which follows from the idea in Remark 7.7.

Proposition 9.14 *Suppose G is the real points of a simply connected, simply laced reductive algebraic group defined over $\mathbb{R}$. Fix an admissible cover $\widetilde{G}$ of G (Definition 3.4) and write $\widetilde{G}_d$ for the preimage of the derived group G_d in $\widetilde{G}$. Then*

$$Ind_{Z(\widetilde{G})\widetilde{G_d}}^{\widetilde{G}}(\pi)$$

is a multiple of an irreducible representation of $\widetilde{G}$. Every irreducible representation of $\widetilde{G}$ is obtained this way.

The same result holds even if $\mathbb{G}$ is not simply laced, provided no simple factor of G is isomorphic to $Sp(4n, \mathbb{R})$, $Sp(n, n)$ or $Spin(2p+1, 2q)$.

10 Cayley Transforms and the Cross Action

Throughout this section $\widetilde{G}$ is an admissible cover of a simply laced, real reductive linear group G. Fix a block $\mathcal{B}$ of genuine representations of $\widetilde{G}$ and a genuine regular character $\gamma = (H, \Gamma, \lambda)$ with $\overline{\pi}(\gamma) \in \mathcal{B}$.

Suppose $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is a noncompact imaginary root for γ, i.e. $\alpha \in \Delta^{i,-}_{\frac{1}{2}}(\gamma)$. We define the *Cayley transform of γ through α* as in [V2, Section 4]; see also [RT3, Section 5]. We write $c^\alpha(\gamma) = (H^\alpha, c^\alpha(\Gamma), c^\alpha(\lambda))$. If G is simply laced this is always singled valued, unlike the linear case. As in the linear case $\overline{\pi}(c^\alpha(\gamma)) \in \mathcal{B}$.

We need a few basic facts about $c^\alpha(\gamma)$, cf. [V1, Definition 8.3.6]. Let $\mathfrak{h}^\alpha$ be the Lie algebra of $\mathfrak{h}$. Then $\mathfrak{h} \cap \mathfrak{h}_\alpha$ is of codimension 1 in $\mathfrak{h}$ and $\mathfrak{h}_\alpha$. Furthermore α corresponds to a real root $\Delta(\mathfrak{h}^\alpha, \mathfrak{g})$, which we also denote α, and $c^\alpha(\lambda)$ is determined by:

$$\begin{aligned} c^\alpha(\lambda)|_{\mathfrak{h}\cap\mathfrak{h}^\alpha} &= \lambda|_{\mathfrak{h}\cap\mathfrak{h}^\alpha} \\ \langle c^\alpha(\lambda), \alpha^\vee\rangle &= \langle \lambda, \alpha^\vee\rangle. \end{aligned} \tag{10.1}$$

Now suppose α is a half-integral real root satisfying the parity condition, i.e. $\alpha \in \Delta^{r,-}_{\frac{1}{2}}(\gamma)$. Then the *Cayley transform* $c_\alpha(\gamma)$ *of* γ *through* α is defined. Again $c^\alpha(\gamma)$ is always single-valued.

An ordered sequence $\alpha_1, \dots, \alpha_k$ of elements of $\Delta^{i,-}_{\frac{1}{2}}(\gamma)$ is called *admissible for* γ if the iterated Cayley transform

$$c^{\alpha_k} \circ \cdots \circ c^{\alpha_1}(\gamma) \tag{10.2}$$

is well-defined. According to [V4, Corollary 5.9], the iterated Cayley transforms of (10.2) depends only on the span $\mathfrak{s}$ of the roots in the admissible sequence. A subspace $\mathfrak{s}$ of the span of $\Delta^{i,-}_{\frac{1}{2}}(\gamma)$ is called admissible if it is spanned by an admissible sequence, and we write $c^{\mathfrak{s}}(\gamma)$ for the corresponding Cayley transform.

We define admissible sequences and subspaces for $\Delta^{r,-}_{\frac{1}{2}}$ similarly. In this case the iterated Cayley transform is of the form $c_{\alpha_k} \circ \cdots \circ c_{\alpha_1}(\gamma)$ and we write $c_{\mathfrak{s}}(\gamma)$.

We need to make precise the sense in which a Cayley transform of γ depends only on the K conjugacy class of γ. This is treated in [V4, Definition 7.10], and we recall the details here. To begin, we introduce the abstract Cartan subalgebra and Weyl group as in [V4, 2.6]. Thus we fix once and for all a Cartan subalgebra $\mathfrak{h}_a$, a set of positive roots Δ^+_a of $\Delta_a = \Delta(\mathfrak{g}, \mathfrak{h}_a)$, and let $W_a = W(\mathfrak{g}, \mathfrak{h}_a)$. Suppose $\mathfrak{h}$ is any Cartan subalgebra of $\mathfrak{g}$, and Δ^+ is a set of positive roots of $\Delta(\mathfrak{g}, \mathfrak{h})$. From this we obtain a unique automorphism $\phi : \mathfrak{h}_a \to \mathfrak{h}$, satisfying $\phi^*(\Delta(\mathfrak{g}, \mathfrak{h})) = \Delta(\mathfrak{g}, \mathfrak{h}_a)$, and also an isomorphism $\phi : W_a \simeq W$. If λ is a regular element of $\mathfrak{h}$ we let ϕ_λ be the isomorphism constructed this way, where $\Delta^+ = \Delta^+(\lambda)$.

Now fix a subspace $\mathfrak{s}_a \subset \mathfrak{h}_a$ and a genuine regular character $\gamma = (H, \gamma, \lambda)$ for $\widetilde{G}$. We say that $\mathfrak{s}_a$ is an (imaginary) admissible subspace for γ if $\mathfrak{s} :=$

$\phi_\lambda(\mathfrak{s}_a)$ (with ϕ_λ defined above) is an admissible subspace spanned by elements of $\Delta^{i,-}_{\frac{1}{2}}(\gamma)$. Given such an $\mathfrak{s}_a$, we define

$$c^{\mathfrak{s}_a}(\gamma) = c^{\mathfrak{s}}(\gamma). \tag{10.3}$$

It is clear that that K conjugacy class of $c^{\mathfrak{s}_a}(\gamma)$ depends only on the class of γ. By making the obvious modifications, we may speak of (real) admissible subspaces $\mathfrak{s}_a \subset \mathfrak{h}_a$ for γ, and define $c_{\mathfrak{s}_a}(\gamma)$. Again this definition descends to the level of K conjugacy classes of regular characters.

Here are some basic properties of Cayley transforms.

Proposition 10.4 *Let γ be a genuine regular character for $\widetilde{G}$, an admissible cover of a aimply laced real linear reductive group.*

(1) Suppose $\alpha_1, \ldots, \alpha_n \in \Delta^{i,-}_{\frac{1}{2}}(\gamma)$. Then $\{\alpha_1, \ldots, \alpha_n\}$ is admissible if and only if $\langle \alpha_i, \alpha_j^\vee \rangle = 0$ for all $i \neq j$.

(2) Suppose $\mathfrak{s} \subset \mathfrak{h}_a$ is an imaginary admissible subspace for γ. Then $\mathfrak{s}$ is a real admissible subspace for $c^{\mathfrak{s}}(\gamma)$ and $c_{\mathfrak{s}}(c^{\mathfrak{s}}(\gamma)) = \gamma$. (An analogous statement holds if $\mathfrak{s}$ is assumed to be a real admissible subspace for γ.)

(3) Suppose $\mathfrak{s}, \mathfrak{s}' \subset \mathfrak{h}_a$ are imaginary admissible subspaces for γ. Then $c^{\mathfrak{s}}(\gamma) = c^{\mathfrak{s}'}(\gamma)$ if and only if $\mathfrak{s} = \mathfrak{s}'$. (An analogous statement holds if $\mathfrak{s}$ and $\mathfrak{s}'$ are assumed to be real admissible subspaces for γ.)

Proof. This follow from [RT3, Propositions 10.1 and 10.2]. □

We now recall the cross action. Let $W(\lambda)$ be the Weyl group of $\Delta(\lambda)$. Given $w \in W(\lambda)$, we define a new regular character $w \times \gamma = (\widetilde{H}, w \times \Gamma, w\lambda)$. The definition is due to Vogan; in the setting of $\widetilde{G}$ (rather than G), the definition is given in [RT3, Section 4]. It follows from the proof of [V1, Theorem 9.2.11], for instance, that if $w \in W(\lambda)$ then $\overline{\pi}(w \times \gamma)$ and $\overline{\pi}(\gamma)$ belong to the same block.

We define the cross action of the abstract Weyl group as in [V4, Definition 4.2]. Suppose $w \in W_a$ and $\phi_\lambda(w) \in W(\lambda)$. Then define

$$w_a \times \gamma = \phi_\lambda(w)^{-1} \times \gamma. \tag{10.5}$$

Following [RT3] we also need the cross action by all of W_a. Fix a dominant regular element $\lambda_a \in \mathfrak{h}_a$. Let R_a be the root lattice of Δ_a. Choose a set $\mathcal{F}$ of dominant representatives of $(W_a\lambda_a+R_a)/R_a$. The map $w\to w\lambda$ induces an isomorphism $W_a/W_a(\lambda_a)\simeq(W_a\lambda_a+R_a)/R_a$. For $w\in W_a$ write ν_w for the corresponding element of $\mathcal{F}$, so $\nu_w=w\lambda+\mu(w)$ for some $\mu(w)\in R_a$.

Suppose $\lambda\in\mathfrak{h}^*$ is conjugate to some $\nu\in\mathcal{F}$. Then $\lambda=\phi_\lambda(\nu_y)$ for some $y\in W_a$. Define

$$w\times\lambda=\phi_\lambda(\nu_{wy}). \tag{10.6}$$

Note that ν_{wy} and $w\nu_y$ differ by a sum of roots.

Now suppose $\gamma=(H,\Gamma,\lambda)$ is a regular character with infinitesimal character in $\mathcal{F}$. For $w\in W_a$ define $w\times\Gamma$ as in [RT3, Section 4], and $w\times(H,\Gamma,\lambda)=(H,w\times\Gamma,w\times\lambda)$. This has infinitesimal character ν_{wy}. If $w\in W_a(\lambda_a)$ this agrees with the usual definition. Note that for $w\in W_a\setminus W_a(\lambda_a)$ this definition depends on the choice of $\mathcal{F}$.

Remark 10.7 This definition differs slightly from that of [RT3, Definition 4.1]. In the present setting, we use the root lattice R_a, and [RT3] instead uses the the weight lattice. It is easy to see that the results of Section 3 and 4 of [RT3] carry over with obvious changes. In particular, in our setting the elements $\mu(y,w_a)$ of [RT3, Definition 4.1] are now in R_a.

Note that if $w\in W_a\setminus W_a(\lambda_a)$ then γ and $w\times\gamma$ have distinct infinitesimal characters. This implies

$$\overline{\pi}(\gamma)\not\simeq\overline{\pi}(w\times\gamma)\quad(w\notin W_a(\lambda_a)). \tag{10.8}$$

The corresponding result would not be true with the version of the cross action of [RT3], using the weight lattice instead of the root lattice. Formally (10.8) makes some proofs below more transparent, and this explains why we deviate from the setting of [RT3].

Next we investigate how the cross action interacts with Cayley transforms. To begin, note that there is a natural candidate for an action of the Weyl group on admissible subspaces: if $\mathfrak{s}\subset\mathfrak{h}_a$ is an admissible subspace for $\gamma=(\widetilde{H},\Gamma,\lambda)$ spanned by $\alpha_1,\dots,\alpha_k$, and $w\in W_a(\mathfrak{g},\mathfrak{h})$, define $w\mathfrak{s}$ to be the subspace spanned by $w\alpha_1,\dots,w\alpha_k$.

Lemma 10.9 *Let $\gamma=(\widetilde{H},\Gamma,\lambda)$ be a genuine regular character for $\widetilde{G}$. Fix $w\in W_a$ and $\mathfrak{s}\subset\mathfrak{h}_a$. Then $\mathfrak{s}$ is an imaginary admissible subspace for γ if and only if $w\mathfrak{s}$ is an imaginary admissible subspace for $w\times\gamma$. In this case*

$$c^{w\mathfrak{s}}(w\times\gamma)=w\times c^{\mathfrak{s}}(\gamma).$$

Similarly $\mathfrak{s}$ is a real admissible subspace for γ if and only if $w\mathfrak{s}$ is a real admissible subspace for $w \times \gamma$. In this case

$$c_{w\mathfrak{s}}(w \times \gamma) = w \times c_{\mathfrak{s}}(\gamma).$$

Proof. If $w \in W_a(\lambda_a)$ the proof of [V4, Proposition 5.20(b)] generalizes easily to this situation. Because of the rigidity of Corollary 7.6 the proof in our setting in general is in fact much easier.

Write $\gamma = (H, \Gamma, \lambda)$. Consider the first case. With the obvious notation we have to show

This notation looks pretty bad to me. I suggest cutting everything in the proof after the first paragraph.

(10.10) $$(H^{\mathfrak{s}}, c^{\mathfrak{s}}(w \times \Gamma), c^{\mathfrak{s}}(w \times \lambda)) = (H^{\mathfrak{s}}, w \times c^{\mathfrak{s}}(\Gamma), w \times c^{\mathfrak{s}}(\lambda)).$$

By Corollary 7.6 it is enough to check

(10.11)(a) $$c^{\mathfrak{s}}(w \times \lambda) = w \times c^{\mathfrak{s}}(\lambda)$$

(10.11)(b) $$c^{\mathfrak{s}}(w \times \Gamma)|_{Z(\widetilde{G})} = w \times c^{\mathfrak{s}}(\Gamma)|_{Z(\widetilde{G})}.$$

Part (a) follows from a straightforward check from the definitions. For the second, since $c^{\mathfrak{s}}$ does not change the central character, it is enough to show $w \times \Gamma|_{Z(\widetilde{G})} = \Gamma|_{Z(\widetilde{G})}$ (we also apply this to $c^{\mathfrak{s}}(\Gamma)$). By [RT3, Definition 4.1] $w \times \Gamma$ and Γ differ by a sum of roots (cf. Remark 10.7), so this is immediate.

The second case is similar. □

For any regular character $\gamma = (H, \Gamma, \lambda)$ define the cross stabilizer of γ to be

(10.12) $$W(\gamma) = \{w \in W(\lambda) \cap W(G, H) \mid w \times \gamma \sim \gamma\}.$$

It is important to compute $W(\gamma)$. This requires some notation from [V4, Section 3]. Recall the sets $\Delta^i(\lambda)$ and $\Delta^r(\lambda)$ of integral imaginary roots and real roots, respectively. Let ρ_i, ρ_r be one-half the sum of a set of positive imaginary and real roots, respectively. Let $\Delta^{cx}(\lambda)$ be the set of complex roots in $\Delta(\lambda)$ orthogonal to both ρ_i and ρ_r. (This deviates from our notation above slightly, since $\Delta^{cx}(\lambda)$ is generally a proper subset of all complex roots.) Let

(10.13) $$\begin{aligned} W^i(\lambda) &= W(\Delta^i(\lambda)) \\ W^r(\lambda) &= W(\Delta^r(\lambda)) \\ W^{cx}(\lambda) &= W(\Delta^{cx}(\lambda))^\theta \end{aligned}$$

Proposition 10.14 *Recall the definitions of* (10.12) *and* (10.13)*. Then*

$$W(\gamma) = W^{cx}(\lambda) \ltimes (W^i(\lambda) \times W^r(\lambda)) \tag{10.15}$$

Proof. The proof of [V4, Proposition 4.14] applies to give

$$W(\gamma) = W^{cx}(\gamma) \ltimes (W^i(\gamma) \times W^r(\gamma)) \tag{10.16}$$

where $W^*(\gamma) = W^*(\lambda) \cap W(\gamma)$. As in [V4, Section 4] we have

$$W(\Delta^{i,+}(\gamma)) \subset W^i(\gamma) \subset \mathrm{Norm}_{W^i(\lambda)}(\Delta^{i,+}(\gamma)) \tag{10.17}$$

By (9.10)(a), $\Delta^{i,+}_{\frac{1}{2}}(\gamma) = \Delta^i(\lambda)$, and so $\Delta^{i,+}(\gamma) = \Delta^i(\lambda)$. Therefore (10.17) becomes

$$W^i(\lambda) \subset W^i(\gamma) \subset \mathrm{Norm}_{W^i(\lambda)}(W^i(\lambda)). \tag{10.18}$$

The outer terms are the same, so $W^i(\gamma) = W^i(\lambda)$. Similarly $W^r(\gamma) = W^r(\lambda)$. This completes the proof. □

We define the abstract version of $W(\lambda)$. The group $W(G,H)$ pulls back via ϕ_λ to a subgroup $W_a(G,H)$ of W_a. Let

$$W_a(\gamma) = \{w \in W_a(\lambda_a) \cap W_a(G,H) \,|\, w \times \gamma \sim \gamma\}. \tag{10.19}$$

where $\lambda_a \in \mathfrak{h}_a = \phi_\lambda^{-1}(\lambda)$. Proposition 10.14 holds with $W_a^{cx}(\lambda_a), W_a^i(\lambda_a)$ and $W_a^r(\lambda_a)$ on the right hand side.

We have defined the cross action of all of W_a, and we could define the cross stabilizer by (10.19) with W_a in place of $W_a(\lambda_a)$ on the right hand side. However if $w \in W_a \setminus W_a(\lambda_a)$ then by (10.8) $w \times \gamma \not\sim \gamma$, so these two definitions agree:

$$W_a(\gamma) = \{w \in W_a \cap W(G,H) \,|\, w \times \gamma \sim \gamma\}. \tag{10.20}$$

11 Duality

Throughout this section let $\widetilde{G}$ be an admissible double cover of a simply laced real reductive linear group G (cf. Section 2 and Definition 3.4). The representation theory of $\widetilde{G}$ is sufficiently rigid to reduce the existence of a duality theory to the duality of bigradings (Definition 9.11). Theorem 11.2 makes this precise.

Definition 11.1 A genuine regular character $\gamma = (\widetilde{H}, \Gamma, \lambda)$ (Definition 7.1) is called *weakly minimal* if $\Delta^{r,-}_{\frac{1}{2}}(\gamma)$ (cf. (9.8)(a)) is empty, i.e. there are no real, half-integral roots satisfying the parity condition. We say an irreducible representation $\pi = \overline{\pi}(\gamma)$ is weakly minimal if γ is weakly minimal. Similarly we say γ and $\overline{\pi}(\gamma)$ are weakly maximal if $\Delta^{i,-}_{\frac{1}{2}}(\gamma)$ is empty.

It is easy to see that every block $\mathcal{B}$ contains a weakly minimal representation: if $\overline{\pi}(\gamma) \in \mathcal{B}$ and $\mathfrak{s}$ is a maximal admissible subspace of $\Delta^{r,-}_{\frac{1}{2}}(\gamma)$ then $\overline{\pi}(c_{\mathfrak{s}}(\gamma)) \in \mathcal{B}$ is weakly minimal.

Theorem 11.2 *Let $\mathcal{B}$ be a block of genuine representations of $\widetilde{G}$ with regular infinitesimal character. Fix a weakly minimal element $\pi = \overline{\pi}(\gamma)$ of $\mathcal{B}$ with $\gamma = (H, \Gamma, \lambda)$. Suppose we are given:*

(a) an admissible cover $\widetilde{G}'$ of a simply laced, real reductive linear group G';

(b) a regular character $\gamma' = (\widetilde{H}', \Gamma', \lambda')$ of $\widetilde{G}'$ (Definition 7.1) with Γ' genuine; and

(c) an isomorphism ϕ from $\mathfrak{h}_d = \mathfrak{h} \cap \mathfrak{g}_d$ to $\mathfrak{h}'_d = \mathfrak{h}' \cap \mathfrak{g}'_d$.

Let θ and θ' denote the Cartan involutions of G and G' respectively. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ and $\Delta' = \Delta'(\mathfrak{g}', \mathfrak{h}')$. We assume

(11.3)(a) $$\phi(\theta(X)) = -\theta'(X) \quad \textit{for all } X \in \mathfrak{h}_d,$$

(11.3)(b) $$\phi^*(\Delta'(\lambda')) = \Delta(\lambda),$$

(11.3)(c) $$\phi^*(\Delta'_{\frac{1}{2}}(\lambda')) = \Delta_{\frac{1}{2}}(\lambda),$$

and (in the terminology of Section 9) ϕ induces an isomorphism

(11.4) $$g_{\frac{1}{2}}(\gamma)^\vee \simeq g_{\frac{1}{2}}(\gamma^\vee).$$

Let $\mathcal{B}'$ be the block containing $\overline{\pi}(\gamma')$. Then there exists a bijection

$$\Phi \;:\; \mathcal{B} \longrightarrow \mathcal{B}'$$

so that $\mathcal{B}'$ is dual to $\mathcal{B}$ with respect to Φ (Definition 4.2).

Note that condition (11.4) is equivalent to (11.3) together with

(11.5)(a) $$\phi^*(\Delta^{i,+}_{\frac{1}{2}}(\gamma')) = \Delta^{r,+}_{\frac{1}{2}}(\gamma)$$

and

(11.5)(b) $$\phi^*(\Delta^{r,+}_{\frac{1}{2}}(\gamma')) = \Delta^{i,+}_{\frac{1}{2}}(\gamma).$$

Before proving this we give a standard form for blocks for $\widetilde{G}$. The corresponding result in the linear case is [V4, Theorem 10.1] where it is much more subtle (because Cayley transforms can be multivalued).

Fix a block $\mathcal{B}$ of genuine representations of $\widetilde{G}$, with regular infinitesimal character $\lambda_a \in \mathfrak{h}_a$, and a weakly minimal element $\overline{\pi}(\gamma) \in \mathcal{B}$. Let $S_i(\gamma)$ be the set of subspaces of the abstract Cartan subalgebra $\mathfrak{h}_a$ which are imaginary admissible subspaces for γ. Recall $W_a(\gamma) = \{w \in W_a \,|\, w \times \gamma \sim \gamma\}$. Note that if $u \in W_a(\gamma)$ and $\mathfrak{s} \in S_i(\gamma)$ then Lemma 10.9 implies that $c^{u\mathfrak{s}}(u \times \gamma) \sim c^{u\mathfrak{s}}(\gamma)$, so $u\mathfrak{s}$ is also an admissible subspace for γ. Therefore $W_a(\gamma)$ acts on on $S_i(\gamma) \times W_a(\lambda_a)$:

(11.6) $$u \cdot (\mathfrak{s}, w) = (u\mathfrak{s}, wu^{-1}) \quad (u \in W_a(\gamma)).$$

The map

(11.7) $$\psi_i(\mathfrak{s}, w)(\gamma) = c^{w\mathfrak{s}}(w \times \gamma)$$

is clearly well defined, and factors to $(S_i(\gamma) \times W_a(\lambda_a))/W_a(\gamma)$, finally giving a well-defined map

(11.8) $$\psi_i : (S_i(\gamma) \times W_a(\lambda_a))/W_a(\gamma) \to \mathcal{B}$$

taking a representative $(\mathfrak{s}, w)$ on the left-hand side to $\overline{\pi}(c^{w\mathfrak{s}}(w \times \gamma))$. On the other hand, if $\overline{\pi}(\gamma)$ is a weakly maximal element of $\mathcal{B}$, we may define $S_r(\gamma)$ similarly (as real admissible subspaces), and ultimately obtain a map

(11.9) $$\psi_r : (S_r(\gamma) \times W_a(\lambda_a))/W_a(\gamma) \to \mathcal{B}.$$

Proposition 11.10 *Let $\widetilde{G}$ be an admissible cover (Definition 3.4) of a simply laced, real reductive linear group G (Section 2). Let $\mathcal{B}$ be a block of genuine representations of $\widetilde{G}$ with regular infinitesimal character λ_a. If $\overline{\pi}(\gamma)$ is a weakly minimal element of $\mathcal{B}$, the map*

$$\psi_i : (S_i(\gamma) \times W_a(\lambda_a))/W_a(\gamma) \to \mathcal{B}$$

of (11.8) *is a bijection. If* $\overline{\pi}(\gamma)$ *is a weakly maximal element of* $\mathcal{B}$*, the map*

$$\psi_r : (S_r(\gamma) \times W_a(\lambda_a))/W_a(\gamma) \to \mathcal{B}$$

of (11.9) *is a bijection.*

Proof. We only consider the first case, the second is similar. Fix a weakly minimal element $\overline{\pi}(\gamma) \in \mathcal{B}$.

We first prove injectivity. Suppose $\psi_i(\mathfrak{s}, w) = \psi_i(\mathfrak{s}', w')$. Then

$$c^{w\mathfrak{s}}(w \times \gamma) \sim c^{w'\mathfrak{s}'}(w' \times \gamma).$$

By repeated applications of Proposition 10.4(2)-(3), and the discussion around the sentence following (10.3), we conclude

$$c_{w'\mathfrak{s}'}c^{w\mathfrak{s}}(w \times \gamma) \sim w' \times \gamma. \tag{11.11}$$

Since γ is weakly minimal, in order for $c_{w'\mathfrak{s}'}c^{w\mathfrak{s}}$ to be defined we must have $w\mathfrak{s} = w'\mathfrak{s}'$. Again by Proposition 10.4(2)-(3), $c_{w'\mathfrak{s}'}c^{w\mathfrak{s}}(w \times \gamma) = c_{w\mathfrak{s}}c^{w\mathfrak{s}}(w \times \gamma) = w \times \gamma$, so we have

$$w \times \gamma \sim w' \times \gamma. \tag{11.12}$$

Then $(\mathfrak{s}', w') = (u\mathfrak{s}, wu^{-1})$ with $u = (w')^{-1}w \in W_a(\gamma)$, and injectivity follows.

For surjectivity, fix $\overline{\pi}(\delta) \in \mathcal{B}$. If $\mathfrak{s}$ is a maximal admissible subspace of the span of $\Delta^{r,-}_{\frac{1}{2}}(\delta)$ then $c_{\mathfrak{s}}(\delta)$ is weakly minimal. Recall the analog of [V4, Lemma 8.10]: if γ, γ' are weakly minimal elements such that $\overline{\pi}(\gamma)$ and $\overline{\pi}(\gamma')$ are in $\mathcal{B}$, then there exists $w \in W_a(\lambda_a)$ so that $w \times \gamma = \gamma'$. (This is proved in the same way as [V4, Lemma 8.10], using [RT3, Theorem 7.8] in place of [V4, Theorem 8.8].) We thus conclude that $c_{\mathfrak{s}}(\delta) = w \times \gamma$ for some w, i.e. $\overline{\pi}(\delta) = \overline{\pi}(c^{\mathfrak{s}}(w \times \gamma)) = \psi_i(w^{-1}\mathfrak{s}, w)$. This completes the proof of Proposition 11.10. □

Using the proposition, we can define the bijection of Theorem 11.2. Fix $\gamma = (H, \Gamma, \lambda)$ and $\gamma' = (\widetilde{H}', \Gamma', \lambda')$ as in the statement of the theorem. The hypotheses of (11.5) implies that ϕ induces bijections

$$\begin{gathered} \Delta^i(\lambda) \leftrightarrow \Delta^r(\lambda') \\ \Delta^i_{\frac{1}{2}}(\lambda) \leftrightarrow \Delta^r_{\frac{1}{2}}(\lambda'). \end{gathered} \tag{11.13}$$

By Proposition 9.9 this gives a bijection

$$\Delta^{i,-}_{\frac{1}{2}}(\gamma) \leftrightarrow \Delta^{r,-}_{\frac{1}{2}}(\gamma') \tag{11.14}$$

Thus we obtain a bijection

$$S_i(\gamma) \leftrightarrow S_r(\gamma') \tag{11.15}$$

which we will denote by

$$\mathfrak{s} \mapsto \mathfrak{s}'. \tag{11.16}$$

Meanwhile the hypotheses of Theorem 11.2 also imply $\Delta(\lambda_a) \simeq \Delta(\lambda'_a)$ and thus

$$W_a(\lambda_a) \simeq W_a(\lambda'_a). \tag{11.17}$$

We write these isomorphisms

$$\alpha \mapsto \alpha' \tag{11.18}$$

and

$$w \mapsto w'. \tag{11.19}$$

It is clear that the isomorphism $W_a(\lambda_a) \simeq W_a(\lambda'_a)$ interchanges $W^i_a(\lambda)$ and $W^r_a(\lambda')$, $W^r_a(\lambda)$ and $W^i_a(\lambda')$, and takes $W^{cx}_a(\lambda)$ to $W^{cx}_a(\lambda')$. Thus Proposition 10.14 implies $W_a(\gamma) \simeq W_a(\gamma')$. Therefore there is a natural isomorphism

$$(S_i(\gamma) \times W_a(\lambda_a))/W_a(\gamma) \leftrightarrow (S_r(\gamma) \times W_a(\lambda_a))/W_a(\gamma'). \tag{11.20}$$

The following result is an immediate consequence.

Proposition 11.21 *Retain the hypotheses and notation of Theorem 11.2, and recall the bijections of* (11.16), (11.18), *and* (11.19). *The map*

$$\begin{gathered}\Phi \,:\, \mathcal{B} \longrightarrow \mathcal{B}' \\ \psi_i(\mathfrak{s}, w) \mapsto \psi_r(\mathfrak{s}', w').\end{gathered}$$

is a bijection. Given $\overline{\pi}(\delta) \in \mathcal{B}$ write $\Phi(\overline{\pi}(\delta)) = \overline{\pi}(\delta')$. Then
(a) If $\alpha \in \Delta^{i,-}_{\frac{1}{2}}(\delta)$, we have

$$\Phi(\overline{\pi}(c^{\alpha}(\delta))) = \overline{\pi}(c_{\alpha'}(\delta')). \tag{11.22}$$

(b) If $w \in W_a(\lambda_a)$ we have

$$\Phi(\overline{\pi}(w \times \delta)) = \overline{\pi}(w' \times \delta'). \tag{11.23}$$

The proposition isolates most of the key properties needed to establish that the bijection is a character multiplicity. But the Kazhdan-Lusztig algorithm of [RT3] admits a few more complications that we need to address before completing the proof of Theorem 11.2. The most serious complication is that the algorithm does not work within a fixed block of representations, but rather several (with possibly difference infinitesimal characters). So we need to extend the bijection of Proposition 11.21 to a suitable union of blocks in such a way that the key properties listed in the corollary still hold.

As in Section 9, let $W_{\frac{1}{2}}(\lambda) = W(\Delta(2\lambda))$. The group $W_{\frac{1}{2}}(\lambda)$ takes the place of the integral Weyl group in the Kazhdan-Lusztig algorithm of [RT3]. (In [RT3], the more general definition of the extended integral Weyl group, denoted $\widetilde{W}(\lambda)$, is given. It reduces to $W_{\frac{1}{2}}(\lambda)$ in the simply laced case.) Note that the hypotheses of Theorem 11.2 imply that ϕ induces an isomorphism of $W_{\frac{1}{2}}(\lambda) \simeq W_{\frac{1}{2}}(\lambda')$. We also obtain an isomorphism

$$W_{\frac{1}{2}}(\lambda_a) \simeq W_{\frac{1}{2}}(\lambda'_a), \tag{11.24}$$

which we again write as

$$w \to w'. \tag{11.25}$$

Let $\mathcal{B}$ be a block as in Theorem 11.2, with infinitesimal character $\lambda_a \in \mathfrak{h}_a$. Choose $\mathcal{F}(\lambda_a)$ as in Section 10, and use this to define the cross action of W_a. Define $W_{a,\frac{1}{2}}(\lambda_a)$ as in Section 9. Set

$$\mathbf{B} = \{\overline{\pi}(w \times \delta) \mid w \in W_{a,\frac{1}{2}}(\lambda_a), \overline{\pi}(\delta) \in \mathcal{B}\}. \tag{11.26}$$

By [RT3, Theorem 7.8], $\mathbf{B}$ is a (disjoint) union of blocks the infinitesimal characters of which are a subset $\mathcal{F}(\lambda_a)$. Similarly we may make analogous choices and define

$$\mathbf{B}' = \{\overline{\pi}(w' \times \delta') \mid w' \in W_{a,\frac{1}{2}}(\lambda'_a), \overline{\pi}(\delta') \in \mathcal{B}'\} \tag{11.27}$$

Proposition 11.28 *Retain the hypotheses and notation of Theorem 11.2, and recall the notation of* (11.26) *and* (11.27). *There is a bijection*

$$\Phi \,:\, \mathbf{B} \longrightarrow \mathbf{B}'$$

extending the bijection of Proposition 11.21. Given $\overline{\pi}(\delta) \in \mathbf{B}$, write $\Phi(\overline{\pi}(\delta)) = \overline{\pi}(\delta')$. Then *(a) If* $\alpha \in \Delta^{i,-}_{\frac{1}{2}}(\delta)$*, we have*

$$\Phi(\overline{\pi}(c^\alpha(\delta))) = \overline{\pi}(c_{\alpha'}(\delta')). \tag{11.29}$$

(b) If $w \in W_{\frac{1}{2}}(\lambda_a)$ *we have*

$$\Phi(\overline{\pi}((w \times \delta)) = \overline{\pi}(w' \times \delta'). \tag{11.30}$$

Proof. Following the definition of the bijection of Proposition 11.21, we need only establish a version of Proposition 11.10 for $\mathbf{B}$.

Fix a weakly minimal element $\overline{\pi}(\gamma) \in \mathbf{B}$ and write $\gamma = (H, \Gamma, \lambda)$. Suppose $\mathfrak{s} \in S_i(\gamma)$ and $w \in W_{a,\frac{1}{2}}(\lambda_a)$. Define

$$\psi(\mathfrak{s}, w) = \overline{\pi}(c^{w\mathfrak{s}}(w \times \gamma)). \tag{11.31}$$

By (10.20), $\psi_i(\mathfrak{s}, w) = \psi_i(u\mathfrak{s}, wu^{-1})$ for $u \in W(\gamma)$. Let $W(\gamma)$ act on $S_i(\gamma) \times W_{a,\frac{1}{2}}(\lambda_a)$ as in (11.6). The proof of Propositions 11.10 and 11.21 now carry over essentially unchanged to show that ψ_i induces a bijection:

$$\psi_i : (S_i(\gamma) \times W_{a,\frac{1}{2}}(\lambda_a))/W(\gamma) \longrightarrow \mathbf{B}. \tag{11.32}$$

Arguing in the same way, we obtain a bijection

$$\psi_r : (S_r(\gamma) \times W_{a,\frac{1}{2}}(\lambda'_a))/W(\gamma) \longrightarrow \mathbf{B}'. \tag{11.33}$$

The proposition follows from the natural bijection between the left hand sides. □

We are now in a position to work within the extended Hecke algebra formalism of [RT3, Section 9]. The operators defined in [RT3, Definition 9.4] generate a $\mathbb{Z}$-algebra $\mathcal{H}$ resembling the finite Hecke algebra of $W_{a,\frac{1}{2}}(\lambda)$; moreover, that definition produces an $\mathcal{H}$ module $\mathcal{M}$ with an integral basis indexed by the elements of $\mathbf{B}$. Likewise we obtain a $\mathcal{H}$ module $\mathcal{M}'$ with an integral basis indexed by the elements of $\mathbf{B}'$. We may define a dual module $\mathcal{M}^*$; since $\mathcal{H}$ is nonabelian, a little care is required and we adopt the convention of [V4, Definition 13.3] or [RT3, Equation 11.3]. The dual module $\mathcal{M}^*$ also has an integral basis indexed by $\mathbf{B}$, and consequently the bijection $\Phi : \mathbf{B} \to \mathbf{B}'$ gives a $\mathbb{Z}$-linear isomorphism

$$\Phi \ : \ \mathcal{M}^* \longrightarrow \mathcal{M}'. \tag{11.34}$$

As in the proofs of [V4, Theorem 13.13] and [RT3, Theorem 11.1], the existence of the duality between elements of $\mathcal{B}$ and $\mathcal{B}'$ follows from the assertion that the map of Equation (11.34) is in fact an $\mathcal{H}$ module isomorphism. Again like the proofs of [V4, Theorem 13.13] and [RT3, Theorem 11.1], this follows formally from the definition of the $\mathcal{H}$-module structure given in [RT3, Definition 9.4] and the key symmetry properties summarized in parts (a) and (b) of Proposition 11.28. (As an example of the formal calculations involved one may consult the proof of [RT3, Theorem 11.1].) This completes the proof of Theorem 11.2. □

12 Definition of the dual regular character

Throughout this section we fix an admissible two-fold cover $\widetilde{G}$ (Definition 3.4 of a real reductive linear group G (cf. Sections 2 and 3). Fix a block $\mathcal{B}$ of genuine representations of $\widetilde{G}$ with regular infinitesimal character. Let $\gamma = (H, \Gamma, \lambda) \in \mathcal{B}$ be a weakly minimal element. We will construct

1. a complex reductive Lie algebra $\mathfrak{g}'$,
2. a Cartan involution θ' of $\mathfrak{g}'$ specifying a real form $\mathfrak{g}'_0$ of $\mathfrak{g}'$,
3. a group G' with Lie algebra $\mathfrak{g}'_0$ in the class defined in Section 2,
4. an admissible cover $\widetilde{G'}$ of G',
5. a genuine regular character γ' of $\widetilde{G'}$

such that the block containing $\overline{\pi}(\gamma')$ is dual to $\mathcal{B}$.

12.1 Definition of $\mathfrak{g}'$

Recall (cf. Section 6) $\Gamma(2\lambda) \in \mathfrak{h}$. Let

$$h = \exp(2\pi i \Gamma(2\lambda))$$

and set

$$\mathfrak{g}' = \mathrm{Kernel}(\mathrm{Ad}(h)).$$

Lemma 12.1

$$\Delta(\mathfrak{g}',\mathfrak{h}) = \Delta_{\frac{1}{2}}(\lambda). \tag{12.2}$$

Proof. We may identify $\Delta(\mathfrak{g}',\mathfrak{h})$ with

$$\{\alpha \in \Delta \mid \langle \alpha, \Gamma(2\lambda)\rangle \in \mathbb{Z}\}$$

which by (6.2)(d) equals

$$\{\alpha \in \Delta \mid \langle 2\lambda, \alpha^\vee\rangle \in \mathbb{Z}\}$$

which is precisely $\Delta_{\frac{1}{2}}(\lambda)$. □

Note $\mathfrak{g}' = \mathfrak{g}$ if and only if λ is half-integral (Definition 9.4), and in general λ is half-integral for $\mathfrak{g}'$.

12.2 Definition of $\mathfrak{g}_0'$ and G'

Proposition 12.3 *Suppose θ' is a Cartan involution of $\mathfrak{g}'$ satisfying*

$$\theta'|_{\mathfrak{h}} = -\theta|_{\mathfrak{h}}. \tag{12.4)(a}$$

Let $\mathfrak{g}_0'$ be the corresponding real form of $\mathfrak{g}$. Let G' be a real reductive linear group (cf. Section 2) with Lie algebra $\mathfrak{g}_0'$. Let H' be the Cartan subgroup of G' corresponding to $\mathfrak{h} \subset \mathfrak{g}'$. Assume $\widetilde{G'}$ is an admissible cover of G', and $\gamma' = (\widetilde{H}', \Gamma', \lambda')$ is a genuine regular character of $\widetilde{G}'$ satisfying

$$\langle \lambda, \alpha^\vee\rangle \equiv \langle \lambda', \alpha^\vee\rangle \pmod{\mathbb{Z}} \quad \text{for all } \alpha \in \Delta_{\frac{1}{2}}(\lambda). \tag{12.4)(b}$$

Then the block containing $\overline{\pi}(\gamma')$ is dual to $\mathcal{B}$.

Fix a genuine regular character γ. For the proof it is convenient to keep track of what data the sets of roots defined in Section 9 depend on. For example $\Delta^i(\gamma)$ depends on λ and $\theta|_{\mathfrak{h}}$, while $\Delta^{i,\pm}_{\frac{1}{2}}(\gamma)$ depends on λ and θ. Accordingly we write

$$\begin{aligned} \Delta^i(\gamma) &= \Delta^i(\lambda, \theta|_{\mathfrak{h}}) \\ \Delta^r(\gamma) &= \Delta^r(\lambda, \theta|_{\mathfrak{h}}) \\ \Delta^{i,\pm}_{\frac{1}{2}}(\gamma) &= \Delta^{i,\pm}_{\frac{1}{2}}(\lambda, \theta). \end{aligned} \tag{12.5}$$

By Proposition 9.9 in fact we have

$$\begin{aligned}\Delta^{i,\pm}_{\frac{1}{2}}(\gamma) &= \Delta^i(\lambda,\theta|_{\mathfrak{h}})\\ \Delta^{r,\pm}_{\frac{1}{2}}(\gamma) &= \Delta^r(\lambda,\theta|_{\mathfrak{h}}).\end{aligned} \tag{12.6}$$

Proof. By (12.4)(b) $\Delta(\mathfrak{g}',\mathfrak{h})(\lambda') = \Delta(\mathfrak{g},\mathfrak{h})(\lambda)$ and $\Delta(\mathfrak{g}',\mathfrak{h})(2\lambda') = \Delta(\mathfrak{g},\mathfrak{h})(2\lambda)$.

$$\begin{aligned}\Delta^{i,+}_{\frac{1}{2}}(\gamma) &= \Delta^i(\mathfrak{g},\mathfrak{h})(\lambda) \quad \text{(by (9.10)(b))}\\ &= \Delta^r(\mathfrak{g}',\mathfrak{h})(\lambda') \quad \text{(by (12.4)(a))}\\ &= \Delta^{r,+}_{\frac{1}{2}}(\gamma') \quad \text{(by (9.10)(a))}.\end{aligned} \tag{12.7}$$

The equality $\Delta^{r,+}_{\frac{1}{2}}(\gamma) = \Delta^{i,+}_{\frac{1}{2}}(\gamma')$ follows by the dual argument.

The conditions of Theorem 11.2 now hold, with ϕ the identity map, and this implies the Proposition. □

We want to choose a Cartan involution θ' of $\mathfrak{g}'$ satisfying (12.4)(a), and such that there exists $\gamma' = (H,\Gamma',\lambda')$ satisfying (12.4)(b). There may be many θ' satisfying (12.4)(a). If γ' exists by Proposition 9.9 we have

$$\Delta^{i,+}_{\frac{1}{2}}(\gamma') = \Delta^i(\lambda'). \tag{12.8}$$

The left hand side only depends on λ' and θ', so we write

(12.9)

$$\begin{aligned}\Delta^{i,+}_{\frac{1}{2}}(\lambda',\theta') &= \Delta^{i,+}_{\frac{1}{2}}(\gamma')\\ &= \{\alpha\in\Delta(\mathfrak{g},\mathfrak{h}) \mid \langle 2\lambda',\alpha^\vee\rangle\in\mathbb{Z}, \theta'(\alpha)=\alpha, \alpha \text{ non-compact}\}\end{aligned}$$

accordingly. On the other hand note that $\Delta^i(\lambda')$ only depends on λ' and $\theta'_{|}\mathfrak{h}$, so write

$$\begin{aligned}\Delta^i(\lambda',\theta'|_{\mathfrak{h}}) &= \Delta^i(\lambda')\\ &= \{\alpha \mid \langle\lambda',\alpha^\vee\rangle\in\mathbb{Z}, \theta'(\alpha) = -\alpha\}\end{aligned} \tag{12.10}$$

We now construct θ' so (12.8) holds, which will turn out to be sufficient for the existence of γ'.

Lemma 12.11 *There exists a Cartan involution θ' of $\mathfrak{g}'$ so that*

(12.12)(a) $$\theta'|_{\mathfrak{h}} = -\theta|_{\mathfrak{h}},$$

and

$$\text{(12.12)(b)} \qquad \Delta^{i,+}_{\frac{1}{2}}(\lambda',\theta') = \Delta^i(\lambda',\theta'|_{\mathfrak{h}'}).$$

Example 12.13 Suppose G is split and H is the split Cartan subgroup. Then we may take

$$\theta' = Ad(exp(\pi i\lambda)).$$

Condition (12.12)(a) is immediate, and (b) follows from (9.10). See Example 14.3.

Proof. This is similar to [V4, Theorem 11.1(a)], although in fact somewhat easier since we are working entirely on (a subalgebra of) $\mathfrak{g}$, and not on the dual Lie algebra. We start with the Cartan involution of a quasisplit group, and then modify it accordingly.

Fix a set of positive roots of $\Delta^r_{\frac{1}{2}}(\lambda,\theta|_{\mathfrak{h}})$, and let ρ_r be one-half the sum of the positive roots.

By [V4, Lemma 10.9] applied to $-\theta|_{\mathfrak{h}}$ we may choose an involution θ^0 of $\mathfrak{g}'$ such that $\theta^0|_{\mathfrak{h}} = -\theta|_{\mathfrak{h}}$ and

$$\theta^0(X_\alpha) = -X_\alpha$$

for every simple root α of $\Delta^i_{\frac{1}{2}}(\lambda,\theta^0|_{\mathfrak{h}})$ (this is the Cartan involution of a quasisplit group). Then for all $\alpha\in\Delta^i_{\frac{1}{2}}(\lambda',\theta^0|_{\mathfrak{h}})$ we have

$$\text{(12.14)} \qquad \theta^0(X_\alpha) = (-1)^{\langle\rho_r^\vee,\alpha\rangle}X_\alpha$$

(Note that ρ_r is one-half the sum of the positive roots of $\Delta^r_{\frac{1}{2}}(\lambda,\theta|_{\mathfrak{h}}) = \Delta^i_{\frac{1}{2}}(\lambda,\theta^0|_{\mathfrak{h}})$.)

We now need an analog of [V4, Proposition 10.8(b)], giving an element of $\mathfrak{h}$ with which to modify θ^0.

Lemma 12.15 *There is an element $y\in\mathfrak{h}$ satisfying*

$$\text{(12.16)(a)} \qquad \theta^0(y) = -y$$

$$\text{(12.16)(b)} \qquad \langle\alpha,y\rangle\in\mathbb{Z} \quad \textit{for all } \alpha\in\Delta_{\frac{1}{2}}(\lambda)$$

$$\text{(12.16)(c)} \qquad (-1)^{\langle 2\lambda,\alpha^\vee\rangle} = (-1)^{\langle\alpha,y+\rho_r^\vee\rangle} \quad \textit{for all } \alpha\in\Delta^r_{\frac{1}{2}}(\lambda,\theta|_{\mathfrak{h}}).$$

We defer the proof, which proceeds by reduction to the linear case, to Section 15, and continue with the proof of Lemma 12.11. Let

$$\theta' = \theta^0 \circ Ad(\exp(\pi i y))$$

with y as in Lemma 12.15. By (12.16)(a) and (b), θ' is an involution. Moreover, for all $\alpha \in \Delta^i_{\frac{1}{2}}(\lambda', \theta'|_{\mathfrak{h}}) = \Delta^i_{\frac{1}{2}}(\lambda', \theta^0|_{\mathfrak{h}})$,

$$\begin{aligned}
\theta'(X_\alpha) &= \theta^0(Ad(\exp(\pi i y))(X_\alpha) \\
&= \theta^0(\alpha(\exp(\pi i y))X_\alpha) \\
&= e^{\pi i \langle y, \alpha^\vee\rangle}\theta^0(X_\alpha) \\
&= e^{\pi i(\langle y, \alpha^\vee\rangle + \langle \rho_R^\vee, \alpha^\vee\rangle)}(X_\alpha) \quad \text{by (12.14)} \\
&= (-1)^{\langle 2\lambda, \alpha^\vee\rangle} X_\alpha \quad \text{by (12.16)(c).}
\end{aligned}$$

Therefore $\theta'(X_\alpha) = \pm X_\alpha$, with equality if and only if $\alpha \in \Delta^i(\lambda, \theta'|_{\mathfrak{h}})$. This is precisely (12.12)(b). This completes the proof of Lemma 12.11. ☐

Definition 12.17 Let $\mathfrak{g}'_0$ be the real form of $\mathfrak{g}'$ defined by θ'.

12.3 Definition of $\widetilde{G'}$ and γ'

With this choice of θ' we now show that we may construct γ' satisfying (12.4)(b).

Proposition 12.18 *Assume G' is the real points of a connected, simply connected, algebraic group, and suppose $\widetilde{G'}$ is an admissible two-fold cover of G' (Definition 3.4). Then there is a genuine character γ' of $\widetilde{G'}$ satisfying* (12.4)(b), *and therefore the block containing $\overline{\pi}(\gamma)$, is dual to $\mathcal{B}$.*

Proof. We apply Corollary 8.12. Notice that if α is an imaginary root then (12.12)(b) gives (8.6)(a). Next suppose $\alpha \in \Delta_{\frac{1}{2}}(\lambda)$ is complex with respect to θ'. Then α is complex with respect to θ and

$$\begin{aligned}
\langle \lambda, \alpha^\vee + \theta'(\alpha^\vee)\rangle &= \langle \lambda, \alpha^\vee - \theta(\alpha^\vee)\rangle \\
&= -\langle \lambda, \alpha^\vee + \theta(\alpha^\vee)\rangle + 2\langle \lambda, \alpha\rangle.
\end{aligned}$$

The final term is an integer since $\alpha \in \Delta_{\frac{1}{2}}(\lambda)$. By (8.6)(b) applied to $\gamma = (H, \Gamma, \lambda)$ the right hand side is an integer if $\langle \alpha, \theta(\alpha^\vee)\rangle = 0$, and in $\mathbb{Z} + \frac{1}{2}$ otherwise. Since $\theta'(\alpha^\vee) = -\theta(\alpha^\vee)$ this gives (8.6)(b) for λ and θ'.

We may therefore apply Corollary 8.12 to deduce the existence of γ' satisfying (12.4)(b). $\square$

13 The Main Theorem

Theorem 13.13 *Suppose G is a simply laced, real reductive linear group (cf. Section 2). Let $\widetilde{G}$ be an admissible two-fold cover of G (Definition 3.4). Let $\mathcal{B}$ be a block of genuine representations of $\widetilde{G}$ with regular infinitesimal character λ.*

Let $\mathfrak{g}' \subset \mathfrak{g}$ be the Lie algebra constructed in Section 12.1. Recall the root system of $\mathfrak{g}'$ is $\Delta_{\frac{1}{2}}(\lambda)$ (9.3). Let $\mathfrak{g}'_0$ be the real form of $\mathfrak{g}'$ constructed in Section 12.2.

Suppose G' is a real reductive linear groups with $\mathrm{Lie}(G') \simeq \mathfrak{g}'_0$ and $\widetilde{G'}$ is an admissible two-fold cover of G'. Then there is a block $\mathcal{B}'$ of genuine representations of $\widetilde{G}'$, with regular, half-integral infinitesimal character, such that $\mathcal{B}'$ is dual to $\mathcal{B}$ (Definition 4.2).

There is a real reductive linear group G' with $\mathrm{Lie}(G') = \mathfrak{g}'_0$ such that there exists an admissible cover $\widetilde{G}'$ of G'.

Proof. All but the last statement is a reformulation of Proposition 12.3. The final statement is Proposition 3.7. $\square$

Remark 13.14 We have assumed that the infinitesimal character is regular. There are several possibilities for formulating a version of Definition 4.2 in the case of singular infinitesimal character. The subtlety is an appropriate definition of standard modules in the singular case. For applications to Langlands' ideas, a geometric perspective is required. This is explained in [ABV], and is made explicit in [RT2, Remark 4.20]. The latter reference gives enough details to extend Theorem 13.13 to a statement in the singular case, and so we omit the details here.

14 Examples

14.1 Example: $SL(2, \mathbb{R})$

The unique nontrivial two-fold cover of G is an admissible cover. At infinitesimal character $\rho/2$ (a typical half-integral infinitesimal character) there are four irreducible genuine representations: the two oscillator representations, each of which has two irreducible summands. It is easy to see that at this infinitesimal character $\widetilde{G}$ has two genuine blocks with distinct central characters, each containing two irreducible representations. Each block is self-dual. See [RT1, Section 4].

14.2 Example: $GL(n, \mathbb{R})$

A two-fold cover $\widetilde{G}$ of $G = \mathrm{GL}(n, \mathbb{R})$ is admissible if it restricts to the (unique) nontrivial cover of $\mathrm{SL}(n, \mathbb{R})$; assume this is the case. Suppose λ is a half-integral infinitesimal character. Then $\Delta_{\frac{1}{2}}(\lambda)$ is of type $A_p \times A_q$. If n is even $\widetilde{G}$ has a unique block at infinitesimal character λ; if n is odd, there are two isomorphic blocks. Fix such a block $\mathcal{B}$. Consider an admissible cover $\widetilde{G}'$ of $G' = \mathrm{U}(p, q)$; there are three, corresponding to the three double covers of $\mathrm{U}(p) \times \mathrm{U}(q)$, and the exact one we choose is not important. Fix an infinitesimal character λ' for which $\widetilde{G}'$ has genuine discrete series. This implies that $\Delta_{\frac{1}{2}}(\lambda)$ is of type $A_p \times A_q$. Then $\widetilde{G}'$ has a unique block $\mathcal{B}'$ at infinitesimal character λ', and $\mathcal{B}$ is dual to $\mathcal{B}'$.

Two cases of the previous paragraph are worth keeping in mind. If λ is integral $\mathcal{B}$ consists of a single irreducible principal series, G' is compact, the cover $\widetilde{G}'$ splits, and $\mathcal{B}'$ consists of a single finite-dimensional representation of the compact linear group $\widetilde{G}'$. On the other hand, consider $\lambda = \rho/2$. Then each block $\mathcal{B}$ for $\widetilde{G}$ has an interesting unitary representation, say π, that is as small as the infinitesimal character permits. (In other words, π might be called unipotent.) The group G' is either isomorphic to $\mathrm{U}(n/2, n/2)$ (if n is even) or $\mathrm{U}\left((n+1)/2, (n-1)/2\right)$ if n is odd. In either case $\widetilde{G}$ is quasisplit and hence has genuine large discrete series. Each such is characterized by its infinitesimal character. Fix λ' for which such a large discrete series π' exists. Let $\mathcal{B}'$ denote the block containing π'. Then $\mathcal{B}$ is dual to $\mathcal{B}'$ and the duality maps π to π'. For further details see [RT3, Part II].

14.3 Example: Minimal Principal Series of Split Groups

Suppose G is the split real form of a simply connected reductive complex group $G(\mathbb{C})$. Suppose $\widetilde{G}$ be an admissible cover of G and H is a split Cartan subgroup. Fix $\lambda \in \mathfrak{h}^*$. Suppose π is a genuine principal series representation of $\widetilde{G}$ with infinitesimal character λ. By Remark 7.7 there is such a representation, determined by its central character. Let $h = exp(\pi i \lambda)$ and

$$G'(\mathbb{C}) = Cent_{G'(\mathbb{C})}(h^2).$$

Let $\theta' = int(h)$ considered as an involution of $G'(\mathbb{C})$. Note that $G'(\mathbb{C}) = G(\mathbb{C})$ if λ is half-integral. Let G' be the corresponding real form, and let $\widetilde{G}'$ be an admissible cover. Let $K'(\mathbb{C}) = G'(\mathbb{C})^{\theta}$. Thus

$$\Delta(K'(\mathbb{C}), H(\mathbb{C})) = \Delta(\lambda).$$

The Cartan subgroup of G' corresponding to $\mathfrak{h}$ is compact, and G' has a discrete series representation with infinitesimal character λ, determined by its central character. Then π and π' have dual bigradings, and the map $\pi \to \pi'$ extends to a duality of blocks.

For example if λ is integral the principal series representation is irreducible. Dually G' is compact, the trivial cover of G' is admissible, and π' is a finite dimensional representation of $\widetilde{G}'$.

If π is any genuine irreducible representation of $\widetilde{G}$ then we may apply a sequence of Cayley transforms to obtain a minimal principal series representation. This reduces the computation of the dual of $\mathcal{B}(\pi)$ to the previous case; in particular $G'(\mathbb{C})$ and G' are computed from the infinitesimal character as above.

14.4 Example: Principal Series of Split Groups (continued)

The duality of Theorem 13.13 is for a single block, as in [V4]. If G is the real points of a connected reductive algegraic group this duality can be promoted, roughly speaking, to a duality on all blocks simultaneously. See [ABV] for details. It would be very interesting to do this also in the nonlinear case. We limit our discussion here to minimal principal series of (simple) split groups.

For simplicity we assume G is the split real form of a connected, semisimple complex group, and let $\widetilde{G}$ be its admissible cover. Fix an infinitesimal character for G. Then the genuine minimal principal series representations of G with this infinitesimal character are parametrized by the genuine characters of $Z(\widetilde{G})$.

There are a finite number $\pi_1, \dots, \pi_n$ of such representations, generating distinct blocks $\mathcal{B}_1, \dots, \mathcal{B}_n$. It follows from Section 11 that there are natural bijections between these blocks, and the corresponding representations of the extended Hecke algebra are isomorphic. Here are the number of such blocks for the simple groups [ABPTV, Section 6, Table 1]:

$$\begin{aligned} A_{2n}, E_6, E_8 &: 1 \\ A_{2n+1}, D_{2n+1}, E_7 &: 2 \\ D_{2n} &: 4 \end{aligned} \tag{14.1}$$

In fact it can be shown that the group of *outer automorphisms* of $\widetilde{G}$ acts transitively on $\{\pi_1, \dots, \pi_n\}$.

14.5 Example: Discrete Series

This is dual to the previous section. Suppose G is a real form of a connected, semisimple group, which contains a compact Cartan subgroup. Let $\widetilde{G}$ be the admissible cover of G, and fix an infinitesimal character λ for which $\widetilde{G}$ has a genuine discrete series representation.

The number of genuine discrete series representations of $\widetilde{G}$ with infinitesimal character λ depends on the real form. If G is quasisplit it can be shown, for example by a case-by-case analysis, that the genuine discrete series representations of $\widetilde{G}$ with infinitesimal character λ are in bijection with the genuine principal series representation of the split real form of G discussed in the previous section. However if G is not quasisplit there may be fewer genuine discrete series of $\widetilde{G}$.

For example suppose $G(\mathbb{C}) = SL(2n, \mathbb{C})$. As in Example 14.2 if $G = SU(p, q)$ then $\widetilde{G}$ has 1 genuine discrete series representation with infinitesimal character λ, if $p \neq q$, and 2 if $p = q$.

14.6 Example: Genuine Discrete Series for E_7

There are three non-compact real forms of E_7: split, Hermitian ($\mathfrak{k} = \mathbb{R} \times D_5$) and "quaternionic" ($\mathfrak{k} = A_1 \times A_5$). All have a compact Cartan subgroup.

We label these $E_7(s), E_7(h)$ and $E_7(q)$, respectively. Then at the appropriate infinitesimal character (for example $\rho - \frac{1}{2}\lambda_i$ where λ_i is an appropriate fundamental weight), $E_7(s)$ and $E_7(h)$ have two genuine discrete series representations. On the other hand $E_7(q)$ has only 1 genuine discrete series representation.

It is worth noting that $Z(\widetilde{G}) = \mathbb{Z}/4\mathbb{Z}$ in the split and Hermitian cases, and $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ in the quaternionic real form. Let π be a genuine discrete series representation of $\widetilde{G}$. If G is split or quaternionic π and π^* have distinct central characters, and are therefore the two genuine discrete series representations. If G is quaternionic then $\pi \simeq \pi^*$.

14.7 Example: Genuine discrete series for D_n

Finally we specialize the discussion in Example 14.5 to type D_n.

Write $\lambda = (\lambda_1, \dots, \lambda_n)$ in the usual coordinates. Then λ is half-integral if

$$\lambda_i \in \frac{1}{2}\mathbb{Z} \quad \text{for all i}$$

or

$$\lambda_i \in \pm\frac{1}{4} + \mathbb{Z} \quad \text{for all i.}$$

In the first case $\Delta(\lambda)$ is of type $D_p \times D_q$, $G = Spin(2p, 2q)$, and $\widetilde{K} \simeq Spin(2p) \times Spin(2q)$. Write $\lambda = (a_1, \dots, a_p, b_1 \dots, b_q)$ with $a_i \in \mathbb{Z}$ and $b_j \in \mathbb{Z} + \frac{1}{2}$. Then $\widetilde{G}$ has discrete series representations with Harish-Chandra parameter (with the obvious notation) $(a_1, \dots, a_{p-1}, \epsilon a_p; b_1, \dots, b_{q-1}, \epsilon b_q)$ with $\epsilon = \pm 1$. If $p = q$ it also has two more, with Harish-Chandra parameter $(b_1, \dots, b_{p-1}, \epsilon b_p; a_1, \dots, a_{p-1}, \epsilon a_p)$. These two or four genuine discrete series representations have distinct central characters.

In the second case $\Delta(\lambda)$ is of type A_{n-1} and $\widetilde{K}$ is a two-fold cover of $U(n)$. In this case G the real form of $Spin(2n, \mathbb{C})$ corresponding to the real form $\mathfrak{so}^*(2n)$ of $\mathfrak{so}(2n, \mathbb{C})$, and a two-fold cover of $SO^*(2n)$.

15 Proof of Lemma 12.15

We give the proof of Lemma 12.15. We reduce to a linear group, using the following lemma.

Let $\widetilde{G}$ be an admissible cover of G. Since we will be working on both $\widetilde{G}$ and G we change notation and let $\tilde{\gamma} = (\widetilde{H}, \widetilde{\Gamma}, \tilde{\lambda})$ be a genuine character. We have

$$\Delta^r(\tilde{\lambda}) \subset \Delta^r_{\frac{1}{2}}(\tilde{\lambda}) = \Delta^r(2\tilde{\lambda}).$$

We want to construct an element y related to the cograding $(-1)^{\langle 2\tilde{\lambda}, \alpha^\vee\rangle}$ of $\Delta^r_{\frac{1}{2}}(\tilde{\lambda})$. The idea is to construct a character $\gamma = (H, \Gamma, 2\tilde{\lambda})$ of the linear group such that the cograding of the real *integral* roots for γ is the same as the cograding of the real *half-integral* roots for $\tilde{\gamma}$, and apply [V4]. That is we would like to choose γ so that $\alpha \in \Delta^r(2\tilde{\lambda})$ does not satisfy the parity condition if and only if $\alpha \in \Delta^r(\tilde{\lambda})$.

Because of the ρ-shifts we need a slightly weaker result.

Lemma 15.1 *There exists a character* $\gamma = (H, \Gamma, \lambda)$ *satisfying*

(15.2)(a) $$\Delta(\lambda) = \Delta_{\frac{1}{2}}(\tilde{\lambda}).$$

and

(15.2)(b) $$\Delta^{r,+}(\gamma) = \Delta^r(\tilde{\lambda}).$$

The second condition can be stated: if $\alpha \in \Delta^r$ satisfies $\langle \lambda, \alpha^\vee\rangle$, then

(15.3) $$\alpha \text{ does not satisfy the parity condition } \Leftrightarrow \langle \lambda, \alpha^\vee\rangle \in 2\mathbb{Z}.$$

We first show this implies Lemma 12.15.

Proof of Lemma 12.15. Write $\delta^\vee(\gamma)$ for the cograding of $\Delta^r(\lambda)$ defined by γ. By the Lemma for all $\alpha \in \Delta^r(\lambda)$ we have

(15.4)(a) $$\delta^\vee(\gamma)(\alpha) = (-1)^{\langle 2\lambda, \alpha^\vee\rangle}$$

By [V4, Proposition 10.8(b)] there is an element $\tau \in \mathfrak{h}^*$ satisfying $\theta^0(\tau) = -\tau$, $\langle \tau, \alpha^\vee\rangle \in \mathbb{Z}$ for all $\alpha \in \Delta^r(\lambda)$, and

(15.4)(b) $$\delta^\vee(\gamma)(\alpha) = (-1)^{\langle \tau + \rho_r, \alpha^\vee\rangle}.$$

Consider the isomorphism Φ of (6.1). Note that $\Phi(\rho_r) = \rho_r^\vee$, and let $y = \Phi(\tau)$. By (6.2)(d) we have $\langle \alpha, y\rangle = \langle \tau, \alpha^\vee\rangle \in \mathbb{Z}$, and

(15.4)(c) $$\delta^\vee(\gamma)(\alpha) = (-1)^{\langle \alpha, y + \rho_r^\vee\rangle}.$$

Together with (a) this proves Lemma 12.15.

□

To construct γ it is natural to try to take $\lambda = 2\tilde{\lambda}$; this would satisfy (15.2)(b). On the other hand $\tilde{\Gamma}^2$ factors to a character of H^0, and it is natural to take $\Gamma = \tilde{\Gamma}^2$ on H^0. Because of the shifts these two choices are incompatible. We use this as a starting point in the next definition.

Let $\rho = \rho(\widetilde{\lambda})$ be one-half the sum of the roots satisfying $\langle \lambda, \alpha^\vee \rangle > 0$. Define $\rho_i = \rho(\widetilde{\lambda})$ and $\rho_{i,c} = \rho_{i,c}(\widetilde{\lambda})$ similarly, as in Definition 7.1.

Lemma 15.5 *There exists a regular character* $\gamma = (H, \Gamma, \lambda)$ *satisfying*

(15.6)(a) $$d\Gamma = 2d\widetilde{\Gamma} + \rho - \rho_i + 2\rho_{i,c}$$

(15.6)(b) $$\Gamma(m_\alpha) = \epsilon_\alpha(-1)^{\langle \rho, \alpha^\vee \rangle} \quad \textit{for all } \alpha \in \Delta^r(\lambda)$$

(15.6)(c) $$\lambda = 2\tilde{\lambda} + \rho$$

Remark 15.7 We may replace ρ in (15.6)(a-c) with any element γ such that $\gamma - \rho$ exponentiates to a character of H^0. This will be clear from the proof. In particular if ρ exponentiates to H^0 we could remove ρ from the right hand side of each term of (15.6)(a-c), and in particular take $\lambda = 2\tilde{\lambda}$.

Assuming this for the moment is is easy to see it satisfies the condition of Lemma 15.1, and therefore proves Lemma 15.1.

Proof of Lemma 15.1. Part (a) is immediate from (15.6)(c). Suppose α is a real root and $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$. Then

$$\begin{aligned} \Gamma(m_\alpha) &= \epsilon_\alpha(-1)^{\langle \rho, \alpha^\vee \rangle} \quad (15.6)(b) \\ &= \epsilon_\alpha(-1)^{\langle \lambda - 2\tilde{\lambda}, \alpha^\vee \rangle} \quad (15.6)(c) \\ &= \begin{cases} \epsilon_\alpha(-1)^{\langle \lambda, \alpha^\vee \rangle} & \langle \tilde{\lambda}, \alpha^\vee \rangle \in \mathbb{Z} \\ -\epsilon_\alpha(-1)^{\langle \lambda, \alpha^\vee \rangle} & \langle \tilde{\lambda}, \alpha^\vee \rangle \in \mathbb{Z} + \frac{1}{2} \end{cases} \end{aligned}$$

which gives (15.2)(b). □

Before turning to the proof of Lemma 15.5 it is convenient to introduce some notation and prove two preliminary results. For now fix a θ-stable Cartan subgroup H and a set of positive roots Δ^+. Let $\Delta^+_{cx} = \Delta^+ \cap \Delta_{cx}$ and $\rho_{cx} = \frac{1}{2}\sum_{\alpha \in \Delta^+_{cx}} \alpha$.

Recall $H = TA$ with $T = H^\theta$ and $A = \exp(\mathfrak{a}_0)$. Complexifying we have $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$, and $H(\mathbb{C}) = T(\mathbb{C})A(\mathbb{C})$ where $T(\mathbb{C}) = \exp(\mathfrak{t})$, $A(\mathbb{C}) = \exp(\mathfrak{a})$.

Note that $T(\mathbb{C})$ is the identity component of $H(\mathbb{C})^\theta$, and $T(\mathbb{C})\cap A(\mathbb{C})$, which is a 2-group, may be non-trivial.

Let B be the subgroup of H generated by $\{m_\alpha \,|\, \alpha \in \Delta^r(\tilde\lambda)\}$. It is an elementary 2-group. Writing $m_\alpha = \exp(\pi i\alpha^\vee) \in A(\mathbb{C})$ we see B is a subgroup of $A(\mathbb{C})^\theta$. It is straightforward to check that $B\cap H^0 = B\cap T(\mathbb{C}) = B\cap T(\mathbb{C})\cap A(\mathbb{C})$. By [V4, Lemma 3.14(a)] the map $m_\alpha \to \epsilon_\alpha$ extends to a character ϵ of B.

Let S be a subset of the complex roots such that the set of all complex roots is $\{\pm\beta, \pm\theta\beta \,|\, \beta \in S\}$. For $h \in A(\mathbb{C})^\theta$ let

$$\zeta_{cx}(h) = \prod_{\beta\in S} \beta(h). \tag{15.8}$$

Lemma 15.9

(1) The character ζ_{cx} is independent of the choice of S, and satisfies $\zeta_{cx}(wh) = \zeta_{cx}(h)$ for all $w \in W(G(\mathbb{C}), H(\mathbb{C}))^\theta$, $h \in A(\mathbb{C})^\theta$.

(2) $\zeta_{cx}(h) = \epsilon(h)$ for all $h \in B$

(3) The character $\rho_{cx}|_{\mathfrak{t}}$ (respectively $\rho_{cx}|_{\mathfrak{a}}$) exponentiates to a character ρ^T_{cx} of $T(\mathbb{C})$ (respectively ρ^A_{cx} of $A(\mathbb{C})$), and for all $h \in A(\mathbb{C})\cap T(\mathbb{C})$,

$$\zeta_{cx}(h) = \rho^T_{cx}(h)/\rho^A_{cx}(h). \tag{15.10}$$

(4) Suppose that for all complex roots $\beta > 0$ implies $\theta\beta < 0$. Then $\zeta_{cx}(\exp(X)) = e^{\rho_{cx}(X)}$ for $X \in \mathfrak{a}(\mathbb{C}), \exp(X) \in A(\mathbb{C})^\theta$.

Proof. The first assertion is immediate since w permutes the set of complex roots.

Let $d = \sum_X \langle\beta, \alpha^\vee\rangle$ where X is the the set of positive roots which become non-compact imaginary roots on the Cayley transform H_α (cf. Section 10). According to ([V1], Lemma 8.3.9) $\epsilon_\alpha = (-1)^d$. A root β of H is imaginary for H_α if $s_\alpha\theta\beta = \beta$, in which case $s_\alpha\beta$ is also imaginary for H_α. If $\langle\beta, \alpha^\vee\rangle \neq 0$ is even β and $s_\alpha\beta$ are both compact or both non-compact for H_α, otherwise one is compact and one is non-compact. Therefore the contribution of $\{\pm\beta, \pm\theta\beta\}$ to d is $\langle\beta, \alpha^\vee\rangle \pmod 2$, and we can replaced X with

$$\{\beta \in S \,|\, s_\alpha(\beta) = \theta\beta, \langle\beta, \alpha^\vee\rangle \text{ odd}\}. \tag{15.11}$$

On the other hand $\zeta_{cx}(m_\alpha) = (-1)^e$ where $e = \sum_S \langle\beta, \alpha^\vee\rangle$. If $\langle\beta, \alpha^\vee\rangle = 0$ then β does not contribute to the sum; if $s_\alpha\beta \neq \pm\theta\beta$ the total contribution of

$\beta, s_\alpha\beta$ is even. It is easy to see the case $s_\alpha\beta = -\theta\beta$ does not arise. Therefore we can replace S with the set (15.11). This proves (2).

We may assume $S \subset \Delta^+$, and write write $S = S_+ \cup S_-$ where $S_\pm = \{\beta \in S \mid \pm\theta\beta > 0\}$. If $\beta \in S_+$ then $\frac{1}{2}(\beta+\theta\beta)|_{\mathfrak{t}} = \beta$ and $\frac{1}{2}(\beta+\theta\beta)|_{\mathfrak{a}} = 0$. If $\beta \in S_-$ then $\frac{1}{2}(\beta-\theta\beta)|_{\mathfrak{t}} = 0$ and $\frac{1}{2}(\beta-\theta\beta)|_{\mathfrak{a}} = \beta$. The first assertion of (3) is now clear, and for $h \in A(\mathbb{C}) \cap T(\mathbb{C})$,

$$\rho^T_{cx}(h) = \prod_{S_+} \beta(h), \quad \rho^A_{cx}(h) = \prod_{S_-} \beta(h)$$

This proves (3), and (4) follows easily. □

Next we have a small technical matter to take care of. Recall $\widetilde{\Gamma}$ is a character of $Z(\widetilde{H})$. A multiple of it extends uniquely to a finite dimensional representation of $\widetilde{H}$, which we denote $\widetilde{\mathbf{\Gamma}}$. See Section 7.

Choose a set Σ of simple roots of Δ^r. Suppose $\widetilde{h} \in Z(\widetilde{H})$ and $h = p(\widetilde{h}) \in B$. Write $h = m_{\alpha_1} \dots m_{\alpha_n}$ with $\alpha_1, \dots, \alpha_n \in \Sigma$, and choose inverse images $\widetilde{m}_{\alpha_i}$ of the m_{α_i} in $\widetilde{G}$.

Lemma 15.12 $\widetilde{\Gamma}(\widetilde{h})^2 = \prod_i \widetilde{\mathbf{\Gamma}}(\widetilde{m}_{\alpha_i})^2$

(More precisely: the matrix on the right hand side is the scalar on the left hand side times the identity matrix.)

This isn't obvious since the map $\widetilde{h} \to \widetilde{\mathbf{\Gamma}}(\widetilde{h})^2$ is not a representation of $\widetilde{H}$. What is true is that this map restricts to a (multiple of a) character of $Z(\widetilde{H})$, which factors to $p(Z(\widetilde{H}))$. Even though $h \in Z(\widetilde{H})$ it is not usually the case that $\widetilde{m}_{\alpha_i} \in Z(\widetilde{H})$. (We thank Rebecca Herb for pointing out this issue.)

Proof. It is enough to show the $\widetilde{m}_{\alpha_i}$ commute, for then $\widetilde{\Gamma}$ extends to a character of the abelian group generated by $Z(\widetilde{H})$ and $\{\widetilde{m}_{\alpha_i}\}$, and using this character the formula is obviously true.

The fact that $\{m_{\alpha_i}, h\} = 1$ implies α_i is not orthogonal to an even number of other $\alpha_j \in \Sigma$. This holds for all i. Consider the subdiagram of the Dynkin diagram of Δ^r consisting of the $\alpha_i \in \Sigma$. We conclude every node is adjacent to an even number of other nodes. This implies the nodes are all orthogonal, i.e. the $\widetilde{m}_{\alpha_i}$ commute. □

The remainder of this section will be taken up with the proof of Lemma 15.5. We have to check Condition (7.3)(c), that there is a character Γ with differential (15.6)(a), and that conditions (15.6)(a) and (b) are consistent.

Proof of Lemma 15.5.

We first check Condition (7.3)(c):

$$\begin{aligned} d\Gamma &= 2d\widetilde{\Gamma} + \rho - \rho_i + 2\rho_{i,c} \\ &= 2[\tilde{\lambda} + \rho_i - 2\rho_{i,c}] + \rho - \rho_i + 2\rho_{i,c} \\ &= 2\tilde{\lambda} + \rho + \rho_i - 2\rho_{i,c} \\ &= \lambda + \rho_i - 2\rho_{i,c}. \end{aligned}$$

It is clear that $\widetilde{\lambda}$ and λ define the same set of positive roots, so (a) follows.

We now show that the right hand side of (15.6)(a) is the differential of a character of H^0. The character Γ^2 of $\widetilde{H}$ factors to H^0, and has differential $2d\Gamma$, so it is enough to show $\rho - \rho_i$ is the differential of a character of H^0, i.e. of T^0. In the notation of Lemma 15.9 $\rho - \rho_i = \rho_r + \rho_{cx}$. The first term is trivial on $\mathfrak{t}$ and the second lifts to T^0 by Lemma 15.9(3).

It remains to show that (15.6)(a) and (b) are consistent. That is, recall $\epsilon(m_\alpha) = \epsilon_\alpha(-1)^{\langle\rho,\alpha^\vee\rangle}$ is a character of B, $d\Gamma_0$ defined in (15.6)(a) exponentiates to a character of H^0, and and we need to see these agree on $B \cap H^0 = B \cap T(\mathbb{C}) \cap A(\mathbb{C})$.

As in Lemma 15.12 choose a set Σ of simple roots of Δ^r. Suppose $h \in B \cap H^0$ and write $h = \prod_{j=1}^n m_{\alpha_j}$ with $\alpha_1, \dots, \alpha_n \in \Sigma$. Since $h \in H^0$ we may write $h = \exp(X)$ for some $X \in \mathfrak{t}_0$. We need to show:

(15.13)(a) $$e^{d\Gamma(X)} = \epsilon(h)(-1)^{\langle\rho,\sum_j \alpha_j^\vee\rangle}$$

or, using (15.6)(a)

(15.13)(b) $$e^{2d\tilde{\Gamma}+\rho-\rho_i+2\rho_{i,c}(X)} = \epsilon(h)(-1)^{\langle\rho,\sum_j \alpha_j^\vee\rangle}$$

Consider the left hand side. Since $2\rho_{i,c}$ is a sum of roots it factors to $H(\mathbb{C})$, and the last term is $(2\rho_{i,c})(h)$. Since $h \in A(\mathbb{C})$ this is trivial.

Secondly $e^{2d\widetilde{\Gamma}}(X) = \widetilde{\Gamma}(h)^2$. By Lemma 15.12 this equals $\prod_{j=1}^n \widetilde{\Gamma}(m_{\alpha_j})^2$. Since each α is metaplectic each term is -1, and we conclude $e^{2d\widetilde{\Gamma}}(X) = (-1)^n$.

Write $\rho - \rho_i = \rho_r + \rho_{cx}$, and note that $\rho_r(X) = 0$. By 15.9(3) ρ_{cx} exponentiates to the character ρ_{cx}^T of $T(\mathbb{C})$. The left hand side is therefore

(15.13)(c) $$(-1)^n \rho_{cx}^T(h).$$

For the right hand side write $\rho=\rho_r+\rho_i+\rho_{cx}$, and the right hand equals

$$\epsilon(h)(-1)^{\langle\rho_r,\sum\alpha_j^\vee\rangle}(-1)^{\langle\rho_i,\sum\alpha_j^\vee\rangle}(-1)^{\langle\rho_{cx},\sum\alpha_j^\vee\rangle}. \tag{15.13)(d}$$

For all j we have $\langle\rho_i,\alpha^\vee+j\rangle=0$ and $\langle\rho_r,\alpha_j^\vee\rangle=1$ since α_j is simple. Also $(-1)^{\langle\rho_{cx},\sum\alpha_j^\vee\rangle}=e^{\langle\rho_{cx},\pi i\sum\alpha_j^\vee\rangle}=\rho_{cx}^A(h)$. This gives

$$\epsilon(h)(-1)^n\rho_{cx}^A(h). \tag{15.13)(e}$$

Comparing (15.13)(c) we have to show

$$(-1)^n\rho_{cx}^T(h)=\epsilon(h)(-1)^n\rho_{cx}^A(h) \tag{15.13)(f}$$

which is Lemma (15.9)(3). □

References

[A1] Jeffrey Adams, Nonlinear covers of real groups *Int. Math. Res. Not.*, 75:4031-4047 (2004).

[A2] Jeffrey Adams, Lifting of characters on orthogonal and metaplectic groups, *Duke Math. J.*, **92** (1998), no. 1, 129–178.

[ABV] Jeffrey Adams, Dan Barbasch, and David A. Vogan, Jr. *The Langlands classification and irreducible characters for real reductive groups*, Progress in Mathematics, **104** (1992), Birkhäuser (Boston).

[ABPTV] Jeffrey Adams, Dan M. Barbasch, Annegret Paul, Peter E. Trapa, and David A. Vogan, Jr., Shimura correspondences for real groups, to appear in *JAMS*.

[AH] Jeffrey Adams and Rebecca Herb, Lifting of characters for nonlinear real groups, preprint.

[Bou] *Éléments de mathématique. Groupes et algébres de Lie. Chapitre VI: systémes de racines.* Hermann (Paris), 1968.

[BD] Jean-Luc Brylinski and Pierre Deligne, Central extensions of reductive groups by $\mathbf{K}_2$, *Publ. Math. Inst. Hautes Études Sci.*, **94** (2001), 5–85.

[HT] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[He] Guy Henniart, Une preuve simple des conjectures de Langlands pour $\mathrm{GL}(n)$ sur un corps p-adique, *Invent. Math.*, **139** (2000), no. 2, 439–455.

[Hum] *Introduction to Lie algebras and representations theory*, Graduate Texs in Mathematics**9**, Springer (Berlin), 1972.

[KP] David A. Kazhdan and S. J. Patterson, Metaplectic forms, *Publ. Math. Inst. Hautes Études Sci.*, **59** (1984), 5–85.

[KV] Anthony W. Knapp and David A. Vogan, Jr. *Cohomological induction and unitary representations*, volume 45 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1995.

[L] R. Langlands. *On the Classification of Irreducible Representations of Real Algebraic Groups*. Number 31 in Mathematical Surveys and Monographs. American Mathematical Society, Providence, 1989.

[LRS] G. Laumon, M. Rapoport, and U. Stuhler. $\mathcal{D}$-elliptic sheaves and the Langlands correspondence. *Invent. Math.*, 113(2):217–338, 1993.

[M] Ivan Mirkovic, Classification of irreducible tempered representations, Ph.D. thesis, University of Utah, 1986.

[PR] Vladimir Platonov and Andrei Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, **139** (1994), Academic Press (Boston).

[R1] David A. Renard, Endoscopy for $\mathrm{Mp}(2n, \mathbf{R})$, *Amer. J. Math.*, **121** (1999), no. 6, 1215–1243.

[RT1] David A. Renard and Peter E. Trapa, Irreducible genuine characters of the metaplectic group: Kazhdan-Lusztig algorithm and Vogan duality, *Representation Theory*, **4** (2000), 245–295.

[RT2] David A. Renard and Peter E. Trapa, Irreducible characters of the metaplectic group II. Functoriality. *J. Reine Angew. Math.*, **557** (2003), 121–158.

[RT3] David A. Renard and Peter E. Trapa, Kazhdan-Lusztig algorithms for nonlinear Lie groups and applications to Kazhdan-Patterson lifting, *American Journal of Mathematics*, to appear.

[T] J. Tate. Number theoretic background. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 3–26. Amer. Math. Soc., Providence, R.I., 1979.

[V1] David A. Vogan, Jr., *Representations of Real Reductive Lie Groups*, Progress in Mathematics, **15** (1981), Birkhäuser (Boston).

[V2] David A. Vogan, Jr, Irreducible characters of semisimple Lie groups. I. *Duke Math. J.*, **46** (1979), no. 1, 61–108.

[V3] David A. Vogan, Jr., Irreducible characters of semisimple Lie groups. III. Proof of Kazhdan-Lusztig conjecture, *Invent. Math.*, **71** (1983), no. 2, 381–417.

[V4] David A. Vogan, Jr., Irreducible characters of semisimple Lie groups. IV. Character-multiplicity duality, *Duke Math. J.*, **49** (1982), no. 4, 943–1073.

[V5] David A. Vogan, Jr., Unitarizability of certain series of representations, *Ann. of Math. (2)*, **120** (1984), no. 1, 141–187.

[V6] David A. Vogan, Jr., The local Langlands conjecture, in *Representation theory of groups and algebras*, *Contemp. Math.*, **145** (1993), 305–379, AMS (Providence, RI).