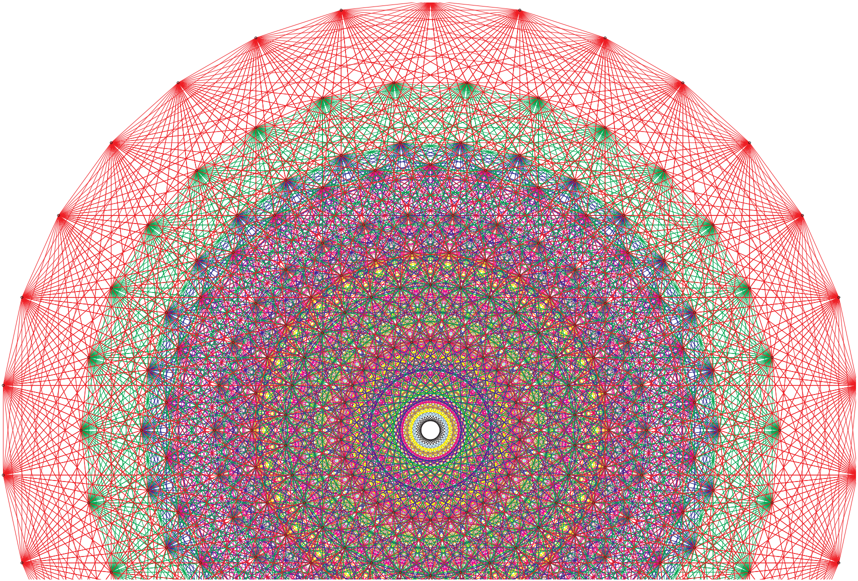
Atlas of Lie Groups and Representations

www.liegroups.org



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Computing The Unitary Dual

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Atlas of Lie Groups and Representations:

Take this idea seriously

Goals of the Atlas Project

- Tools for education: teaching Lie groups to graduate students and researchers

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Fokko du Cloux

Abstract Mathematics	$\rightarrow$	Algorithm	$\rightarrow$	Software
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Representation Theory				

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Mathematics $\longleftarrow$ Computer output

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Outline of the lectures:

Real Reductive Groups
Admissible Representations of Real Reductive Groups
The admissible dual $\hat{G}$
Three pictures of $\hat{G}$
Some geometry: $K(\mathbb{C})$ orbits on $G(\mathbb{C})/B(\mathbb{C})$
An algorithm for computing $\hat{G}$
Structure theory: Cartan subgroups, Weyl groups
Blocks and Vogan Duality
Kazhdan-Lusztig-Vogan polynomials
Character table of G
The E_8 calculation
The future: the unitary dual?

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Lecture IV: Character tables, the E_8 calculation, and the future

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Character of π: $\theta_\pi(g) = Trace(\pi(g))$

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Character table of A_5

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & -1 & 0 & \tau & \overline{\tau} \\ 3 & -1 & 0 & \overline{\tau} & \tau \\ 4 & 0 & 1 & -1 & -1 \\ 5 & 1 & -1 & 0 & 0 \end{bmatrix}$$

τ = Golden Ratio $\frac{1+\sqrt{5}}{2}$

$$\overline{\tau} = \frac{1-\sqrt{5}}{2}$$

Problem: Given a row in the character table of G, construct the corresponding representation.

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For example, if G has generators $g_1, \ldots, g_n$ and relations R, give matrices $A_1, \ldots, A_n$, satisfying relations R (and giving the row).

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Example: $G = W(E_8)$
$|G| = 696,729,600$
Number of representations: 112
Largest dimension: 7,168

Character table of $W(E_8)$

Class		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Size		1	1	120	120	3150	3780	3780	37800	37800	113400	2240	4480	89600	268800	15120
Order		1	2	2	2	2	2	2	2	2	2	3	3	3	3	4
X.1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	+	1	1	-1	-1	1	1	1	-1	-1	1	1	1	1	1	1
X.3	+	8	-8	-6	6	0	4	-4	2	-2	0	5	-4	-1	2	0
X.4	+	8	-8	6	-6	0	4	-4	-2	2	0	5	-4	-1	2	0
X.5	+	28	28	14	14	-4	4	4	-2	-2	-4	10	10	1	1	4
X.6	+	28	28	-14	-14	-4	4	4	2	2	-4	10	10	1	1	4
X.7	+	35	35	21	21	3	11	11	5	5	3	14	5	-1	2	-5
X.8	+	35	35	-21	-21	3	11	11	-5	-5	3	14	5	-1	2	-5
X.9	+	50	50	20	20	18	10	10	4	4	2	5	5	-4	5	10
...																
X.100	+	4200	4200	0	0	104	40	40	0	0	8	-120	15	-12	6	-40
X.101	+	4200	4200	420	420	-24	40	40	4	4	8	-30	-30	15	-3	40
X.102	+	4480	4480	0	0	-128	0	0	0	0	0	-80	-44	-20	4	64
X.103	+	4536	-4536	-378	378	0	60	-60	30	-30	0	-81	0	0	0	0
X.104	+	4536	-4536	378	-378	0	60	-60	-30	30	0	-81	0	0	0	0
X.105	+	4536	4536	0	0	-72	-72	-72	0	0	24	0	81	0	0	-24
X.106	+	5600	-5600	0	0	0	-80	80	0	0	0	-10	-100	2	-4	0
X.107	+	5600	-5600	-280	280	0	-80	80	8	-8	0	20	20	11	2	0
X.108	+	5600	-5600	280	-280	0	-80	80	-8	8	0	20	20	11	2	0
X.109	+	5670	5670	0	0	-90	-90	-90	0	0	6	0	-81	0	0	6
X.110	+	6075	6075	405	405	27	-45	-45	-27	-27	-21	0	0	0	0	-45
X.111	+	6075	6075	-405	-405	27	-45	-45	27	27	-21	0	0	0	0	-45
X.112	+	7168	-7168	0	0	0	0	0	0	0	0	-128	16	-32	-8	0

Example: one matrix from a 27-dimensional representation of $W(E_7)$

```
1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,-1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
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The irreducible representations are parametrized by a lattice in $\mathbb{R}^n$ intersected with a cone
A finite dimensional representation is determined by its character
The Weyl character formula computes $\theta_\pi(g)$

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Conclusion: Everything about representations of a compact group is "known".

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$$\langle \alpha, \alpha^\vee \rangle = 2,\ s_\alpha(\Delta) = \Delta,\ s_{\alpha^\vee}(\Delta^\vee) = \Delta^\vee.$$

$\langle\ ,\ \rangle : X \times X^\vee \to \mathbb{Z}$ is a perfect pairing

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(These are all of them)

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In practice to define $G(\mathbb{C})$: give $\mathfrak{g}$, A

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Simple complex Lie algebra $\overset{1-1}{\longleftrightarrow}$
simple, complex, simply connected groups $\overset{1-1}{\longleftrightarrow}$
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$G_{sc}(\mathbb{C})$: $(\mathbb{C}^\times)^m \times G_1(\mathbb{C}) \times \cdots \times G_n(\mathbb{C})$
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Atlas of Lie Groups
and Representations

www.liegroups.org

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Now: real reductive groups and their representations

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The Weyl group $\mathrm{Norm}_{G(\mathbb{C})}(H(\mathbb{C}))/H(\mathbb{C})$ acts on $H(\mathbb{C})$ (finite reflection group)

It isn't hard to find all involutions of $G(\mathbb{C})$

Example: Assume $G(\mathbb{C})$ is semisimple and the Dynkin diagram has no automorphisms (type $B_n, C_n, G_2, F_4, E_7, E_8$)
Every involution of $G(\mathbb{C})$ is inner

$H(\mathbb{C})$ = Cartan subgroup (maximal, semisimple, abelian; unique up to conjugacy)
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$H(\mathbb{C})_2 = \{h \mid h^2 \in Z\} \simeq (\mathbb{Z}/2\mathbb{Z})^n$

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One other orbit of size 135
Three real forms of E_8: compact, split, and “quaternionic”

EXAMPLES OF INVOLUTIONS

$G(\mathbb{C})$	compact	$\theta = 1$	$G(\mathbb{C})$	$G(\mathbb{R})$
$G(\mathbb{C})$	$G(\mathbb{R})$	θ	$K(\mathbb{C})$	$K(\mathbb{R})$
$GL(n,\mathbb{C})$	$GL(n,\mathbb{R})$	$\theta(g) = {}^t g^{-1}$	$O(n,\mathbb{C})$	$O(n,\mathbb{R})$
$GL(n,\mathbb{C})$	$U(p,q)$	$\theta(g) = JgJ^{-1}$	$GL(p,\mathbb{C})\times GL(q,\mathbb{C})$	$U(p)\times U(q)$
E_8	$E_8(split)$	$*$	$Spin(16,\mathbb{C})/\mathbb{Z}_2$	$Spin(16)/\mathbb{Z}_2$

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$G = G(\mathbb{R})$
V=complex Hilbert space, Hermitian form $\langle\, ,\, \rangle$
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irreducible (no closed invariant subspace)

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Note: G simple non-compact, π unitary $\Rightarrow$ dimension(π)=∞

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c) (another compatability condition, not needed if G is connected)

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Note: $K = K(\mathbb{R})$ (compact) or $K = K(\mathbb{C})$ (complex) are interchangeable)

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Lemma(π, V) (admissible) of G is unitary if and only if (π, V_K) is unitary.

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G= real Lie group

Source of representations:

$$L^2(G) = \text{“}\bigoplus\text{''} m(\pi)\pi$$
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Support of $d\pi$: tempered representations: $\widehat{G}_t$

Discrete part: Discrete Series: $\widehat{G}_d$

$\pi \in \widehat{G}_d \Leftrightarrow \pi \hookrightarrow L^2(G)$ (actual summand)

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Uncountably many π to test (unless G is compact)

Various duals of $SL(2,\mathbb{R})$

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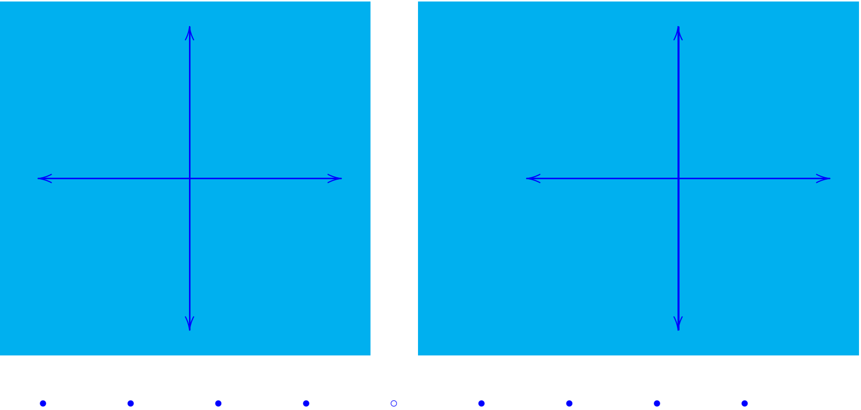

Admissible dual

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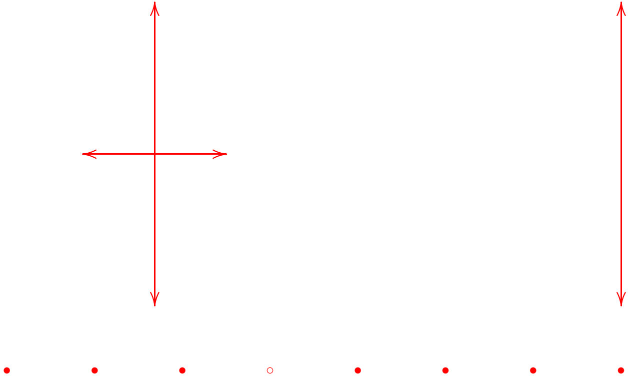

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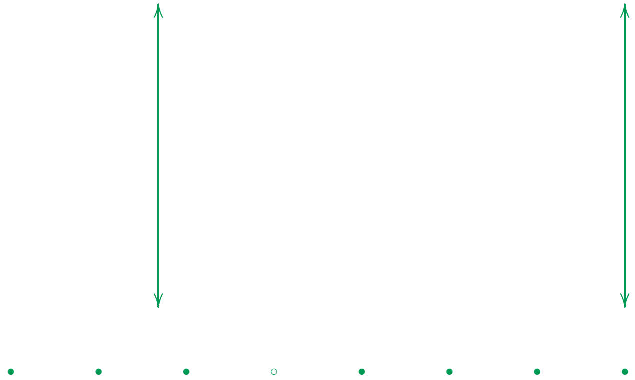

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Example: How many irreducible representations does the split real form of E_8 have at infinitesimal character ρ?

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Answer: 526,471

Admissible Dual

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Next two lectures: Implement $\widehat{G}_a$ on a computer

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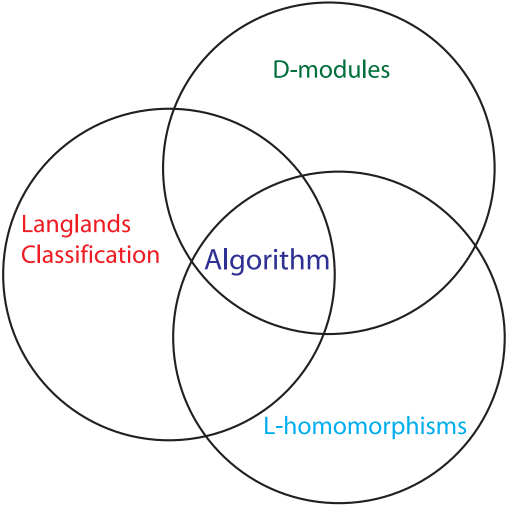
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L-homomorphism: local systems on the space of admissible homomorphism of the Weil group into the dual group

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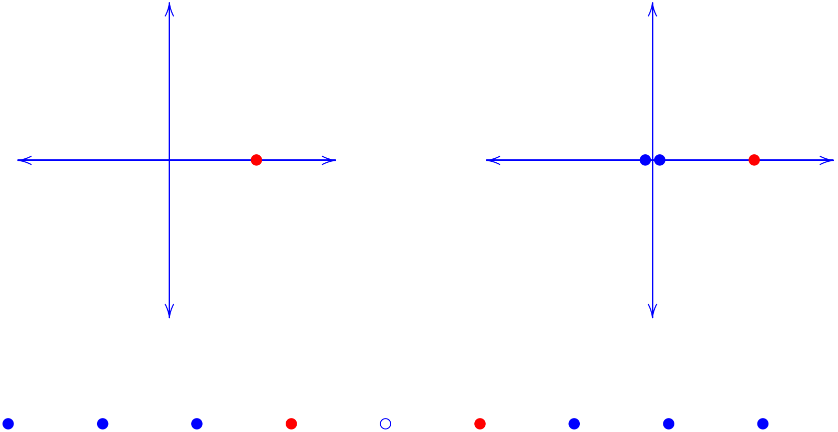
Say π has infinitesimal character $\lambda \in \mathfrak{h}^*$, $\rightarrow \widehat{G}_a(\lambda)$

Example: $G = SL(2,\mathbb{R})$, infinitesimal character $=\rho$



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$S = \{\rho = \alpha/2, \alpha\}$
For these talks: assume $G(\mathbb{C})$ is semisimple and simply connected, $S = \{\rho\}$

So the problem is:

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Example: If G is compact $\widehat{G}_a(\rho) = \{\mathbb{C}\}$.

Atlas Project Members, AIM, July 2007

Known Unitary Duals

red: known   black: not known

Type A: $SL(n,\mathbb{R})$, $SL(n,\mathbb{H})$, $SU(n,1)$, $SU(n,2)$, $SL(n,\mathbb{C})$
$SU(p,q)$ $(p,q \geq 2)$

Type B: $SO(2n,1)$, $SO(2n+1,2)$, $SO(2n+1,\mathbb{C})$

$SO(p,q)$ $(p,q \geq 3)$

Type C: $Sp(4,\mathbb{R})$, $Sp(n,1)$, $Sp(2n,\mathbb{C})$

$Sp(p,q)$ $(p,q \geq 2)$

Type D: $SO(2n+1,1)$, $SO(2n,2)$, $SO(2n,\mathbb{C})$

$SO(p,q)$ $(p,q \geq 3)$, $SO^*(2n)$ $(n \geq 4)$

Type E_6: $E_6(F_4) = SL(3, \text{Cayley})$

$E_6(\text{Hermitian})$, $E_6(\text{split})$, $E_6(\text{quaternionic})$, $E_6(\mathbb{C})$

Type F_4: $F_4(B_4)$

$F_4(\text{split})$, $F_4(\mathbb{C})$

Type G: $G_2(\text{split})$, $G_2(\mathbb{C})$

E_7/E_8: nothing known

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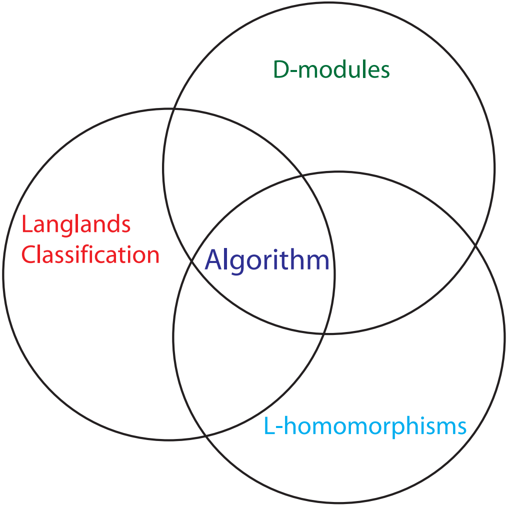
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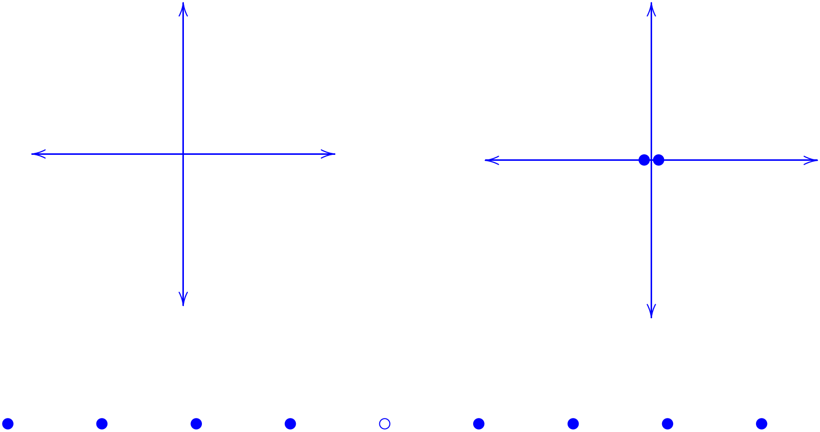

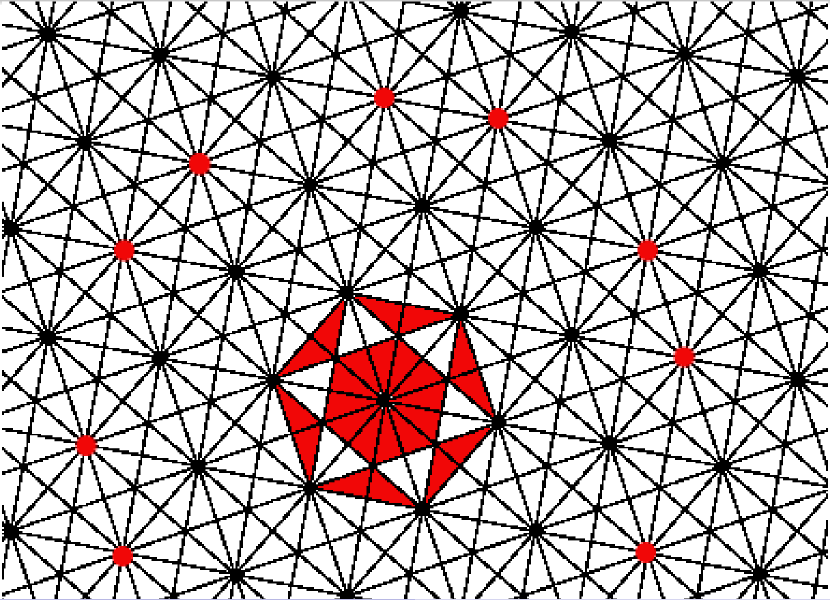

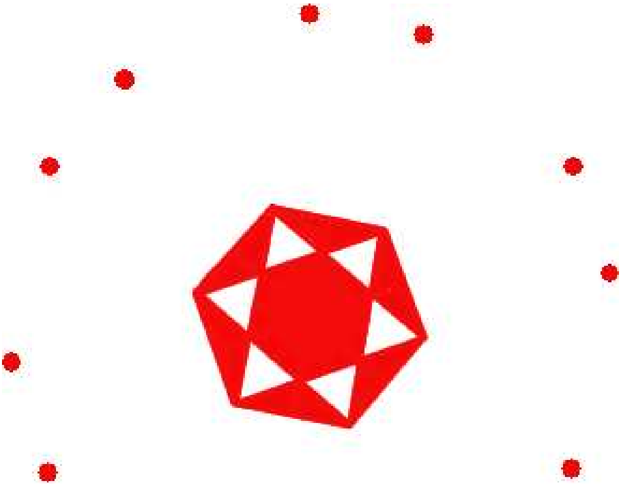

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(like the $R_T(\theta)$'s in Deligne-Lusztig's theory for finite groups)

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$\pi(H(\mathbb{R}), \chi)$ = unique irreducible quotient of $I(H(\mathbb{R}), \chi)$ (choose $N(\mathbb{R})$ properly)

The Langlands Classification

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Theorem: The map $(H(\mathbb{R}),\chi)\rightarrow\pi(H(\mathbb{R}),\chi)$ induces a canonical bijection:

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In particular:

$$|\widehat{G}_a(\rho)| = \sum_i |W/W(G(\mathbb{R}), H(\mathbb{R})_i)||H(\mathbb{R})/H(\mathbb{R})_i|$$

where $H(\mathbb{R})_1, \ldots, H(\mathbb{R})_n$ are representatives of the conjugacy classes of Cartan subgroups.

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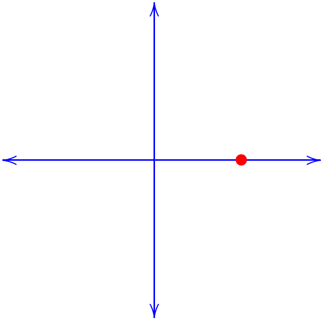

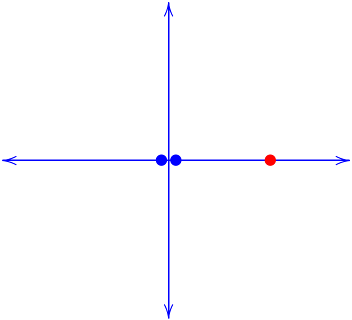

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$$\mathcal{D}(G, K, \rho) = \{(x, \chi)\}/K$$

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= character of $\mathrm{Stab}(x)/\mathrm{Stab}(x)^0$

Theorem: There is a natural bijection

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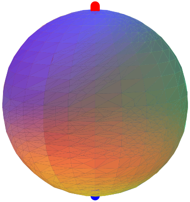

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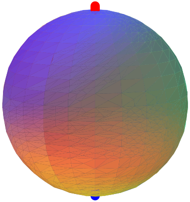

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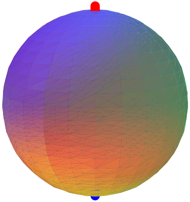

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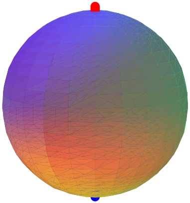

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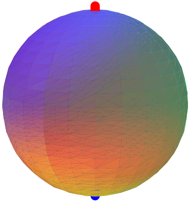

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Isotropy group: $1, 1, \mathbb{Z}/2\mathbb{Z} \to 4$ representations

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Examples:

$G(\mathbb{C})$	type	$G^{\vee}(\mathbb{C})$	type
$GL(n,\mathbb{C})$	A_{n-1}	$GL(n,\mathbb{C})$	A_{n-1}
$SL(n,\mathbb{C})$	A_{n-1}	$PSL(n,\mathbb{C})$	A_{n-1}
$Sp(2n,\mathbb{C})$	C_n	$SO(2n+1,\mathbb{C})$	B_n
$SO(2n,\mathbb{C})$	D_n	$SO(2n,\mathbb{C})$	D_n
$Spin(2n,\mathbb{C})$	D_n	$PSO(2n,\mathbb{C})$	D_n

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where $G_1(\mathbb{R}),\dots,G_n(\mathbb{R})$ are the real forms of G.

Recap

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(1) Character Data:

$$\Pi(G(\mathbb{R}), \rho) \overset{1-1}{\longleftrightarrow} \mathcal{C}(G(\mathbb{R})) = \{(H, \chi)\}/G(\mathbb{R})$$

(2) $\mathcal{D}$-modules (orbits of K on G/B):

$$\Pi(G(\mathbb{R}), \rho) \overset{1-1}{\longleftrightarrow} \mathcal{D}(G, K, \rho) = \{(\mathcal{O}, \tau)\}/K$$

(3) L-homomorphisms (orbits of $G^\vee$ on $\mathcal{H}(G, \rho)$

$$\coprod_{i=1}^{n} \Pi(G_i(\mathbb{R}), \rho) \overset{1-1}{\longleftrightarrow} \mathcal{L}(G^\vee) = \{(\phi, \chi)\}/G^\vee$$

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With appropriate modifications they hold in general (perhaps later).

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$$\Pi_R(G(\mathbb{R}), \mathcal{O})$$

Theorem (Vogan): The intersection of an L-packet and an R-packet is at most one element.

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Corollary $\Pi(G(\mathbb{R}), \rho)$ is parametrized by a subset of pairs

$$(K \text{ orbit on } \mathcal{B}, G^\vee \text{ orbit on } \mathcal{H})$$

via

$$(\mathcal{O}, \Omega^\vee) \rightarrow \Pi_R(G(\mathbb{R}), \mathcal{O}) \cap \Pi_L(G(\mathbb{R}), \Omega^\vee)$$

Which pairs?...

K-orbits on the dual side

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SKETCH OF PROOF

$\mathcal{P} = \{(x, B)\}/G$ ($x^2 = 1$, $B =$ Borel)

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(Note for the experts: Canonical up to characters of $G_{qs}(\mathbb{R})/G_{qs}(\mathbb{R})^0$)

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(Λ is a certain set of infinitesimal characters, S is the set of "strong real forms")

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Vogan Duality

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with lots of wonderful properties. . .

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$SL(2)/PGL(2)$ VIA ATLAS OUTPUT

```
main: type
Lie type: A1 sc s
main: block
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
possible (weak) dual real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
entering block construction ...
2
done
Name an output file (return for stdout, ? to abandon):
0(0,1):  1    (2,*)    [i1]  0
```

EXAMPLE: $Sp(4,\mathbb{R})$

```
main: type
Lie type: C2 sc s
main: block
(weak) real forms are:
0: sp(2)
1: sp(1,1)
2: sp(4,R)
enter your choice: 2
possible (weak) dual real forms are:
0: so(5)
1: so(4,1)
2: so(2,3)
enter your choice: 2
entering block construction ...
10
done
Name an output file (return for stdout, ? to abandon):
 0( 0,6):   1   2    ( 6, *)  ( 4, *)    [i1,i1]  0
 1( 1,6):   0   3    ( 6, *)  ( 5, *)    [i1,i1]  0
 2( 2,6):   2   0    ( *, *)  ( 4, *)    [ic,i1]  0
 3( 3,6):   3   1    ( *, *)  ( 5, *)    [ic,i1]  0
 4( 4,4):   8   4    ( *, *)  ( *, *)    [C+,r1]  1  2
 5( 5,4):   9   5    ( *, *)  ( *, *)    [C+,r1]  1  2
 6( 6,5):   6   7    ( *, *)  ( *, *)    [r1,C+]  1  1
 7( 7,2):   7   6    (10,11)  ( *, *)    [i2,C-]  2  2,1,2
 8( 8,3):   4   9    ( *, *)  (10, *)    [C-,i1]  2  1,2,1
 9( 9,3):   5   8    ( *, *)  (10, *)    [C-,i1]  2  1,2,1
10(10,0):  11  10    ( *, *)  ( *, *)    [r2,r1]  3  1,2,1,2
```

EXAMPLE: E_8

```
real: type
Lie type: E8 sc s
main: blocksizes
              compact quaternionic split
compact       0       0            1
quaternionic  0       3,150        73,410
split         1       73,410       453,060
```

Recap

$G = G(\mathbb{C}), K = G^{\theta},$

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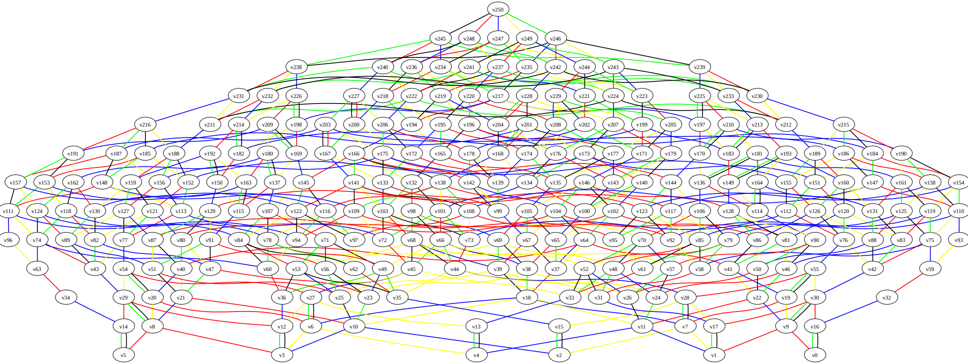
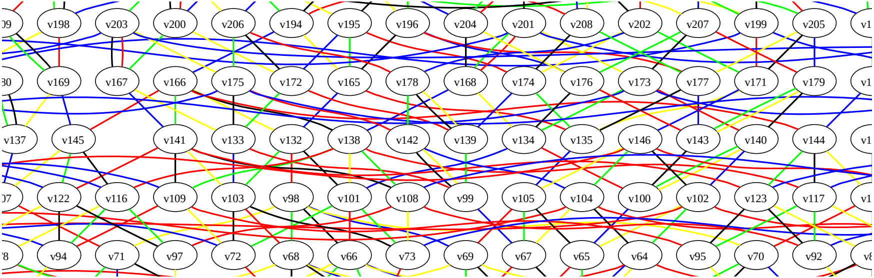
$\mathcal{X} = \{x \in H \mid x^2 = 1\}/H$

Theorem:

$$\mathcal{X} \overset{1-1}{\longleftrightarrow} \coprod_{i=1}^{n} K_i \backslash G/B \overset{1-1}{\longleftrightarrow}$$



$K\backslash G/B$ for $SO(5,5)$



Closeup of $SO(5,5)$ graph

$G^\vee, K_j^\vee, \mathcal{X}^\vee, \ldots$

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$\gamma \rightarrow \pi(\gamma)$ = irreducible module
(quotient of $I(\gamma)$)

EXAMPLE: SL(2)/PGL(2)

$\mathcal{O}$	x	x^2	K	$G_x(\mathbb{R})$	λ	rep	$\mathcal{O}^\vee$	y	y^2	$K^\vee$	$G_y^\vee(\mathbb{R})$	λ	rep
$\cdot$	I	I	G	$SU(2,0)$	ρ	$\mathbb{C}$	$\mathbb{C}^\times$	w	I	$O(2,\mathbb{C})$	$SO(2,1)$	2ρ	PS_+
$\cdot$	-I	I	G	$SU(0,2)$	ρ	$\mathbb{C}$	$\mathbb{C}^\times$	w	I	$O(2,\mathbb{C})$	$SO(2,1)$	2ρ	PS_-
$\{0\}$	t	-I	$\mathbb{C}^\times$	$SU(1,1)$	ρ	DS_+	$\mathbb{C}^\times$	w	I	$O(2,\mathbb{C})$	$SO(2,1)$	ρ	$\mathbb{C}$
$\{\infty\}$	-t	-I	$\mathbb{C}^\times$	$SU(1,1)$	ρ	DS_-	$\mathbb{C}^\times$	w	I	$O(2,\mathbb{C})$	$SO(2,1)$	ρ	sgn
$\mathbb{C}^\times$	w	-I	$\mathbb{C}^\times$	$SU(1,1)$	ρ	$\mathbb{C}$	$\{\infty\}$	t	I	$O(2,\mathbb{C})$	$SO(2,1)$	ρ	DS
$\mathbb{C}^\times$	w	I	$O(2,\mathbb{C})$	$SU(1,1)$	ρ	PS	$\cdot$	I	I	$G^\vee$	$SO(3)$	ρ	$\mathbb{C}$

Character of a representation

Recall if V is finite dimensional $\Theta_\pi(g) = \text{Trace}(\pi(g))$.
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$$\pi(f)v = \int_{G(\mathbb{R})} \pi(g)f(g)v\,dg$$

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Theorem (Harish-Chandra) The distribution θ_π is represented by a conjugation invariant function θ_π (locally integrable, analytic on an open dense subset):

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Generalization of Weyl character formula:

$$\Theta_\pi(g) = \frac{\sum_w (w\chi)(g)}{\Delta(g)}$$

(sum is over $W(G(\mathbb{R}), T(\mathbb{R}))$, Δ = Weyl denominator)

$I = I(\gamma)$ standard module
Every representation can be written as a direct sum of irreducible representations:

$$I(\gamma) = \sum_{\delta \in \mathcal{Z}} m(\delta, \gamma)\pi(\delta)$$

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(really means:)

$$\theta_{I(\gamma)} = \sum_{\delta \in \mathcal{Z}} m(\delta, \gamma)\theta_{\pi}(\delta)$$

Inverting the character formulas

Theorem (…Zuckerman)

There are integers $M(\delta,\gamma)$ so that

$$\pi(\gamma) = \sum_{\delta} M(\delta,\gamma) I(\gamma)$$

(really:)

$$\theta_{\pi(\gamma)} = \sum_{\delta} M(\delta,\gamma)\theta_{I(\gamma)}$$

The $M(\delta,\gamma)$ the Character Table of $G(\mathbb{R})$

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Theorem: $M(\delta, \gamma) = \pm m_g(\gamma, \delta)$

Kazhdan-Lusztig-Vogan polynomials

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$$\delta, \gamma \in \mathcal{Z} \to P_{\delta,\gamma} = \sum a_i q^i$$

$\delta, \gamma \in \mathcal{Z} \rightarrow P_{\delta,\gamma} = \sum a_i q^i$

Theorem (Vogan):

$$\boxed{M(\delta, \gamma) = \pm P_{\delta,\gamma}(1)}$$

Recursion Relations

Change notation: $x, y, x', \cdots \in \mathcal{Z}$
Partial order $<$ on $\mathcal{Z}$
Length function $\ell(x)$

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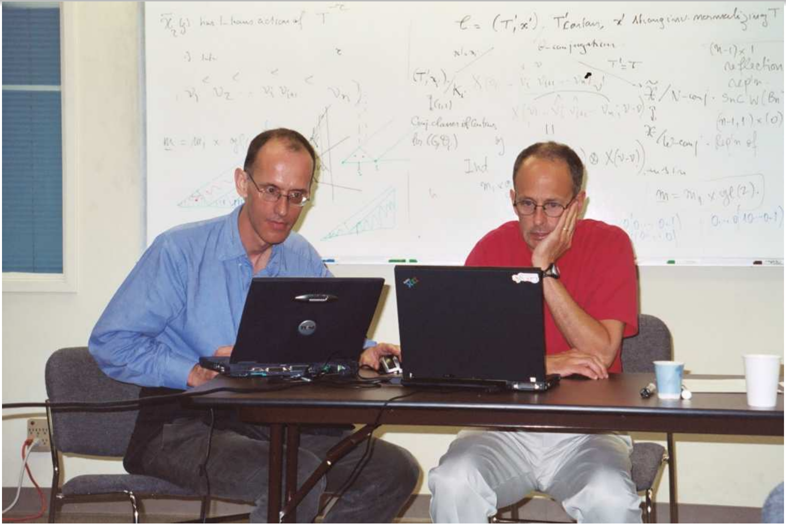
Recursion relations: compute $P_{x,y}$ by upward induction on $\ell(y)$ and downward induction on $\ell(y)$.
Long list of complicated recursion formulas.

The E_8 Calculation

Fokko's software computed KLV polynomials for all exceptional groups except the split real form of E_8.
E_7 takes about 30 seconds.
In order to test the mathematics, the software, and get an idea of our computing needs, we set as our goal:

Compute the KLV polynomials for $E_8(split)$



Fokko du Cloux

```
empty: type
Lie type: E8 sc s
main: blocksizes
              compact quaternionic split
compact       0       0            1
quaternionic  0       3,150        73,410
split         1       73,410       453,060
real: kgb
kgbsize: 320206
```

(I've added the labelling of rows and columns)

There are 320, 206 orbits of K on G/B
The computation goes on in the "block" with 453, 060 parameters.
The KLV matrix has size $453,060 \times 453,060$

Maximal degree: 31

Recursion Relations

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Recursion relations: compute $P_{x,y}$ like this:

$$\begin{array}{lllll}
P_{0,0} & & & & \\
P_{0,1} & \leftarrow P_{1,1} & & & \\
P_{0,2} & \leftarrow P_{1,2} & \leftarrow P_{2,2} & & \\
P_{0,3} & \leftarrow P_{1,3} & \leftarrow P_{2,3} & \leftarrow P_{3,3} & \\
P_{0,4} & \leftarrow P_{1,4} & \leftarrow P_{2,4} & \leftarrow P_{3,4} & \leftarrow P_{4,4} \\
\dots & & & &
\end{array}$$

($P_{3,4}$ is shorthand for all of the $P_{x,y}$ with $\ell(x) = 3, \ell(y) = 5$)

Recursion Relations II

$$P_{x,y} = \sum_{\ell(x')=\ell(x)+1} M(x', y') + \sum_{x''} M(x'', y'')$$

$$(\ell(y') = \ell(y); \ell(y'') = \ell(y) - 1)$$

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P_{0,4} & P_{1,4} & P_{2,4} & P_{3,4} & P_{4,4} & \\
P_{0,5} & P_{1,5} & P_{2,5} & \boxed{P_{3,5}} & P_{4,5} & P_{5,5}
\end{array}$$

($\leq$ 4 terms)

RECURSION RELATIONS

$$P_{x,y} = M(x', y') + xM(x, y') - \sum_{x' \leq z < y'} \mu(z, y') x^{(l(y') - l(z) - 1)/2} M(x', z).$$

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\end{array}$$

Average number of terms for E_8: 150

Recursion Relations: Conclusion

$$P_{x,y} = \sum_{x',y'} c(x',y')M(x',y')$$

for some very complicated constants $c(x',y')$

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We need to keep all $P_{x,y}$ in RAM!
All accessible from a single processor!

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Big Problem: we did not have a good idea of the size of the answer beforehand.

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Each polynomial has ≤ 32 coefficients.

$450,060^2 \times 32 = 6.5$ trillion coefficients =26 trillion bytes

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Store only the distinct polynomials.

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$6 \times 10^9 \times 32 = 200$ billion coefficents, or 800 billion bytes
Plus about 100 billion bytes for the pointers = 900 billion bytes

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Hope: average degree = 20 $\rightarrow$ 35+8=43 billion bytes

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Marc reduced the size of the indices to about 15 billion bytes (by using a lot of information about the nature of the data)

Calculating Modulo n

Noam Elkies: have to think harder

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$2^{32} < 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 = 100$ billion
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This gets us down to about $15 + 4 = 19$ billion bytes

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In fact we can work $\pmod{n}$ for any n.

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Run the program 4 times modulo $n = 251, 253, 255$ and 256

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Dec. 26	255	complete	12 hours

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Dec. 27	253	crash	

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Dec. 27	253	crash	
Jan. 3	253	complete	12 hours

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Combine the answers using the Chinese Remainder Theorem.
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A calculation the size of Manhattan

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Number of coefficients in distinct polynomials: 13,721,641,221 (13.9 billion)

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Stay tuned…