Lifting of Characters for Nonlinear Simply Laced Groups

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1 Introduction

This is an alpha version of this paper. Sorry, the introduction gets written last.

2 Some Notation

Throughout this paper G will be the set of real points of a connected reductive complex Lie group $G(\mathbb{C})$. Let $H(\mathbb{C})$ be a Cartan subgroup of $G(\mathbb{C})$, and let Φ be the set of roots of $H(\mathbb{C})$ in $G(\mathbb{C})$. Let Φ^+ be a set of positive roots, and $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ as usual. We assume that the derived group $G_d(\mathbb{C})$ is acceptable, i.e. ρ exponentiates to a character of $H(\mathbb{C}) \cap G_d(\mathbb{C})$. Let $p : \widetilde{G} \to G$ be a two-fold cover of G. We identify the kernel of p with ± 1.

Let $G_{\mathrm{ad}}(\mathbb{C})$ be the adjoint group, with real points G_{ad}. Write $\mathrm{int}(g)$ for the action of $G_{\mathrm{ad}}(\mathbb{C})$ on $G(\mathbb{C})$, or G_{ad} on G.

The main results only apply if every factor of Φ has one root length or is of type G_2. Many of the auxiliary results of this paper hold in general so we will not assume this except when necessary. We plan to return to the general case in a future paper.

For any subgroup H of G we write $\widetilde{H} = p^{-1}H$. Writing $Z(H)$ for the center, let $Z_0(H) = p(Z(\widetilde{H})) \subset Z(H)$; then $Z(\widetilde{H}) = p^{-1}(Z_0(H))$. In particular $Z_0(G) \subset Z(G)$ plays an important role.

We denote real Lie algebras by Gothic letters $\mathfrak{h}, \mathfrak{g}, \mathfrak{t}, \ldots$, and their complexifications by $\mathfrak{h}(\mathbb{C}), \mathfrak{g}(\mathbb{C}), \mathfrak{t}(\mathbb{C}), \ldots$. We write σ for the action of the non-trivial element of the Galois group on $\mathfrak{g}(\mathbb{C})$ and $G(\mathbb{C})$, so $\mathfrak{g} = \mathfrak{g}(\mathbb{C})^\sigma$ and $G(\mathbb{R}) = G(\mathbb{C})^\sigma$.

Fix a Cartan involution θ of $G(\mathbb{C})$, so that $K = G^\theta$ is a maximal compact subgroup of G. Let H be a θ-stable Cartan subgroup of G. Then $\sigma(\alpha) = -\theta(\alpha)$ for all $\alpha \in \Phi$. Roots are classified as real, imaginary, complex, compact as in [10]. Write $\Phi = \Phi_r \cup \Phi_i \cup \Phi_{cx}$ accordingly. We also have the decomposition of $\Phi_i = \Phi_{i,c} \cup \Phi_{i,n}$ into compact and non-compact roots. If Φ^+ is a set of positive roots, write $\Phi^+_r = \Phi^+ \cap \Phi_r$, and $\Phi^+ i, \Phi^+_{cx}$ similarly. Write $\rho = \rho(\Phi^+) = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$ as usual. Define ρ_r, ρ_i and ρ_{cx} similarly, so $\rho = \rho_r + \rho_i + \rho_{cx}$.

We say Φ^+ is *real* if $\sigma\Phi^+_{cx} = \Phi^+_{cx}$, i.e. if α is a positive non-imaginary root then $\sigma(\alpha) > 0$. This is the set of roots of the nilpotent radical of a real parabolic of G.

Let P, R be the weight and root lattices, respectively, and denote the coweight and coroot lattices $P^\vee$ and $R^\vee$.

Write $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ as usual, and $H = TA$ with $T = H \cap K$ and $A = \exp(\mathfrak{a})$. Typically $\widetilde{H}$ is not abelian, and its center plays an important role.

A simple root system is said to be *simply laced* if all roots have the same length, and an arbitrary root system is simply laced if this holds for each simple factor. More succinctly a root system is simply laced if whenever α, β are non-proportional roots then $\langle \alpha, \beta^\vee \rangle = 0, \pm 1$. We adopt the convention that in this case all roots are long.

We say a root system is *oddly laced* if whenever α, β are non-proportional roots then $\langle \alpha, \beta^\vee \rangle = 0$ or is *odd*. Thus *oddly laced* as shorthand for *each simple factor is simply laced or of type* G_2. We also adopt the convention that in type G_2 all roots are long. The reason for these conventions is Lemma 3.1. The main results in this paper hold for oddly laced groups, although (with the general case in mind) we will only make this assumption when necessary.

3 Admissible Triples

Fix G as in Section 2 and a two-fold cover $p : \widetilde{G} \to G$. Let H be a Cartan subgroup of G. Fix a real or non-compact imaginary root α. Associated to α is the root subgroup M_α, which is locally isomorphic to $SL(2,\mathbb{R})$. As in [2, Definition 3.2] we say α is *metaplectic* if $p^{-1}(M_\alpha)$ is a non-linear group. It is easy to see $SL(2,\mathbb{R})/\pm 1$ has no such cover, so we conclude $M_\alpha \simeq SL(2,\mathbb{R})$

and $p^{-1}(M_\alpha) \simeq \widetilde{SL}(2,\mathbb{R})$, the unique non-trivial two-fold cover.

For the next Lemma see [3] or [2, Lemma 3.3].

Lemma 3.1 *Assume $G(\mathbb{C})$ is simple and simply connected and that $\widetilde{G}$ is non-linear. Fix a θ-stable Cartan subgroup H of G with roots Φ. Then every long real or non-compact imaginary root $\alpha \in \Phi$ is metaplectic. Furthermore G admits a non-linear cover if and only if there is a Cartan subgroup H with a long real or long non-compact imaginary root. If this condition holds the non-linear two-fold cover is unique up to isomorphism.*

It is enough to check this condition on the fundamental or the maximally split Cartan subgroup. If G is oddly laced it is enough to check this condition on any Cartan subgroup.

Definition 3.2 *We say that $p : \widetilde{G} \to G$ is an admissible two-fold cover if for every Cartan subgroup H, every long real or long noncompact imaginary root is metaplectic.*

Equivalently $\widetilde{G}$ is admissible if and only if $\widetilde{G_i}$ is non-linear for every simple factor G_i of G which admits such a cover.

See [2, Definition 3.4]).

As discussed in the Introduction we are going to lift characters from a real form of a quotient of $G(\mathbb{C})$. We need to impose some conditions on this quotient. This will take up the remainder of this section.

Suppose $C \subset Z(G(\mathbb{C}))$ is a finite subgroup stable under the Galois action. Then $\overline{G}(\mathbb{C}) = G(\mathbb{C})/C$ is defined over $\mathbb{R}$, and let $\overline{G}$ be its real points. Write $\overline{p} : G(\mathbb{C}) \to \overline{G}(\mathbb{C})$ for the projection map. Let $\mathcal{O}(G)$ be the conjugacy classes of G. If $g, g' \in G$ are conjugate by $G(\mathbb{C})$ we say g, g' are stably conjugate, and let $\mathcal{O}_{\mathrm{st}}(G)$ be the set of stable conjugacy classes. (This is a naive definition, which agrees with the usual one for strongly regular semisimple elements.) Similar notation applies to other groups.

For $g \in G$ let $\mathcal{O}(G,g)$ be the conjugacy class of g, and $\mathcal{O}_{\mathrm{st}}(G,g) = \{xgx^{-1} \mid x \in G(\mathbb{C}), xgx^{-1} \in G\}$ the stable conjugacy class. Similar notation applies to other groups.

Definition 3.3 *Assume C is a two-group. For $h \in \overline{G}$ let $\phi(h) = s(h)^2$ where $s : \overline{G}(\mathbb{C}) \to G(\mathbb{C})$ is any section.*

Lemma 3.4 *The map ϕ is well defined, and satisfies:*

1. $\phi(\overline{G}) \subset G$ *and* $\phi(\overline{G}^0) \subset G^0$,

2. ϕ *induces a map* $\mathcal{O}(\overline{G}^0) \to \mathcal{O}(G^0)$,

3. ϕ *induces a map* $\mathcal{O}^{st}(\overline{G}) \to \mathcal{O}^{st}(G)$.

Proof. Since C is a two-group it is immediate that $\phi(h)$ is independent of the choice of s.

Suppose $g \in \overline{G}, h \in G$ and $p(h) = g$. Then $\overline{p}(\sigma(h)) = \sigma(\overline{p}(h)) = \overline{p}(h)$, so $\sigma(h) = zh$ for some $z \in C$. Since C is a two-group $\sigma(h)^2 = h^2$, so $\phi(g) \in G$. Furthermore since $p : G^0 \to \overline{G}^0$ is surjective the second assertion in (1) is clear.

Define

$$\phi(\mathcal{O}(\overline{G}^0, g)) = \mathcal{O}(G^0, \phi(g)) \quad (g \in \overline{G}^0). \tag{3.5}$$

If $x \in \overline{G}^0$ choose $y \in G^0$ with $\overline{p}(y) = x$. Then $\phi(xgx^{-1}) = y\phi(g)y^{-1}$ so this is well defined, proving (2). Similarly define

$$\phi(\mathcal{O}_{\mathrm{st}}(\overline{G}, g)) = \mathcal{O}_{\mathrm{st}}(G, \phi(g)) \quad (g \in \overline{G}) \tag{3.6}$$

Suppose $x \in \overline{G}(\mathbb{C})$ and $xgx^{-1} \in \overline{G}$. Choose $y \in G(\mathbb{C})$ with $\overline{p}(y) = x$. Then $\phi(xgx^{-1}) = y\phi(g)y^{-1}$, so again ϕ is well-defined. □

It is worthwhile noting that ϕ does *not* define a map $\mathcal{O}(\overline{G}) \to \mathcal{O}(G)$. Suppose we try to define $\phi(\mathcal{O}(\overline{G}, g)) = \mathcal{O}(G, \phi(g))$ as in (3.5). If $x \in \overline{G}$ but $x \notin \overline{G}^0$ then there is no guarantee we can find $y \in G$ such that $\phi(y) = x$, so it may not hold that $\phi(g)$ is conjugate to $\phi(xgx^{-1})$. (By (3) $\phi(g)$ is *stably* conjugate to $\phi(xgx^{-1})$).

Example 3.7 Let $G = SL(2, \mathbb{R})$ and $\overline{G} = PGL(2, \mathbb{R})$. Let

$$t(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \in G.$$

Let $\overline{t}(\theta)$ be the image of $t(\theta)$ in $\overline{G}$. Then $\overline{t}(\theta)$ is conjugate to $\overline{t}(\theta)$, but $\phi(\overline{t}(\theta)) = t(2\theta)$ is not conjugate to $\phi(\overline{t}(-\theta)) = t(-2\theta)$ unless θ is an integral multiple of $\pi/2$.

We assume throughout this paper that $\mathcal{O}$ is a two-group.

Now fix a Cartan subgroup H of G, and let $\overline{H}(\mathbb{C}) = H(\mathbb{C})/C \subset \overline{G}(\mathbb{C})$, with real points $\overline{H}$. We define $\phi : \overline{H}(\mathbb{C}) \to H(\mathbb{C})$ by the same formula as in Definition 3.3. Equivalently, if we write $\exp : \mathfrak{h}(\mathbb{C}) \to H(\mathbb{C})$, $\overline{\exp} : \mathfrak{h}(\mathbb{C}) \to \overline{H}(\mathbb{C})$, then for $X \in \mathfrak{h}(\mathbb{C})$, $\phi(\overline{\exp}(X)) = \exp(2X)$. It is immediate that

$$\alpha(\phi(h)) = \alpha(h)^2 \quad (\text{for all } \alpha \in \Phi). \tag{3.8}$$

We make repeated use of some structure theory of Cartan subgroups which we now explain. Write $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ and $H = TA$ as in Section 2. Let

$$\Gamma(H) = \exp(i\mathfrak{a}) \cap H = \{\exp(iX) \,|\, X \in \mathfrak{a}, \exp(2iX) = 1\}. \tag{3.9}$$

For $\alpha \in \Phi_R$ (cf. Section 2) let $m_\alpha = \alpha^\vee(-1) = \exp(\pi i \alpha^\vee)$ and define

$$\Gamma_R(H) = \langle m_\alpha \,|\, \alpha \in \Phi_R \rangle \subset \Gamma(H) \cap G_d^0. \tag{3.10)(a}$$

It is well known that

$$H = \Gamma(H) H^0, \quad H \cap G^0 = \Gamma_R(H) H^0. \tag{3.10)(b}$$

Up to conjugacy by G there are unique maximally split and fundamental Cartan subgroups H_s and H_f. Then

$$G = H_s G^0,\ H_f \cap G^0 = H_f^0. \tag{3.10)(c}$$

Let G_d be the derived group of G, and $H_d = H \cap G_d$, with identity components G_d^0 and H_d^0, respectively. Equations (a) and (b) imply

$$G_d = G_d^0, \quad H = \Gamma(H) Z(G) H_d^0. \tag{3.10)(d}$$

We will also apply this to $\overline{G}$ and $\overline{H}$.

Lemma 3.11 *The homomorphism $\phi : \overline{H}(\mathbb{C}) \to H(\mathbb{C})$ has the following properties.*

1. *$\phi(wh) = w\phi(h)$ for all $w \in W(\Phi), h \in \overline{H}(\mathbb{C})$,*
2. *$\phi(\overline{H}^0) = H^0$,*
3. *$\phi(h) = 1$ for all $h \in \Gamma_R(\overline{H})$,*

4. $\phi(\overline{H}\cap\overline{G}^0) = H^0$,

5. $\phi(\Gamma(\overline{H})) = \Gamma(H)\cap C$,

6. $\phi(\overline{H}) = (\Gamma(H)\cap C)H^0$.

Proof. The first two are clear from the definition. For (3) note that $\phi(\overline{\exp}(\pi i\alpha^\vee)) = \exp(2\pi i\alpha^\vee) = 1$ for all $\alpha\in\Phi_R$. Then (4) follows since $\overline{H}\cap\overline{G}^0 = \Gamma_R(\overline{H})\overline{H}^0$.

Let $\mathfrak{h} = \overline{\exp}(iX)\in\Gamma(\overline{H})$ where $X\in\mathfrak{a}$ with $\overline{\exp}(2iX) = 1$. Now $\phi(h) = \exp(2iX)\in\exp(i\mathfrak{a})$. But $\overline{p}(\phi(h)) = \overline{\exp}(2iX) = 1$ so that $\phi(h)\in C\cap\exp(i\mathfrak{a})\subset\Gamma(H)\cap C$. Conversely, let $z_G = \exp(iY)\in C\cap\Gamma(H)$ where $Y\in\mathfrak{a}$ with $\exp(2iY) = 1$. Since $z_G\in C$, $\overline{p}(z_G) = \overline{\exp}(iY) = 1$. Let $z = \overline{\exp}(iY/2)$. Then $z\in\Gamma(\overline{H})$ and $\phi(z) = z_G$. Finally (6) follows since $\overline{H} = \Gamma(\overline{H})\overline{H}^0$. □

It is a standard fact that the character of an irreducible genuine representation of $\widetilde{G}$, considered as a function on the regular semisimple elements, vanishes off of $Z(\widetilde{H})$ (for example see [1, §II]). Our character identities involve the image of ϕ in H, so we would like to know that $\phi(\overline{H})\subset Z_0(H)$, with image as large as possible.

Let H_s be a maximally split Cartan subgroup of G, and assume $\phi(\overline{H}_s)\subset Z_0(H_s)$. Let $C_s = \Gamma(H_s)\cap C$ so that by Lemma 3.11 $\phi(\overline{H}_s) = C_sH_s^0$. By assumption $C_s\subset Z_0(H_s)\cap Z(G)$, so (cf. (4.1)(g)) $\widetilde{C_s}\subset \mathrm{Cent}_{\widetilde{G}}(\widetilde{G^0})\cap Z(\widetilde{H})$. Since H_s is maximally split $G = H_sG^0$, which implies $\mathrm{Cent}_{\widetilde{G}}(\widetilde{G}^0)\cap Z(\widetilde{H}_s) = Z(\widetilde{G})$. Therefore $C_s\subset Z_0(G)$.

Now for a general θ-stable Cartan subgroup $H = TA$, there is a maximally split Cartan $H_s = T_sA_s$ so that that $A\subset A_s$. Then $\Gamma(H)\subset\Gamma(H_s)$, so that $\phi(\overline{H}) = (C\cap\Gamma(H))H^0 = (C_s\cap\Gamma(H))H^0$. That is, if we use C_s in place of C we have the same images $\phi(\overline{H})$ for every Cartan subalgebra $\mathfrak{h}$. Thus we may as well assume that $C\subset Z_0(G)$.

Lemma 3.12 *Assume that $C\subset Z_0(G)$. Then for every Cartan subgroup $\phi(\Gamma(\overline{H}))\subset Z_0(G)$ and $\phi(\overline{H})\subset Z_0(G)H^0\subset Z_0(H)$. Furthermore $\phi(Z(\overline{G}))\subset Z_0(G)$.*

Proof. The first part follows from the above discussion. Let H_s be maximally split with roots Φ. Let $z\in Z(\overline{G})\subset\overline{H}_s, \widetilde{z}\in Z(\widetilde{H}_s)$ such that $p(\widetilde{z}) = \phi(z)$. Then for every $\alpha\in\Phi$,

$$\alpha(\widetilde{z}) = \alpha(\phi(z)) = \alpha(z)^2 = 1$$

by (3.8). Thus $\widetilde{z} \in \mathrm{Cent}_{\widetilde{G}}(\widetilde{G}^0) \cap Z(\widetilde{H}_s) = Z(\widetilde{G})$ as above. $\square$

To reiterate, so far we have assumed C is a σ-invariant subgroup of $G(\mathbb{C})$, $C \subset Z_0(G)$, and $z^2 = 1$ for all $z \in C$. For the definition of transfer factors we need to impose a further technical condition on C. Suppose $\widetilde{\chi}$ is a genuine character of $Z(\widetilde{G})$. Then $\widetilde{\chi}^2$ factors to a character of $Z_0(G)$. Let

$$Z_2(G) = \{z \in Z_0(G) \,|\, z^2 = 1\}. \tag{3.13)(a}$$

Then $\widetilde{\chi}^2$ restricted to $Z_2(G)$ is independent of the choice of $\widetilde{\chi}$: if $p(\widetilde{z}) = z$ then $\widetilde{\chi}^2(z) = 1$ if $\widetilde{z}^2 = 1$, or -1 otherwise.

Since G is acceptable $e^\rho(z)$ is well defined for $z \in Z_0(G) \cap G_d$. Let

$$\zeta_2(z) = \widetilde{\chi}^2(z) e^\rho(z) \quad z \in Z_2(G) \cap G_d^0 \subset Z(G_d^0). \tag{3.13)(b}$$

This is independent of the choices of $\widetilde{\chi}$, H and Φ^+.

Definition 3.14 *An admissible triple is a set $(\widetilde{G}, G, \overline{G})$ where:*

1. *G is the set of real points of a connected reductive complex Lie group $G(\mathbb{C})$ such that the derived group $G_d(\mathbb{C})$ is acceptable, and every simple factor of the root system of $G(\mathbb{C})$ has one root length or is of type G_2.*

2. *$p : \widetilde{G} \to G$ is an admissible two-fold cover (Definition 3.2).*

3. *$\overline{G}$ is the set of real points of $\overline{G}(\mathbb{C}) = G(\mathbb{C})/C$ where C is a finite central subgroup of G satisfying the following conditions:*

   (a) $c^2 = 1$ for all $c \in C$,

   (b) $C \subset Z_0(G) = p(Z(\widetilde{G}))$,

   (c) $\zeta_2(z) = 1$ for all $z \in C \cap G_d^0$ (cf. (3.13)(b)).

Assume that $(\widetilde{G}, G, \overline{G})$ is an admissible triple. Let $H = TA$ be a Cartan subgroup of G. Let $M = \mathrm{Cent}_G(A)$ be a cuspidal Levi subgroup. Then $\widetilde{M} \subset \widetilde{G}$ and $\overline{M} = \mathrm{Cent}_{\overline{G}}(\overline{A}) \subset \overline{G}$ are also cuspidal Levi subgroups. We will show in Section **??** that $(\widetilde{M}, M, \overline{M})$ is also an admissible triple. This means our constructions will be compatible with parabolic induction.

Suppose $\widetilde{G}$ and G satisfy conditions (1) and (2). Then $(\widetilde{G}, G, \overline{G})$ is an admissible triple for $\overline{G}(\mathbb{C}) = G(\mathbb{C})/C$ for any subgroup C satisfying conditions (3)(a-c). For example, $C = \{1\}$ satisfies this condition, so that $(\widetilde{G}, G, G)$ is an admissible triple. By Lemma 3.11(6) taking C bigger makes the image of $\phi(\overline{H}) \subset Z_0(H)$ bigger, and in general, we get the best lifting results when we take C as large as possible. However, if we take C as large as possible for G, it may not be the maximal choice for Levi subgroups of G. Thus we do not specify C beyond the requirements of Definition 3.14.

Example 3.15 Let $H = TA$ be a maximally split Cartan subgroup of G. We will show in Lemma 3.17 below that it is not always possible to pick C satisfying the conditions of Definition 3.14 such that $\phi(\overline{H}) = Z_0(H)$. Let $M = C_G(A)$. We will show it is possible to pick a finite central subgroup C_M of M satisfying the conditions of Definition 3.14 for M such that $\phi(\overline{H}) = Z_0(H)$ where ϕ is defined using $\overline{M}(\mathbb{C}) = M(\mathbb{C})/C_M$. Recall $H = \Gamma(H)H^0$. Thus $\widetilde{H} = \widetilde{\Gamma}(H)\widetilde{H}^0$ where $\widetilde{\Gamma}(H) = p^{-1}\Gamma(H)$. Since $\widetilde{H}^0 \subset Z(\widetilde{H})$, we have $Z(\widetilde{H}) = Z(\widetilde{\Gamma}(H))\widetilde{H}^0$. Now $\Gamma(H) \subset Z(M)$ so that $Z(\widetilde{\Gamma}(H)) \subset C_{\widetilde{M}}(\widetilde{M}^0) \cap Z(\widetilde{H}) = Z(\widetilde{M})$. Thus $Z_0(H) = \Gamma_0(H)H^0$ where $\Gamma_0(H) = p(Z(\widetilde{\Gamma}(H))) \subset Z_0(M) \cap \Gamma(H)$. Since $\Gamma_0(H)$ is a two-group, there is $C_M \subset \Gamma_0(H)$ such that $Z_0(H) = C_M H^0$ and $C_M \cap H^0 = \{1\}$. But $C_M \cap M_d^0 \subset C_M \cap H^0 = \{1\}$, so it is easy to see that C_M satisfies the conditions of Definition 3.14 for M and gives $\phi(\overline{H}) = C_M H^0 = Z_0(H)$.

The following lemma, which is an immediate consequence of (5.12) is useful for producing admissible triples.

Lemma 3.16 *Let H be a maximally split Cartan subgroup of G. Then $\zeta_2(c) = 1$ for all $c \in Z_0(G) \cap \Gamma_R(H)$.*

Lemma 3.17 *Suppose that $G(\mathbb{C})$ is simple and simply connected, and let H be a maximally split Cartan subgroup of G. Then we can choose C so that $\phi(\overline{H}) = Z_0(H)$ except when $G = SU(n,n)$ where n is even, $G = Spin(p,q)$ where p and q are even with $p > q \geq 2$ or when $G = Spin^*(2n)$ where $n \equiv 0$ mod 4.*

Proof.

Clearly if $Z_0(H)$ is connected we can take $C = \{1\}$. We know that $Z_0(H) = Z_0(G)H^0 = Z(G)H^0$ since G is connected. But since $H = \Gamma_R(H)H^0$ where every element of $\Gamma_R(H)$ is two group, it is easy to see that $Z(G)H^0 = Z_e(G)H^0$ where $Z_e(G)$ is the group of elements in $Z(G)$ with even order. Thus $Z_0(H)$ is connected when $Z_e(G) = \{1\}$. This will be the case for all real forms when Φ is of type A_{2n}, E_6, E_8, or G_2. Thus we may as well assume that Φ is type A_{2n-1}, D_n, or E_7. In these cases $Z_e(G) = Z(G)$. We may also assume that G is a real form such that $Z_0(H) = Z(G)H^0$ is not connected. In particular, we can assume that G is not compact or complex. Suppose that $Z(G) \subset \Gamma_R(H)$. Then $Z(G)$ is a two-group, and by Lemma 3.16 we can take $C = Z(G)$, giving us $\phi(\overline{H}) = Z(G)H^0 = Z_0(G)$ as desired. This will always be the case if G is the split real form.

Suppose that $\Phi = A_{2n-1}$. If G is type AI, then G is split. If G is type AII, then H is connected. If G is type $AIII$, H is connected unless $G = SU(n,n)$. In this case $Z_0(H) = H$ has two components. Now $Z(G) =< z >$ is cyclic of order $2n$ and so $Z_2(G) =< z_0 >$ where $z_0 = z^n \in \Gamma_R(H)$ is the unique element of order two. Thus the biggest C we can take is $C = \{1, z_0\}$. When n is odd, $z_0 \notin H^0$ so that $\phi(\overline{H}) = CH^0 = H$, but when n is even, $z_0 \in H^0$ so that $\phi(\overline{H}) = CH^0 = H^0$ is a proper subgroup of $Z_0(H)$.

Suppose that $\Phi = D_n$. If G is type DI, then $G = Spin(p,q), p \geq q, p + q = 2n$, and Φ_R is type D_q. H is connected unless $q \geq 2$ and H is split if $p = q$. Thus we can assume that $p > q \geq 2$. If p, q are odd, then $Z(G) = Z_2 \subset \Gamma_R(H)$. If p, q are even, then $Z(G) = \{1, z_0, z_1, z_0z_1\}$ where $z_0 \in \Gamma_R(H)$, but $z_1 \notin \Gamma(H)$. Thus the biggest C we can take is $C = \{1, z_0\}$. But $z_1 \notin CH^0$ so that CH^0 is a proper subgroup of $Z_0(H)$. If G is type $DIII$, then $G = Spin^*(2n)$. H is connected if n is odd. Suppose that $n = 2k$ is even. Then $Z(G) = \{1, z_0, z_1, z_0z_1\}$ as above where $z_1 \in \Gamma_R(H)$, but $z_0 \notin \Gamma(H)$ so the biggest C we can take is $C = \{1, z_1\}$. If k is even, then $z_0 \notin CH^0$ while $z_0 \in CH^0$ when k is odd.

Finally, if $\Phi = E_7$, then either G is split or $Z_0(H)$ is connected. □

4 Structure Theory

Assume $G_d(\mathbb{C})$ is simply connected and τ is an automorphism of G. Then τ stabilizes $G_d = G_d^0$, and by Lemma 3.1 it lifts uniquely to an automorphism $\widetilde{\tau}$ of the (unique) admissible cover $\widetilde{G_d}$ of G_d. Suppose $z \in Z(G_d)$ and $\tau(z) = z$. Choose an inverse image $\widetilde{z}$ of z and set

$$\{\tau, z\} = \widetilde{\tau}(\widetilde{z})\widetilde{z}^{-1}. \tag{4.1)(a}$$

Then $p(\{\tau, z\}) = 1$ so $\{\tau, z\} = \pm 1$. It is independent of the choice of $\widetilde{z}$.

Note that the real points G_{ad} of $G_{ad}(\mathbb{C})$ act by conjugation on G, and apply this discussion to $\tau = \text{int}(g)$ for $g \in G_{ad}$. Then $\text{int}(g)$ fixes $Z(G_d)$

pointwise, and we may therefore define

(4.1)(b) $$\{g, z\} = \{\mathrm{int}(g), z\} \quad (g \in G_{ad}, z \in Z(G_d)).$$

This defines a continuous pairing $\{,\} : G_{ad} \times Z(G_d) \to \mathbb{Z}/2\mathbb{Z}$. By continuity $\{g, z\} = 1$ for all $g \in G^0_{ad}$.

Let $\mathcal{R}(G) = G/Z(G)G^0$ and $\mathcal{R}_{\mathrm{ad}}(G) = \mathcal{R}(G_{ad}) = G_{ad}/G^0_{ad}$. We obtain a canonical pairing

(4.1)(c) $$\{,\} : \mathcal{R}_{\mathrm{ad}}(G) \times Z(G_d) \to \mathbb{Z}/2\mathbb{Z}.$$

By duality we obtain a canonical homomorphism

(4.1)(d) $$\gamma_{\mathrm{ad}} : \mathcal{R}_{\mathrm{ad}} \to \mathrm{Hom}(Z(G_d), \mathbb{Z}/2\mathbb{Z}).$$

We now drop the assumption that $G_d(\mathbb{C})$ is simply connected. Let $G_{sc}(\mathbb{C})$ be the simply connected cover of $G_d(\mathbb{C})$, with corresonding real form G_{sc}. The preceding discussion gives a canonical map:

(4.1)(e) $$\gamma_{\mathrm{ad}} : \mathcal{R}_{\mathrm{ad}} \to \mathrm{Hom}(Z(G_{sc}), \mathbb{Z}/2\mathbb{Z}).$$

Assume that we are given an admissible cover $\widetilde{G}$ of G. Suppose g, h are commuting elements of G. Choose inverse images $\widetilde{g}, \widetilde{h}$ of g, h in $\widetilde{G}$. Analogously to (4.1)(a) define

(4.1)(f) $$\begin{aligned} \{g, h\} &= \mathrm{int}(\widetilde{g})(\widetilde{h})\widetilde{h}^{-1} \\ &= \widetilde{g}\widetilde{h}\widetilde{g}^{-1}\widetilde{h}^{-1}. \end{aligned}$$

As in (4.1)(a) $\{g, h\} = \pm 1$, independent of the choices, and we say $\{g, h\}$ is the commutator of g, h (with respect to the cover $\widetilde{G}$). It is continuous in each factor so

(4.1)(g) $$\{Z(G), G^0\} = 1.$$

It follows that we obtain a pairing

(4.1)(h) $$\{,\} : \mathcal{R}(G) \times Z(G^0_d) \to \mathbb{Z}/2\mathbb{Z}$$

depending on the cover $\widetilde{G}$. As in (4.1)(d) by duality this gives a homomorphism

(4.1)(i) $$\gamma : \mathcal{R}(G) \to \mathrm{Hom}(Z(G^0_d), \mathbb{Z}/2\mathbb{Z}).$$

For later use we note a simple consequence of (4.1)(g):

$$Z(\widetilde{G}) = \widetilde{Z(G)} \quad \text{if } G \text{ is connected.} \tag{4.2}$$

In general we only have $Z(\widetilde{G}) \subset \widetilde{Z(G)}$.

Remark 4.3 Unlike γ_{ad}, which is canonical, γ appears to depend on the choice of an admissible cover $\widetilde{G}$ of G. In fact by (4.4) below γ is independent of this choice. However the existence of the map γ depends on the *existence* of an admissible cover.

The map $G \to G_{\text{ad}}$ factors to an inclusion $\mathcal{R}(G) \hookrightarrow \mathcal{R}_{\text{ad}}(G)$. Also $Z(G_d^0)$ is a quotient of $Z(G_{\text{sc}})$, and there is a natural inclusion $\text{Hom}(Z(G_d^0), \mathbb{Z}/2\mathbb{Z}) \hookrightarrow \text{Hom}(Z(G_{sc}), \mathbb{Z}/2\mathbb{Z})$. It is easy to see the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{R}_{\text{ad}}(G) & \xrightarrow{\gamma_{\text{ad}}} & \text{Hom}(Z(G_{sc}), \mathbb{Z}/2\mathbb{Z}) \\ \uparrow & & \uparrow \\ \mathcal{R}(G) & \xrightarrow{\gamma} & \text{Hom}(Z(G_d^0), \mathbb{Z}/2\mathbb{Z}) \end{array} \tag{4.4}$$

Proposition 4.5 *Assume G is oddly laced. Then γ_{ad} and γ are injective.*

Proof. It is enough to prove the first statement, so assume $G_d(\mathbb{C})$ is simply connected. It is easy to reduce to the case $G(\mathbb{C})$ simple, so we assume $G(\mathbb{C})$ is simple and simply connected. If G is of type G_2 then $G = G_{ad}$ and $\mathcal{R}_{ad}(G) = 1$ so this case is obvious. Assume G is simply laced.

First assume G contains a compact Cartan subgroup T, which implies $Z(G) = Z(G(\mathbb{C}))$. The standard isomorphism $P/R \simeq \text{Hom}(Z(G), \mathbb{C}^\times)$ gives an isomorphism

$$P \cap \frac{1}{2}R/R \simeq \text{Hom}(Z(G), \mathbb{Z}/2\mathbb{Z}). \tag{4.6}$$

Let $\Phi = \Phi(G(\mathbb{C}), T(\mathbb{C}))$. By [11, Proposition 9.5(a)]

$$\mathcal{R}_{\text{ad}}(G) \simeq \text{Norm}_W(\Phi_{I,c})/W(\Phi_{I,c}) \tag{4.7}$$

Let λ be the differential of a genuine character of $\widetilde{T}$. It is not hard to see that using the isomorphism (4.7)

$$\gamma_{\text{ad}} : w \to w\lambda - \lambda \in P \cap \frac{1}{2}R/R \simeq \text{Hom}(Z(G), \mathbb{Z}/2\mathbb{Z}). \tag{4.8}$$

It is enough to show

$$w\lambda - \lambda \in R \Rightarrow w \in W(\Phi_{I,c}). \tag{4.9}$$

This follows most easily from a straightforward case-by-case check. We sketch a more conceptual proof below.

Now suppose G does not contain a compact Cartan subgroup. We claim that if G is $SL(2n+1,\mathbb{R}), SL(n,\mathbb{H})$ or of type E_6 then $\mathcal{R}_{\mathrm{ad}}(G)=1$ and there is nothing to prove. Let Γ be the Galois group of $\mathbb{C}/\mathbb{R}$. It is well known (for example see [4]) there is an injection

$$\alpha : \mathcal{R}_{\mathrm{ad}}(G) \hookrightarrow H^1(\Gamma, Z(G(\mathbb{C}))). \tag{4.10}$$

It is easy to compute the right hand side; it is trivial for $SL(2n+1,\mathbb{R})$ or any real form of E_6. We leave to the reader verification that $\mathcal{R}_{\mathrm{ad}}(G)=1$ when G is the adjoint form of $SL(n,\mathbb{H})$.

Suppose $G=SL(2n,\mathbb{R})$. It is easy to see $\mathcal{R}_{\mathrm{ad}}(G)\simeq \mathbb{Z}/2\mathbb{Z}$ in this case. Write x,y for the elements of $p^{-1}(-I)\subset Z(\widetilde{G})$. It is not hard to see that if $g\in G_{ad}/G^0_{ad}$ then the lift $\widetilde{\tau}$ of $\mathrm{int}(g)$ to $\widetilde{SL}(2n,\mathbb{R})$ satisfies $\widetilde{\tau}(x)=y$ (cf. Example 4.13). This proves this case. The only remaining case is $Spin(p,q)$ with p,q odd, in which case $\mathcal{R}_{\mathrm{ad}}(G)=\mathbb{Z}/2\mathbb{Z}$. This is similar to $SL(2n,\mathbb{R})$ and we leave the details to the reader. □

Remark 4.11 The injectivity of γ imposes a strong constraint on the existence of an admissible cover of G if $G_d(\mathbb{C})$ is not simply connected, or (essentially the same thing) if $G_d(\mathbb{C})$ is simply connected but $\mathcal{R}(G)$ is non-trivial.

Example 4.12 If $G_d(\mathbb{C})$ is adjoint then G has an admissible cover only if $\mathcal{R}(G)=1$.

Example 4.13 It is obvious that $G=PGL(2n+1,\mathbb{R})\simeq SL(2n+1,\mathbb{R})$, and therefore G has an admissible cover. In this case $\mathcal{R}(G)=1$.

If $G=PGL(2n,\mathbb{R})$ then $\mathcal{R}(G)\simeq\mathbb{Z}/2\mathbb{Z}$, so G does not have an admissible cover. We make this explicit.

Note that $G^0\simeq SL(2n,\mathbb{R})/\pm I$. Let $\widetilde{SL}(2n,\mathbb{R})\to SL(2n,\mathbb{R})$ be the admissible cover. Write $Z(\widetilde{SL}(2n,\mathbb{R}))=\{1,z,x,y\}$ with $p(z)=I, p(x)=p(y)=-I$. It is easy to compute that $Z(\widetilde{SL}(2n,\mathbb{R}))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ if n is even, or $\mathbb{Z}/4\mathbb{Z}$ if n is odd.

Suppose $p : \widetilde{G} \to G$ is an admissible cover of G. Then $\widetilde{G^0}$ is a quotient of $\widetilde{SL}(2n,\mathbb{R})$ by a central two element subgroup A. If $A = \{1,z\}$ then $\widetilde{G^0} = SL(2n,\mathbb{R})$ is not admissible. If n is odd then x, y have order 4, so there is no other such subgroup A: in this case G^0 does not have an admissible two-fold cover.

Now suppose n is even, and $A = \{1,x\}$ or $\{1,y\}$. Choose $g \in G\backslash G^0$ and $\widetilde{g} \in p^{-1}(g)$. Then $\mathrm{int}(\widetilde{g}) \in \mathrm{Aut}(\widetilde{G^0})$ lies over $\mathrm{int}(g) \in \mathrm{Aut}(G^0)$. However $\mathrm{int}(g)$ lifts uniquely to an automorphism $\widetilde{\tau}$ of $\widetilde{SL}(2n,\mathbb{R})$, and $\widetilde{\tau}$ satisfies $\widetilde{\tau}(x) = y$. Therefore $\widetilde{\tau}$ does not factor to $\widetilde{G^0}$. This contradiction proves that no such admissible cover exists.

Note that what is subtle is that if n is even then G^0 has an admissible cover (take $\widetilde{SL}(2n,\mathbb{R})/\{1,x\}$), but this cover does not extend to G.

Corollary 4.14 *Assume G is oddly laced. Then*

$$Z(\widetilde{Z(G)})\widetilde{G^0} = Z(\widetilde{G})\widetilde{G^0} \tag{4.15}$$

Proof. It is obvious that $Z(\widetilde{G}) \subset Z(\widetilde{Z(G)})$ so one inclusion is obvious, and it is enough to show $Z(\widetilde{Z(G)}) \subset Z(\widetilde{G})\widetilde{G^0}$. Suppose $\widetilde{z} \in Z(\widetilde{Z(G)})$ and let $z = p(\widetilde{z}) \in Z(G)$. Then $\{z, g\} = 1$ for $g \in Z(G)$ and (by (4.1)(g)) G^0. So $\tau_z : g \to \{z, g\}$ is an element of $\mathrm{Hom}(\mathcal{R}(G), \mathbb{Z}/2\mathbb{Z})$. By Proposition 4.5 the Pontrijagin dual of γ (4.1)(i) is a surjection $Z(G_d^0) \to \mathrm{Hom}(\mathcal{R}(G), \mathbb{Z}/2\mathbb{Z})$. Choose an element $z' \in Z(G_d^0)$ with image τ_z. Then $\{zz', g\} = 1$ for all $g \in \mathcal{R}(G)$. That is, letting $\widetilde{z}' \in \widetilde{G^0}$ be an inverse image of z', $\widetilde{z}\widetilde{z}' \in Z(\widetilde{G})$, and $\widetilde{z} \in Z(\widetilde{G})\widetilde{G^0}$. □

We conclude this section with another application of Proposition 4.5. Suppose $\widetilde{G}$ is an admissible cover of G, and G contains a relatively compact Cartan subgroup H. Let

$$\begin{aligned}
&W_0 = \{w \in W \mid w : \mathfrak{h} \to \mathfrak{h} \text{ exponentiates to } \widetilde{H^0}\}\\
&W_2 = \{w \in W_0 \mid w\widetilde{z} = \widetilde{z} \quad \text{for all } \widetilde{z} \in Z(\widetilde{G}) \cap \widetilde{H^0}\}\\
&W_* = \mathrm{Norm}_W(\Phi_{I,c}).
\end{aligned} \tag{4.16}$$

We have obvious inclusions

$$W(\Phi_{I,c}) \subset W(G,H) \subset W_2 \subset W_0 \subset W_*. \tag{4.17}$$

By [11, Proposition 9.5(a)] we have

(4.18)(a) $$\mathcal{R}_{\text{ad}}(G) \simeq W_*/W(\Phi_{I,c})$$

(4.18)(b) $$\mathcal{R}(G) \simeq W(G,H)/W(\Phi_{I,c})$$

Lemma 4.19 $W_2 = W(G,H)$.

Proof. Let

(4.20) $$\begin{aligned} A &= Z_0(G) \cap H^0 \cap G_d^0 \\ &= Z_0(G) \cap G_d^0 \end{aligned}$$

since (cf. 3.10)(c) $H \cap G^0 = H^0$. Let $w \in W_0$. It is clear that

(4.21) $$w \in W_2 \Leftrightarrow w\widetilde{z} = \widetilde{z} \quad \text{for all } z \in \widetilde{A}.$$

If $w \in W_0$ corresponds to $g \in G_{\text{ad}}$ via (4.18)(a), we have

(4.22) $$w\widetilde{z} = \{g, p(\widetilde{z})\}\widetilde{z}.$$

In other words it is enough to show

(4.23) $$\{g, z\} = 1 \quad \text{for all } z \in A \Leftrightarrow g \in G.$$

Assume that $G_d(\mathbb{C})$ is simply connected. Then using the pairing on $\mathcal{R}_{\text{ad}}(G) \times Z(G_d)$ (4.1)(c) we need to show

(4.24) $$A^{\perp} = \mathcal{R}(G).$$

It is easy to see that $A = \mathcal{R}(G)^{\perp}$ so we want to show

(4.25) $$(\mathcal{R}(G)^{\perp})^{\perp} = \mathcal{R}(G).$$

This follows from the injectivity of γ_{ad}.

The argument if $G_d(\mathbb{C})$ is not simply connected is a slight generalization of this. Write $Z(G_d^0) = Z(G_{\text{sc}})/B$ for some B. Then $\mathcal{R}(G) \subset B^{\perp} \subset \mathcal{R}_{\text{ad}}(G)$. The pairing $\mathcal{R}_{\text{ad}}(G) \times Z(G_{sc}) \to \mathbb{Z}/2\mathbb{Z}$ descends to a pairing $B^{\perp} \times Z(G_d^0) \to \mathbb{Z}/2\mathbb{Z}$, and $B^{\perp} \hookrightarrow \text{Hom}(Z(G_d^0), \mathbb{Z}/2\mathbb{Z})$. The preceding argument now applies with $B^{\perp}$ in place of $\mathcal{R}_{\text{ad}}(G)$.

By Lemma 5.34

(4.26)(a) $$e^{\rho(\Phi_1^+) - \rho(\Phi_2^+)}(ta)\Gamma(H, \Phi_1^+)(h, \widetilde{g}) = \Gamma(H, \Phi_2^+)(h, \widetilde{g}).$$

□

4.1 Cartan Subgroups

We need some structural facts about Cartan subgroups. Fix an admissible cover $\widetilde{G}$ of G, and a Cartan subgroup H of G.

Suppose τ is an automorphism of G_d^0 and $\alpha \in \Phi(G,H)$ is a long real root. We assume $\tau(H \cap G_d^0) = H \cap G_d^0$ and $\tau(\alpha) = \alpha$. Let M_α be as in Section 3; we have $\tau(M_\alpha) = M_\alpha$. Let $T_\alpha \simeq S^1$ be a τ-stable compact Cartan subgroup of M_α. Then for all $t \in T_\alpha$, $\tau(t) = t$ or t^{-1}. Let $m_\alpha = \alpha^\vee(-1) \in Z(M_\alpha)$.

Proposition 4.27 *Suppose α is a long real root. Then*

(4.28)(a)
$$\{\tau, m_\alpha\} = \begin{cases} 1 & \tau(t) = t \quad \textit{for all } t \in T_\alpha \\ -1 & \tau(t) = t^{-1} \quad \textit{for all } t \in T_\alpha \end{cases}$$

For all $h \in H$,

(4.28)(b)
$$\{h, m_\alpha\} = sgn(\alpha(h)).$$

If $\beta \in \Phi_r(G,H)$ then

(4.28)(c)
$$\{m_\alpha, m_\beta\} = (-1)^{\langle \alpha, \beta^\vee \rangle}.$$

Proof. We can compute $\tau(m_\alpha)$ by working in T_α. If τ acts trivially on T_α then the same holds for the action of $\widetilde{\tau}$ on $\widetilde{T_\alpha}$. On the other hand if $\tau(t) = t^{-1}$ then $\widetilde{\tau}(\widetilde{t}) = \widetilde{t}^{-1}$ for all $\widetilde{t} \in \widetilde{T_\alpha}$. Since $\widetilde{G}$ is admissible, m_α has order 4, and we conclude $\widetilde{\tau}(m_\alpha) = -m_\alpha$.

It is a standard fact (essentially a calculation in $GL(2,\mathbb{R})$) that for all $h \in H, t \in T_\alpha$ $hth^{-1} = t^{\mathrm{sgn}(\alpha(h))}$. This proves (4.28)(b), and (4.28)(c) follows from this and the identity $\alpha(m_\beta) = (-1)^{\langle \alpha, \beta^\vee \rangle}$. □

Proposition 4.29 *Assume G is oddly laced. Then*

(4.30)
$$Z(\widetilde{H}) = Z(\widetilde{G})\widetilde{H^0}$$

We first prove

Lemma 4.31 *In the setting of the Proposition we have*

(4.32)
$$Z(\widetilde{H}) \subset \widetilde{Z(G)}\widetilde{H^0}.$$

Proof. It is enough to show if $h \in H$, $\{h, m_\alpha\} = 1$ for all real roots α then $h \in Z(G)H^0$. (This is where the oddly laced condition appears: otherwise we only have this identity for the long roots.) By (4.28)(b) it is enough to show

$$\alpha(h) > 0 \text{ for all } \alpha \in \Phi_r(G,H) \text{ implies } h \in Z(G)H^0 \tag{4.33}$$

This is a straightforward calculation using roots and weights. Choose a basis of the root lattice of the form

$$\alpha_1, \dots, \alpha_m, \alpha_{m+1}, \dots, \alpha_{m+n}, \alpha_{m+n+1}, \dots \alpha_{m+n+2r}$$

where α_i is real for $1 \le i \le m$, imaginary for $m+1 \le i \le m+n$, and $\theta\alpha_{m+n+2i-1} = -\alpha_{m+n+2i}$ for all $1 \le i \le r$. It is a basic fact about lattices that such a basis exists. Let $\lambda_1^\vee, \dots, \lambda_{m+n+2r}^\vee$ be the dual basis of coweights.

For each i choose $x_i \in \mathbb{C}$ so that $e^{x_i} = \alpha_i(h)$. If $i \le m$ we may assume $x_i \in \mathbb{R}$ since $\alpha_i(h) > 0$. We may also assume $x_{m+n+2i} = \overline{x}_{m+n+2i-i}$ for all $1 \le i \le r$. Let $X = \sum_i x_i \lambda_i^\vee$, $h_1 = \exp(X)$. It follows easily that $X \in \mathfrak{h}$, $h_1 \in H^0$, and $\alpha(h) = \alpha(h_1)$ for all roots α. Let $z = hh_1^{-1}$; this is contained in G, and also $Z(G(\mathbb{C}))$, since $\alpha(z) = 1$ for all roots α. Therefore $z \in Z(G)$, and therefore $h = zh_1 \in Z(G)H^0$. □

Proof of the Proposition. The statement is equivalent to

$$Z_0(H) = Z_0(G)H^0. \tag{4.34}$$

It is enough to show $Z_0(H) \subset Z_0(G)H^0$ (the reverse inclusion is obvious). We first prove this for a maximally split Cartan subgroup H_s of G. Suppose $h \in Z_0(H)$. By the Lemma $h = zy$ with $z \in Z(G), y \in H^0$. Then $y \in Z_0(H)$, so $z = hy^{-1} \in Z_0(H)$. It is enough to show $z \in Z_0(G)$. We have $\{z, g\} = 1$ for all $g \in H$, and also for $g \in G^0$ by (4.1)(g) Since H is maximally split $G = HG^0$, so $z \in Z_0(G)$.

The general case will be proved after Lemma 4.45, using Cayley transforms. □

4.2 Cayley Transforms

Suppose H is a Cartan subgroup and α is a real or non-compact imaginary root. We define the Cayley transform H_α of H of H with respect to α as in

[10]. Then H_α has a non-compact imaginary or real root β, and the Cayley transform of H_α with respect to β is H.

In order to emphasize the symmetry of the situation we change notation and let $H_\alpha = T_\alpha A_\alpha$ be a Cartan subgroup with a real root α. Then we let $H_\beta = T_\beta A_\beta$ be its Cayley transform, with non-compact imaginary root β.

We have elements $Z_\alpha \in \mathfrak{p}$ and $Z_\beta \in \mathfrak{k}$, and let $B_\alpha = \exp(\mathbb{R}Z_\alpha) \simeq \mathbb{R}^+ \subset H_\alpha$, and $B_\beta = \exp(\mathbb{R}Z_\beta) \simeq S^1 \subset H_\beta$.

It follows easily from [10, 8.3.4 and 8.3.13] that we have

$$\begin{aligned} &(H_\alpha \cap H_\beta)B_\alpha \subset H_\alpha \quad \text{index 1 or 2} \\ &(H_\alpha \cap H_\beta)B_\beta = H_\beta \end{aligned} \tag{4.35}$$

The root α takes positive real values on $(H_\alpha \cap H_\beta)B_\alpha$. We say α is type I if the first inclusion in (4.35) is an equality. Otherwise α is of type II, in which case H_α is generated by the left hand side and an element t satisfying $\alpha(t) = -1$.

We say β is type I if $s_\beta \notin W(G, H_\beta)$, and type II otherwise. Then α, β are both of type I or both of type II.

Definition 4.36 *Suppose χ_α is a character of H_α, and χ_β is a character of H_β. We say χ_α is a Cayley transform of χ_β, and vice-versa, if*

$$\chi_\alpha(h) = \chi_\beta(h) \quad (h \in H_\alpha \cap H_\beta) \tag{4.37)(a}$$

$$Ad^*(g)(d\chi_\alpha) = d\chi_\beta \textit{ for some } g \in Aut(\mathfrak{g}). \tag{4.37)(b}$$

To be precise, in (b) the element g satisfies $\mathrm{Ad}(g)(\mathfrak{h}_\beta(\mathbb{C})) = \mathfrak{h}_\alpha(\mathbb{C})$, and the identity is between elements of $\mathfrak{h}_\beta^*$.

Here is a convenient alternative characterization of Cayley transforms.

Lemma 4.38 *In the setting of the Definition, χ_α is a Cayley transform of χ_β, and vice versa, if and only if* (4.37)(a) *holds, and*

$$\langle d\chi_\alpha, \alpha^\vee \rangle = \pm \langle d\chi_\beta, \beta^\vee \rangle. \tag{4.39}$$

Lemma 4.40

(1) Fix a character χ of H_α. There is a Cayley transform χ_β of χ if and only if

$$\langle d\chi, \alpha^\vee \rangle \in \mathbb{Z} \tag{4.41}$$

and

$$\chi(m_\alpha) = (-1)^{\langle d\chi, \alpha^\vee\rangle}. \tag{4.42}$$

Assume these hold. Then there are two choices of Cayley transform, given by (4.37)(a) *and*

$$\chi^\pm_\beta(\exp(xZ_\beta)) = e^{\pm i\langle d\chi,\alpha^\vee\rangle x} \tag{4.43}$$

The characters $\chi^\pm_\beta$ are conjugate by $W(G, H_\beta)$ if and only if α is type I.
(2) Fix a character χ of H_β. Define χ_α restricted to $(H_\alpha \cap H_\beta)B_\alpha$ by (4.37)(a), *and*

$$\chi_\alpha(\exp(xZ_\alpha)) = e^{\langle d\chi,\beta^\vee\rangle x}. \tag{4.44}$$

If β is of type I this defines a character of H_α. If β is of type II define $\chi^\pm_\alpha$ to be the two extensions of χ_α to H_α.

Sketch of proof. This is elementary from the identities [10, 8.3.4 and 8.3.13]. See [10][Lemma 8.3.7 and 8.3.15], where the setting is a regular character and the construction differs from this one by a ρ-shift. For this reason proof of the Lemma is in fact much easier than those in [10]. □

We now consider Cayley transforms for $\widetilde{G}$. The analogue of (4.35) is:

Lemma 4.45 *Assume G is oddly laced and $\widetilde{G}$ is an admissible cover of G. Then*

$$(Z(\widetilde{H}_\alpha) \cap Z(\widetilde{H}_\beta))\widetilde{B_\beta} = Z(\widetilde{H}_\beta) \tag{4.46(a)}$$

$$(Z(\widetilde{H}_\alpha) \cap Z(\widetilde{H}_\beta))\widetilde{B_\alpha} = Z(\widetilde{H}_\alpha) \tag{4.46(b)}$$

Proof. We first prove (a). We have to show

$$(Z_0(H_\alpha) \cap Z_0(H_\beta))B_\beta = Z_0(H_\beta). \tag{4.47}$$

Since B_β is connected, $B_\beta = Z_0(B_\beta) \subset Z_0(H_\beta)$, and the inclusion $\subset$ is clear. For the opposite inclusion suppose $g \in Z_0(H_\beta)$. By (4.35) write $g = hb$ with $h \in H_\alpha \cap H_\beta, b \in B_\beta$. Then $h = gb^{-1} \in Z_0(H_\beta)$ since both g and b are in $Z_0(H_\beta)$. It is enough to show $h \in Z_0(H_\alpha)$.

Note that $\{h, x\} = 1$ for $x \in H_\alpha \cap H_\beta$ (since $h \in Z_0(H_\beta)$). Since B_α is connected $B_\alpha = Z_0(B_\alpha) \subset Z_0(H_\alpha)$, and therefore $\{h, x\} = 1$ for $x \in B_\alpha$.

Therefore $\{h,x\}=1$ for $x\in(H_\alpha\cap H_\beta)B_\alpha$. If α is type I the right hand side is H_α and we are done. Otherwise choose t satisfying $\alpha(t)=-1$, so that H_α is generated by $(H_\alpha\cap H_\beta)B_\alpha$ and t.

It is enough to show $\{h,t\}=1$. If this is not the case replace h with hm_α and b with bm_α (note that $m_\alpha\in H_\alpha\cap H_\beta$ and B_β). By (4.28)(b) $\{m_\alpha,t\}=-1$, so $\{hm_\alpha,t\}=1$.

For (b) the inclusion $\subset$ is immediate since B_α is connected. On the other hand suppose $g\in H_\alpha$ but $g\notin(H_\alpha\cap H_\beta)B_\alpha$. Then $\alpha(g)<0$, and by (4.28)(b) $\{m_\alpha,g\}=-1$, so $g\notin Z_0(H_\alpha)$. This proves the reverse inclusion. □

We may now complete the proof of Proposition 4.29.

Proof. We have already shown this for the most split Cartan subgroup. By repeated use of the Cayley transform it is enough to show that in the previous setting

$$Z(\widetilde{H}_\alpha)=Z(\widetilde{G})\widetilde{H}^0_\alpha\Rightarrow Z(\widetilde{H}_\beta)=Z(\widetilde{G})\widetilde{H}^0_\beta \tag{4.48}$$

By the Lemma we have

$$\begin{aligned}Z(\widetilde{H}_\beta)&=(Z(\widetilde{H}_\alpha)\cap Z(\widetilde{H}_\beta))\widetilde{B_\beta}\\&=(Z(\widetilde{G})\widetilde{H}^0_\alpha\cap Z(\widetilde{H}_\beta))\widetilde{B_\beta}\\&=Z(\widetilde{G})(\widetilde{H}^0_\alpha\cap Z(\widetilde{H}_\beta))\widetilde{B_\beta}\\&=Z(\widetilde{G})\widetilde{H}^0_\beta\end{aligned}\tag{4.49}$$

For the last equality we have used that $(H^0_\alpha\cap H_\beta)B_\beta\subset H^0_\beta$, which implies $(\widetilde{H}^0_\alpha\cap Z(\widetilde{H}_\beta))\widetilde{B_\beta}\subset\widetilde{H}^0_\beta$. □

If $\widetilde{G}$ is an admissible cover of a oddly laced group G, we define Cayley transforms for genuine characters as in Definition 4.36, using $Z(\widetilde{H}_\alpha),Z(\widetilde{H}_\beta)$ in place of H_α,H_β, and Lemma 4.45 in place of (4.35). By 4.29 we can restate the definition of Cayley transform in this setting.

Lemma 4.50 *Assume G is oddly laced and $\widetilde{G}$ is an admissible cover of G. In the setting of Lemma 4.45, $\widetilde{\chi}_\alpha$ and $\widetilde{\chi}_\beta$ are each others Cayley transforms if and only if*

$$\widetilde{\chi}_\alpha(z)=\widetilde{\chi}_\beta(z)\quad\text{for all } z\in Z(\widetilde{G}) \tag{4.51)(a}$$

$$Ad^*(g)(d\chi_\alpha)=d\chi_\beta\text{ for some } g\in Aut(\mathfrak{g}). \tag{4.51)(b}$$

Lemma 4.52 *Assume G is oddly laced and $\widetilde{G}$ is an admissible cover of G.*
(1) Suppose $\widetilde{\chi}_\alpha$ is a genuine character of $Z(\widetilde{H}_\alpha)$. A Cayley transform $\widetilde{\chi}_\beta$ of $\widetilde{\chi}_\alpha$ exists if and only if

$$\langle d\widetilde{\chi}_\alpha, \alpha^\vee\rangle \in \mathbb{Z} + \frac{1}{2}. \tag{4.53}$$

Assume this holds. Define $\widetilde{\chi}_\beta$ by (4.37)(a) *and*

$$\widetilde{\chi}_\beta(\widetilde{\exp}(xZ_\beta)) = e^{\pm\langle d\widetilde{\chi}_\alpha,\alpha^\vee\rangle ix}. \tag{4.54)(a}$$

If α is type II either sign is allowed. If α is type I let $\widetilde{m}_\beta = \widetilde{\exp}(\pi Z_\beta) \in Z(\widetilde{H}_\alpha) \cap Z(\widetilde{H}_\beta)$. Note that $\widetilde{\chi}_\beta(m_\beta) = \pm i$. Then there is a unique choice of sign, satisfying

$$e^{\pm i\pi\langle d\widetilde{\chi}_\alpha,\alpha^\vee\rangle} = \widetilde{\chi}(m_\beta). \tag{4.54)(b}$$

(2) Suppose $\widetilde{\chi}_\beta$ is a genuine character of $Z(\widetilde{H}_\beta)$. Then there are two choices of Cayley transform χ_α of χ_β, given by (4.37)(a) *and*

$$\widetilde{\chi}_\alpha(\widetilde{\exp}(xZ_\alpha)) = e^{\langle d\widetilde{\chi}_\beta,\beta^\vee\rangle x.} \tag{4.54)(c}$$

Proof. For (1) it is immediate from the definition that $\widetilde{\chi}_\beta$ is defined by (4.37)(a) and (4.54)(a). If α is type II then $Z(\widetilde{H}_\alpha) \cap Z(\widetilde{H}_\beta) \cap \widetilde{B_\beta} = 1$, and there is no further condition. If α is type I then $\widetilde{m}_\beta \in Z(\widetilde{H}_\alpha) \cap Z(\widetilde{H}_\beta) \cap \widetilde{B_\beta}$, which gives condition (4.54)(b). Case (2) is similar, and easier. □

Remark 4.55 It is easy to see that in the cases when there are two Cayley transform $\widetilde{\chi}_\pm$ in Lemma 4.52, $\widetilde{\chi}_\pm$ are conjugate by $\widetilde{G}$. In [10] and elsewhere Cayley transforms are used to construct "pseudocharacters", in which one is concerned only about G-conjugacy classes. In this sense Cayley transforms for $\widetilde{G}$ are single-valued.

5 Characters of Cartan Subgroups.

Let $(\widetilde{G}, G, \overline{G})$ be an admissible triple. The definition of transfer factors involves certain factors defined for each Cartan. In this section we define these and establish their properties. We apply this to the definition of transfer factors in Section 6.

Let H be a Cartan subgroup of G and let $\Phi = \Phi(G, H)$. Define the character $\zeta_{\text{cx}} = \zeta_{\text{cx}}(G, H)$ of $\Gamma(\overline{H})$ as in [2, Section 15]. That is $\zeta_{\text{cx}} = \prod_S \alpha$ where S is a set of complex roots such that the set of all complex roots is $\{\pm\alpha, \pm\sigma\alpha \mid \alpha \in S\}$. This is independent of the choice of S, factors to G_{ad}, and satisfies

$$\text{(5.1)(a)} \qquad \zeta_{\text{cx}}(g) = \zeta_{\text{cx}}(wg) \quad (w \in W(\overline{G}, \overline{H}), g \in \Gamma(\overline{H}))$$

$$\text{(5.1)(b)} \qquad \zeta_{\text{cx}}(h) = e^{\rho_{cx}}(h) = e^{\rho-\rho_r}(h) \quad (h \in \Gamma(\overline{H}))$$

for any real set of positive roots (Section 2).

Suppose Φ^+ is a set of positive roots and $\widetilde{\chi}$ is a genuine character of $Z(\widetilde{G})$. Since $G_d(\mathbb{C})$ is acceptable ρ exponentiates to a character of $(H \cap G_d)^0$, and $\widetilde{\chi}^2 e^\rho$ is defined on $(H \cap G_d)^0$. By Definition 3.14(3c) this character factors to $\overline{H}_d^0$, where $\overline{H}_d = \overline{H} \cap \overline{G}_d$.

Definition 5.2 *Let H be a Cartan subgroup of G. Choose*

1. *a real set of positive roots Φ^+ (see Section 2),*
2. *a genuine character $\widetilde{\chi}$ of $Z(\widetilde{H})$,*
3. *a character χ of $\overline{H}$.*

Assume these satisfy:

$$\text{(5.3)(a)} \qquad \chi(h) = (\widetilde{\chi}^2 e^\rho)(h), \quad (h \in \overline{H}_d^0)$$

$$\text{(5.3)(b)} \qquad \chi(h) = \zeta_{cx}(h) \quad (h \in \Gamma_R(\overline{H})).$$

Fix $(\Phi^+, \widetilde{\chi}, \chi)$ satisfying these conditions. Suppose $\widetilde{g} \in \widetilde{H}, h \in \overline{H}$ satisfy $p(\widetilde{g}) = \phi(h)$. Then we define

$$\text{(5.4)} \qquad \Gamma(\widetilde{\chi}, \chi)(\widetilde{g}, h) = \frac{\widetilde{\chi}(\widetilde{g})}{\chi(h)}$$

Let

$$\text{(5.5)(a)} \qquad \mathcal{S}(H, \Phi^+) = \{(\widetilde{\chi}, \chi)\} \text{ satisfying (5.3)(a) and (b),}$$

$$\text{(5.5)(b)} \qquad \mathcal{T}(H, \Phi^+) = \{\Gamma(\widetilde{\chi}, \chi) \mid (\widetilde{\chi}, \chi) \in \mathcal{S}(H, \Phi^+)\},$$

$$\text{(5.5)(c)} \qquad \mathcal{S}(H) = \cup_{\Phi^+} \mathcal{S}(H, \Phi^+),$$

$$\text{(5.5)(d)} \qquad \mathcal{T}(H) = \cup_{\Phi^+} \mathcal{T}(H, \Phi^+)$$

(the unions are over real sets of positive roots).

We begin with some elementary properties.

Lemma 5.6 *Fix real Φ^+ and $(\widetilde{\chi},\chi)\in\mathcal{S}(H,\Phi^+)$.*
(1) Suppose μ is a character of $Z_0(H)$. Then $(\widetilde{\chi}\mu,\chi(\mu\circ\phi))\in\mathcal{S}(H,\Phi^+)$ and

$$\Gamma(\widetilde{\chi}\mu,\chi(\mu\circ\phi))=\Gamma(\widetilde{\chi},\chi). \tag{5.7}$$

(2) Suppose Φ_1^+ is another real set of positive roots. There exists a character $\tau=\tau(\Phi^+,\Phi_1^+)$ such that $(\widetilde{\chi},\chi\tau)\in\mathcal{S}(H,\Phi_1^+)$, and

$$\Gamma(\widetilde{\chi},\chi)(\widetilde{g},h)=\Gamma(\widetilde{\chi},\chi\tau)(\widetilde{g},h) \tag{5.8}$$

for all $h\in\Gamma(\overline{H})Z(\overline{G}),\widetilde{g}\in\widetilde{H}$ satisfying $p(\widetilde{g})=\phi(h)$.

Proof. The first part is straightforward. For the second define $\tau(h)=e^{\rho(\Phi^+)-\rho(\Phi_1^+)}(h)$ for $h\in\overline{H}^0_d$, and $\tau(h)=1$ for $h\in\Gamma(\overline{H})Z(\overline{G})$. It is easy to see that the fact that both Φ^+ and Φ_1^+ are real implies $e^{\rho(\Phi^+)-\rho(\Phi_1^+)}(h)=1$ for all $h\in\Gamma(\overline{H})Z(\overline{G})\cap\overline{H}^0_d$, so τ is well defined. It follows easily that it has the desired properties. $\square$

If ψ is a one-dimensional representation of $\overline{G}$ define

$$\psi\cdot\Gamma(\widetilde{\chi},\chi)=\Gamma(\widetilde{\chi},\chi\psi|_{\overline{H}}) \tag{5.9}$$

This defines an action of the characters of $\overline{G}$ on $\mathcal{T}(H,\Phi^+)$.

Lemma 5.10 *The space $\mathcal{T}(H,\Phi^+)$ is non-empty, and the action of the group of characters of $\overline{G}$ on $\mathcal{T}(H,\Phi^+)$ is transitive. If H is maximally split it is simply transitive.*

Proof. Fix any genuine character $\widetilde{\chi}$ of $Z(\widetilde{H})$. For existence it is enough to show

$$\zeta_{\mathrm{cx}}(h)=e^{\rho}\widetilde{\chi}^2(h)\quad h\in\Gamma_R(\overline{H})\cap\overline{H}^0_d. \tag{5.11}$$

Note that $p^{-1}(\overline{H}^0_d)\subset p^{-1}(\overline{H}^0)=H^0C\subset H^0Z_0(G)=Z_0(H)$ by 3.14(3b)) and Proposition 4.29. Pulling back to G it is therefore enough to show

$$\zeta_{\mathrm{cx}}(h)=e^{\rho}\widetilde{\chi}^2(h)\quad h\in\Gamma_R(H)\cap Z_0(H). \tag{5.12}$$

Any element $h \in \Gamma_R(H)$ may be written $h = \prod_{i=1}^n m_{\alpha_i}$ where $\{\alpha_1, \dots, \alpha_n\}$ are simple roots for Φ_R^+. If $h \in Z_0(H)$ then as in [2, Section 14] $\widetilde{\chi}^2(h) = (-1)^n$. Therefore

$$\widetilde{\chi}^2(h) = e^{\rho_r}(h) \quad h \in \Gamma_R(H) \cap Z_0(H). \tag{5.13}$$

We conclude that

$$\begin{aligned} \widetilde{\chi}^2 e^{\rho}(h) &= e^{\rho_r}(h) e^{\rho_r + \rho_{cz} + \rho_i}(h) \\ &= e^{\rho_{cx}}(h) \end{aligned} \tag{5.14}$$

since $e^{2\rho_r}(h) = e^{\rho_i}(h) = 1$. Then (5.12) follows from (5.1)(b)

Suppose $(\widetilde{\chi}_1, \chi_1), (\widetilde{\chi}_2, \chi_2) \in \mathcal{S}(H, \Phi^+)$. Define a character of $\Gamma(\overline{H})Z(\overline{G})$ by

$$\psi_0(h) = \frac{\chi_1}{\chi_2}(h) \frac{\widetilde{\chi}_1}{\widetilde{\chi}_2}(\phi(h)). \tag{5.15}$$

If $h \in \Gamma_R(\overline{H})$ then by Lemmma 3.11(3) $\phi(h) = 1$, and by (5.3)(b) $\chi_1(h) = \chi_2(h)$, so $\psi_0(h) = 1$. If $h \in \Gamma(\overline{H})Z(\overline{G}) \cap \overline{H}_d^0$ then

$$\psi_0(h) = \frac{(\widetilde{\chi}_1^2 e^{\rho})(h)}{(\widetilde{\chi}_2^2 e^{\rho})(h)} \frac{\widetilde{\chi}_2(\phi(h))}{\widetilde{\chi}_1(\phi(h))} = 1 \tag{5.16}$$

Therefore ψ_0 is trivial on $\Gamma_R(\overline{H})(\Gamma(\overline{H})Z(\overline{G}) \cap \overline{H}_d^0)$. By (3.10)(b) $\Gamma_R(\overline{H})\overline{H}_d^0 = \overline{H}_d$, and $\Gamma_R(\overline{H})(\Gamma(\overline{H})Z(\overline{G}) \cap \overline{H}_d^0) = \Gamma(\overline{H})Z(\overline{G}) \cap \overline{H}_d$. Therefore $\overline{H}/\overline{H}_d = \Gamma(\overline{H})Z(\overline{G})\overline{H}_d/\overline{H}_d \simeq \Gamma(\overline{H})Z(\overline{G})/\Gamma(\overline{H})Z(\overline{G}) \cap \overline{H}_d$, so ψ_0 defines a character of $\overline{H}/\overline{H}_d$. We have a surjection

$$\mathrm{Hom}(\overline{G}/\overline{G}_d, \mathbb{C}) \to \mathrm{Hom}(\overline{H}/\overline{H}_d, \mathbb{C}) \tag{5.17}$$

dual to the injection $\overline{H}/\overline{H}_d \hookrightarrow \overline{G}/\overline{G}_d$, which is an isomorphism if H is maximally split. If we choose any preimage ψ of ψ_0 in (5.17) then it is easy to see that $\Gamma(\widetilde{\chi}_2, \chi_2) = \Gamma(\widetilde{\chi}_1, \chi_1)\psi$. It follows that characters of $\overline{G}$ act transitively on $\mathcal{T}(H, \Phi^+)$, and this action is simply transitive if H is maximally split. □

We now need to choose elements of $\mathcal{T}(H, \Phi^+)$ consistently for all H. We do this by reducing to the most split Cartan subgroup.

Lemma 5.18 *Suppose $(\widetilde{\chi},\chi)\in\mathcal{S}(H)$. Then*

$$\chi(h)=\chi(wh)\quad(w\in W(\overline{G},\overline{H}),h\in\Gamma(\overline{H})Z(\overline{G})) \tag{5.19}$$

We defer the proof to the end of this section.

Fix a maximally split Cartan subgroup $H_s=T_sA_s$ of G and $(\widetilde{\chi}_s,\chi_s)\in\mathcal{S}(H_s)$. Let $\overline{H}_s=\overline{T}_x\overline{A}_s$ be the corresponding Cartan subgroup of $\overline{G}$. If $\overline{H}=\overline{TA}$ is any Cartan subgroup of $\overline{G}$ there exists $x\in\overline{G}$ so that $x\overline{A}x^{-1}\subset\overline{A}_s$, and therefore $x\Gamma(\overline{H})x^{-1}\subset\Gamma(\overline{H}_s)$. Suppose $h\in\Gamma(\overline{H})Z(\overline{G}),\widetilde{g}\in\widetilde{H}$ satisfy $p(\widetilde{g})=\phi(h)$. By Lemmas 3.11(5) and 3.12 $\phi(h)\in Z_0(G)$, so $\widetilde{g}\in Z(\widetilde{G})\subset Z(\widetilde{H_s})$. Define

$$\Gamma(\widetilde{\chi}_s,\chi_s)(h,\widetilde{g})=\frac{\widetilde{\chi}_s(\widetilde{g})}{\chi_s(xhx^{-1})}. \tag{5.20}$$

By the Lemma this is independent of the choice of x.

Proposition 5.21 *Fix a maximally split Cartan subgroup H_s and $(\widetilde{\chi}_s,\chi_s)\in\mathcal{S}(H_s)$. Suppose H is a Cartan subgroup, $\widetilde{\chi}\in X_g(\widetilde{H})$ and Φ^+ is a real set of positive roots for H. Then there exists unique $\chi\in X(\overline{H})$ such that $(\widetilde{\chi},\chi)\in\mathcal{S}(H,\Phi^+)$ and*

$$\Gamma(\widetilde{\chi}_s,\chi_s)(\widetilde{g},h)=\Gamma(\widetilde{\chi},\chi)(\widetilde{g},h) \tag{5.22}$$

for all $h\in\Gamma(\overline{H})Z(\overline{G})$, $\widetilde{g}\in\widetilde{H}$ satisfying $p(\widetilde{g})=\phi(h)$.

We say $(\widetilde{\chi},\chi)\in\mathcal{S}(H,\Phi^+)$ are *compatible with* $(\widetilde{\chi}_s,\chi_s)$ if they satisfy the conditions of the Proposition, in particular

$$\frac{\widetilde{\chi}(\widetilde{g})}{\chi(h)}=\frac{\widetilde{\chi}_s(\widetilde{g})}{\chi_s(xhx^{-1})} \tag{5.23)(a}$$

for all $h,\widetilde{g}$ as in the Proposition, and x as in the preceding discussion. Using the fact that $\phi(h)=\widetilde{g}$ and that $\widetilde{\chi}_s$ and $\widetilde{\chi}$ are both genuine this condition can be expressed

$$\chi(h)=(\widetilde{\chi}/\widetilde{\chi}_s)(\phi(h))\chi_s(xhx^{-1}). \tag{5.23)(b}$$

Using Cayley transforms (Section 4.2) we will reduce to the following Lemma. Suppose H_α is a Cartan subgroup, with real root α, and H_β is the Cayley transform of H_α, with non-compact imaginary root β.

Lemma 5.24 *Suppose Φ^+_α is a real set of positive roots for H_α and $(\widetilde{\chi}_\alpha, \chi_\alpha) \in \mathcal{S}(H_\alpha, \Phi^+_\alpha)$. Suppose Φ^+_β is a real set of positive roots for H_β, and $\widetilde{\chi}_\beta$ is a genuine character of $Z(\widetilde{H}_\beta)$. Then there exists unique χ_β such that $(\widetilde{\chi}_\beta, \chi_\beta) \in X(H_\beta, \Phi^+_\beta)$ and*

$$\Gamma(\widetilde{\chi}_\alpha, \chi_\alpha)(\widetilde{g}, h) = \Gamma(\widetilde{\chi}_\beta, \chi_\beta)(\widetilde{g}, h) \tag{5.25}$$

for all $h \in \Gamma(\overline{H}_\beta)Z(\overline{G})$, $\widetilde{g} \in \widetilde{H}_\beta$ satisfying $p(\widetilde{g}) = \phi(h)$.

In addition assume $c_\alpha(\Phi^+_\alpha) = \Phi^+_\beta$. Then (5.25) holds for all $h \in \overline{H}_\alpha \cap \overline{H}_\beta$.

Proof. To avoid runaway notation let $\overline{H} = \overline{H}_\beta$ for a moment. The character χ_β is determined on $\Gamma(\overline{H})Z(\overline{G})$ by (5.25) and on $\overline{H}^0_d$ by (5.3)(a). Since $\overline{H}^0 \subset Z(\overline{G})\overline{H}^0_d$, by (3.10)(b) $\overline{H} = \Gamma(\overline{H})Z(\overline{G})\overline{H}^0_d$, and uniqueness is immediate.

For existence, by Lemma 5.6 it is enough to prove this for a single choice of Φ^+_α and Φ^+_β. It is not hard to see we can choose Φ^+_α real so that $\Phi^+_\beta = c_\alpha(\Phi^+_\alpha)$ is also real (this implies α is simple for Φ^+_r).

By 5.6(1) again we may choose $\widetilde{\chi}$ arbitrarily. We have $\langle d\widetilde{\chi}, \alpha^\vee\rangle \in \mathbb{Z} + \frac{1}{2}$ and $\langle d\chi, \alpha^\vee\rangle \in \mathbb{Z}$. By Lemma 4.52 choose $\widetilde{\chi}_\beta$ to be a Cayley transform of $\widetilde{\chi}_\alpha$. Choose $g \in \mathrm{Aut}(\mathfrak{g})$ so that $\mathrm{Ad}^*(g)(d\widetilde{\chi}_\alpha) = d\widetilde{\chi}_\beta$. By Lemma 4.40 let χ_β be the (unique) Cayley transform of χ_α satisfying $\mathrm{Ad}^*(g)(d\chi_\alpha) = d\chi_\beta$.

It is enough to show $(\widetilde{\chi}_\beta, \chi_\beta) \in \mathcal{T}(H_\beta, \Phi^+_\beta)$, for it is then obvious from the definition of Cayley transforms that (5.25) holds.

We have

$$\begin{aligned} d\chi_\beta &= \mathrm{Ad}^*(g)(d\chi_\alpha) \\ &= \mathrm{Ad}^*(g)(2d\widetilde{\chi}_\alpha + \rho(\Phi^+_\alpha)) \\ &= 2d\widetilde{\chi}_\beta + \rho(\Phi^+_\beta) \end{aligned} \tag{5.26}$$

by the choice of Φ^+_α and Φ^+_β. This verifies (5.3)(a).

We now verify (5.3)(b). Suppose γ is a real root of $\overline{H}_\beta$. Then $m_\gamma \in \overline{H}_\alpha \cap \overline{H}_\beta$, and $\chi_\beta(m_\gamma) = \chi_\alpha(m_\gamma) = \zeta_{cx}(\overline{G}, \overline{H}_\alpha)(m_\gamma)$, by (5.3)(b) for $(\widetilde{\chi}_\alpha, \chi_\alpha)$. We want to show this equals $\zeta_{cx}(\overline{G}, \overline{H}_\beta)(m_\gamma)$, i.e.

$$\zeta_{cx}(\overline{G}, \overline{H}_\alpha)(m_\gamma) = \zeta_{cx}(\overline{G}, \overline{H}_\beta)(m_\gamma). \tag{5.27}$$

If the simple factor containing α, β is of type G_2 an explicit calculation shows that both sides are 1, so assume G is simply laced.

The root $\widetilde{\gamma}$ of $\overline{H}_\alpha$ corresponding to γ is also real, and $\langle \widetilde{\gamma}, \alpha^\vee\rangle = 0$. Suppose δ is a complex root of $\overline{H}_\alpha$. The corresponding root of $\overline{H}_\beta$ is also complex unless $s_\alpha\delta = -\sigma\delta$ ($s_\alpha\delta = \sigma\delta$ does not occur), in which case it is imaginary. We claim $s_{\widetilde{\gamma}}\delta \neq \pm\sigma\delta$, i.e. $s_{\widetilde{\gamma}}\delta \neq \pm s_\alpha\widetilde{\delta}$. Since $\delta \neq \pm\alpha$ the only possibility is with the minus sign, which would give

$$2\delta = \langle\delta, \widetilde{\gamma}^\vee\rangle\widetilde{\gamma} + \langle\delta, \alpha^\vee\rangle\widetilde{\alpha}. \tag{5.28}$$

By considering rank two root systems it is easy to see this cannot happen in the simply laced case.

Therefore both δ and $s_{\widetilde{\gamma}}(\delta)$ contribute to $\zeta_{cx}(\overline{G}, \overline{H}_\alpha)(m_\gamma)$, and their total contribution is $\delta(m_\gamma)(s_{\widetilde{\gamma}}\delta)(m_\gamma) = \delta(m_\gamma)\delta(m_\gamma) = 1$.

A similar argument holds for complex roots δ of $\overline{H}_\beta$ for which $s_\beta\delta = \sigma\delta$. It follows that the terms in $\zeta_{cx}(\overline{G}, \overline{H}_\alpha)$ and $\zeta_{cx}(\overline{G}, \overline{H}_\beta)$ are the same, with the exception of those just discussed, which are 1. □

For later use we note a consequence of the last part of the proof.

Lemma 5.29 *Suppose $H_1 = T_1A_1$ and $H_2 = T_2A_2$ are Cartan subgroups with $A_1 \subset A_2$. Suppose $\alpha \in \Phi_r(G, H_1)$.*
(1)

$$\zeta_{cx}(G, H_1)(m_\alpha) = \zeta_{cx}(G, H_2)(m_\alpha).$$

(2) Let $M_1 = Cent_G(A_1)$ and suppose $\alpha \in \Phi_r(M_1, H_2)$. Then

$$\zeta_{cx}(G, H_2)(m_\alpha) = \zeta_{cx}(M_1, H_2)(m_\alpha).$$

Proof. The first equality follows from a repeated application of (5.27). The second is similar. Choose a set S of complex roots such that

$$\{\beta \in \Phi_{cx}(G, H_1)\backslash\Phi_{cx}(M, H_1) \,|\, \langle\beta, \alpha^\vee\rangle \neq 0\} = \{\pm\beta, \pm\theta\beta \,|\, \beta \in S\}. \tag{5.30}$$

If $\beta \in S$ then $s_\alpha\beta$ is also complex, is not contained in $\Phi(M_2, H_2)$, and is not equal to $\pm\beta, \pm\sigma\beta$ Then S is can be written as a union over pairs $\{\beta, s_\alpha\beta\}$ and the result follows as in the proof of (5.27). □

Proof of the Proposition. This is now straightforward. If H is maximally split there is nothing to prove. If H is obtained from H_s by a series of Cayley transforms we conclude the result by a repeated application of Lemma 5.24. It is easy to check that if the conditions of the Proposition hold for a Cartan subgroup H, they hold for every G-conjugate of H. Up to conjugacy every Cartan subgroup is conjugate to one obtained by a series of Cayley transforms from H_s, and the result follows. □

Definition 5.31 *Fix* $(\widetilde{\chi}_s, \chi_s) \in \mathcal{S}(H_s)$. *Suppose* H *is a Cartan subgroup and* Φ^+ *is a real set of positive roots. Choose* $\widetilde{\chi} \in X_g(\widetilde{H})$. *Let* $\chi \in X(\overline{H})$ *be the character given by Proposition 5.21. Define*

$$\Gamma(H, \Phi^+) = \Gamma(\widetilde{\chi}, \chi). \tag{5.32}$$

Lemma 5.33 *In the setting of the Definition* $\Gamma(H, \Phi^+)$ *is independent of the choice of* $\widetilde{\chi}$.

This follows easily from Lemma 5.6.

We note some properties of $\Gamma(H, \Phi^+)$ which will be useful later. First of all its dependence on Φ^+ follows easily from Lemma 5.6(2) and its proof.

Lemma 5.34 *Suppose* Φ_1^+, Φ_2^+ *are real sets of positive roots for* H. *Suppose* $\widetilde{g} \in \widetilde{H}, h \in \overline{H}$ *satisfy* $p(\widetilde{g}) = \phi(h)$, *and write* $h = \gamma h_0$ *with* $\gamma \in \Gamma(\overline{H})$ *and* $h_0 \in \overline{H}^0$. *Then*

$$\Gamma(H, \Phi_1^+)(h, \widetilde{g}) = \Gamma(H, \Phi_2^+)(h, \widetilde{g}) e^{\rho(\Phi_2^+) - \rho(\Phi_1^+)}(h_0). \tag{5.35}$$

We obtain a character of the center of $\mathfrak{g}$:

Lemma 5.36 *Given* $\widetilde{\chi} \in X_g(\widetilde{H})$, *let* $\chi \in X(\overline{H})$ *be the character given by Proposition 5.21 and set*

$$\mu = d\chi - 2d\widetilde{\chi} - \rho(\Phi^+) \tag{5.37}$$

Then $\mu|_{\mathfrak{h}_d} = 0$, *and the restriction of* μ *to the center of* $\mathfrak{g}$ *is independent of* H, Φ^+ *and* $\widetilde{\chi}$.

Proof. The fact that $\mu|_{\mathfrak{h}_d} = 0$ is (5.3)(a). On the other hand $\rho(\Phi^+)|_{\mathfrak{z}} = \rho(\Phi_s^+)|_{\mathfrak{z}} = -$, and by (5.22) $2d\widetilde{\chi} - \chi$ and $2d\widetilde{\chi}_s - \chi_s$ have the same restriction to $\mathfrak{z}$. □

We conclude this section with the proof of Lemma 5.18.

Proof. We need to show $\chi(g(wg)^{-1}) = 1$ for $g \in \Gamma(\overline{H})$. This is obvious if $w \in W(\Phi_i)$. If $w \in \Phi_r$ then $g(wg)^{-1} \in \Gamma_R(\overline{H})$. Then $\chi(g(wg)^{-1}) = \zeta_{\rm cx}(g(wg)^{-1}) = 1$ by (5.3)(b) and (5.1)(a).

By [11, Proposition 3.12] $W(\overline{G}(\mathbb{C}), \overline{H}(\mathbb{C}))^\theta$, which contains $W(\overline{G}, \overline{H})$, is generated by $W(\Phi_i), W(\Phi_r)$ and elements of the form $s_\alpha s_{\theta\alpha}$ where $\langle \alpha, \theta\alpha^\vee \rangle = 0$. So it is enough to show $\chi(g(wg)^{-1}) = 1$ for w of this form.

Write $g = \overline{\exp}\pi i X$ with $X \in \mathfrak{a}$. Since $g^2 = 1$, $\alpha(X) \in \mathbb{Z}$. Then

$$(5.38) \qquad g(wg)^{-1} = e^{\pi i \alpha(X)(\alpha^\vee + \theta\alpha^\vee)} \in \Gamma(\overline{H}) \cap \overline{H}_d^0$$

Let $\rho = \rho(\Phi^+)$. Then

$$(5.39) \qquad \begin{aligned} \chi((wg)g^{-1}) &= (\widetilde{\chi}^2 e^\rho)(g(wg)^{-1}) \quad \text{(by (5.3)(a))} \\ &= (-1)^{\alpha(X)\langle 2d\widetilde{\chi}, \alpha^\vee + \theta\alpha^\vee\rangle} e^{\rho - w^{-1}\rho}(g) \end{aligned}$$

By [2, Corollary 8.5] $\langle d\widetilde{\chi}, \alpha^\vee + \theta\alpha^\vee\rangle \in \mathbb{Z}$. The fact that Φ^+ is real implies $\rho - w^{-1}\rho$ is a sum of imaginary roots and terms $\beta - \theta\beta$ with β complex. Therefore $e^{w^{-1}\rho-\rho}(g) = 1$. □

Lemma 5.40 *Let Φ^+ be a σ-stable choice of positive roots. Let $\widetilde{\chi}_i \in X_g(\widetilde{H}), i = 1, 2$, and $\chi_i = \chi(\widetilde{\chi}_i, \Phi^+)$. Let $h \in \overline{H}, \widetilde{g} \in Z(\widetilde{H})$ with $p(\widetilde{g}) = \phi(h)$. Then*

$$\frac{\widetilde{\chi}_1(\widetilde{g})}{\chi_1(h)} = \frac{\widetilde{\chi}_2(\widetilde{g})}{\chi_2(h)}.$$

6 Transfer Factors

In this section we define transfer factors for an admissible triple $(\widetilde{G}, G, \overline{G})$. There are some choices involved; we make these choices consistently for different Cartan subgroups by fixing $(\widetilde{\chi}_s, \chi_s) \in \mathcal{S}(\widetilde{H}_s)$ where H_s is a maximally split Cartan subgroup of G.

Suppose $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ is a θ-stable Cartan subalgebra, with corresponding Cartan subgroups $\widetilde{H}, H$ and $\overline{H}$ of $\widetilde{G}, G$ and $\overline{G}$, respectively. Let $\widetilde{H}', H', \overline{H}'$ be the regular elements of of these groups.

Let Φ^+ be a real set of positive roots of H in G. For $h \in \widetilde{H}', H'$ or $\overline{H}'$ define

$$(6.1)(a) \qquad \Delta^0(\Phi^+, h) = \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}(h))$$

$$(6.1)(b) \qquad \epsilon_r(\Phi^+, h) = sign \prod_{\alpha \in \Phi_r^+} (1 - e^{-\alpha}(h))$$

$$(6.1)(c) \qquad \Delta^1(\Phi^+, h) = \epsilon_r(\Phi^+, h)\Delta^0(\Phi^+, h)$$

$$(6.1)(d) \qquad |\Delta(h)| = |e^\rho(h)\Delta^0(\Phi^+, h)|$$

In the last definition we may write $|\Delta_G|, |\Delta_{\widetilde{G}}|$ or $|\Delta_{\overline{G}}|$ to indicate the group involved.

Here is the definition of transfer factors.

Definition 6.2 *Fix a maximally split Cartan subgroup H_s of G, and $(\widetilde{\chi}_s, \chi_s) \in \mathcal{S}(H_s)$. Let H be a Cartan subgroup of G. Choose a real set positive roots Φ^+ for H. Recall $\Gamma(H, \Phi^+)(h, \widetilde{g})$ is given by Definition 5.31. Suppose $h \in \overline{H}', \widetilde{g} \in \widetilde{H}'$ satisfy $p(\widetilde{g}) = \phi(h)$. Define*

$$\Delta_{\overline{G}}^{\widetilde{G}}(h, \widetilde{g}) = \frac{\Delta^1(\Phi^+, h)}{\Delta^1(\Phi^+, \widetilde{g})}\Gamma(H, \Phi^+)(h, \widetilde{g}). \tag{6.3}$$

Recall that by (5.32)

$$\Delta_{\overline{G}}^{\widetilde{G}}(h, \widetilde{g}) = \frac{\Delta^1(\Phi^+, h)}{\Delta^1(\Phi^+, \widetilde{g})}\frac{\widetilde{\chi}(\widetilde{g})}{\chi(h)} \tag{6.4}$$

for any $(\widetilde{\chi}, \chi) \in \mathcal{S}(H, \Phi^+)$, compatible with $(\widetilde{\chi}_s, \chi_s)$ in the sense of Proposition 5.21.

It is clear that $\Delta_{\overline{G}}^{\widetilde{G}}(h, \widetilde{g})$ is a genuine function of $\widetilde{g}$.

Proposition 6.5 $\Delta_{\overline{G}}^{\widetilde{G}}$ *is independent of the choice of Φ^+, and depends only on the choice of $(\widetilde{\chi}_s, \chi_s) \in \mathcal{S}(H_s)$.*

Proof. We have to show that $\frac{\Delta^1(\Phi^+,h)}{\Delta^1(\Phi^+,\widetilde{g})}$ and $\Gamma(H, \Phi^+)$ satisfy inverse transformation properties with respect to Φ^+.

Fix $\widetilde{g} \in \widetilde{H}'$ and $h \in \overline{H}'$ such that $\phi(h) = p(\widetilde{g})$. By (3.10)(b) write $h = \gamma h_0$ with $\gamma \in \Gamma(\overline{H})$ and $h_0 \in \overline{H}^0$. Suppose Φ_1^+, Φ_2^+ are two choices of real positive roots. By Lemma 5.34

$$\Gamma(H, \Phi_1^+)(h, \widetilde{g}) = \Gamma(H, \Phi_2^+)(h, \widetilde{g})e^{\rho(\Phi_2^+)-\rho(\Phi_1^+)}(h_0). \tag{6.6)(a}$$

By the definition of $\Delta_{\overline{G}}^{\widetilde{G}}$ it is enough to show

$$\frac{\Delta^1(\Phi_1^+, h)}{\Delta^1(\Phi_1^+, \widetilde{g})} = \frac{\Delta^1(\Phi_2^+, h)}{\Delta^1(\Phi_2^+, \widetilde{g})}e^{\rho(\Phi_1^+)-\rho(\Phi_2^+)}(h_0). \tag{6.6)(b}$$

Multiplying (a) and (b) gives the desired equality.

We first consider the ϵ_r term of (6.1)(c). By (3.10)(b) write $h = \gamma h_0$ with $\gamma \in \Gamma(\overline{H})$ and $h_0 \in \overline{H}^0$. Write $h_0 = t_0\exp(X)$ with $t_0 \in \overline{T}^0$ and $X \in \mathfrak{a}$. Since

h is regular $\Phi_r^+(X) = \{\alpha \in \Phi_r : \alpha(X) > 0\}$ is a choice of positive roots for Φ_r. For $\alpha \in \Phi_r$ $e^{-\alpha}(h) = e^{-\alpha}(\gamma)e^{-\alpha(X)}$. Since $\gamma^2 = 1$, $e^{-\alpha}(\gamma) = e^{\alpha}(\gamma) = \pm 1$, so for any choice of Φ^+ we have

$$\text{(6.7)(a)} \qquad \epsilon_r(\Phi^+, h) = \prod_{\alpha \in \Phi_r^+} \operatorname{sgn}(1 - e^{\alpha}(\gamma)e^{-\alpha(X)}).$$

Note that $\operatorname{sgn}(1 - e^{\alpha}(\gamma)e^{-\alpha(X)}) = 1$ unless $e^{\alpha}(\gamma) = 1$ and $\alpha(X) < 0$. Therefore

$$\text{(6.7)(b)} \qquad \begin{aligned} \epsilon_r(h, \Phi^+) &= \prod_{\alpha \in \Phi_r^+ \cap \Phi_r^-(X)} -e^{\alpha}(\gamma) \\ &= (-1)^{|\Phi_r^+ \cap \Phi_r^-(X)|} e^{\rho(\Phi_r^+) - \rho(\Phi_r^+(X))}(\gamma). \end{aligned}$$

Now write $\widetilde{g} = \widetilde{\gamma}\widetilde{g}_0$ with $p(\widetilde{\gamma}) = \phi(\gamma)$ and $p(\widetilde{g}_0) = \phi(h_0)$. By Lemma 3.11(5) $\widetilde{\gamma} \in Z(\widetilde{G})$, so $e^{\alpha}(\widetilde{\gamma}) = 1$. By a similar argument to the preceding we conclude $\epsilon_r(\widetilde{g}, \Phi^+) = (-1)^{|\Phi_r^+ \cap \Phi_r^-(X)|}$. Thus

$$\text{(6.7)(c)} \qquad \frac{\epsilon_r(h, \Phi^+)}{\epsilon_r(\widetilde{g}, \Phi^+)} = e^{\rho(\Phi_r^+) - \rho(\Phi_r^+(X))}(\gamma).$$

Now suppose Φ_1^+, Φ_2^+ are two choices of real positive roots. Letting $\Phi_{r,j}^+ = \Phi_j^+ \cap \Phi_r$ we have

$$\text{(6.7)(d)} \qquad \frac{\epsilon_r(h, \Phi_1^+)}{\epsilon_r(\widetilde{g}, \Phi_1^+)} = e^{\rho(\Phi_{r,1}^+) - \rho(\Phi_{r,2}^+)}(\gamma) \frac{\epsilon_r(h, \Phi_2^+)}{\epsilon_r(\widetilde{g}, \Phi_2^+)}$$

Now consider the Δ^0 term of (6.1)(c). Write $\Phi_1^+ = w\Phi_2^+$ for some $w \in W(\Phi)$, so that

$$\text{(6.7)(e)} \qquad \frac{\Delta^0(\Phi_1^+, h)}{\Delta^0(\Phi_1^+, \widetilde{g})} = \frac{\epsilon(w)e^{\rho(\Phi_2^+) - \rho(\Phi_1^+)}(h)\Delta^0(\Phi_2^+, h)}{\epsilon(w)e^{\rho(\Phi_2^+) - \rho(\Phi_1^+)}(\widetilde{g})\Delta^0(\Phi_2^+, \widetilde{g})}.$$

Using the fact that $\rho(\Phi_2^+) - \rho(\Phi_1^+)$ is a sum of roots, $p(\widetilde{g}) = \phi(h)$ and (3.8) we conclude

$$e^{\rho(\Phi_2^+) - \rho(\Phi_1^+)}(\widetilde{g}) = e^{\rho(\Phi_2^+) - \rho(\Phi_1^+)}(\phi(h)) = e^{2\rho(\Phi_2^+) - 2\rho(\Phi_1^+)}(h).$$

Therefore

$$\text{(6.7)(f)} \qquad \frac{\Delta^0(\Phi_1^+, h)}{\Delta^0(\Phi_1^+, \widetilde{g})} = e^{\rho(-\Phi_2^+) + \rho(\Phi_1^+)}(h) \frac{\Delta^0(\Phi_2^+, h)}{\Delta^0(\Phi_2^+, \widetilde{g})}.$$

Taking the product of (b) and (d) gives

$$\text{(6.7)(g)} \qquad \frac{\Delta^1(\Phi_1^+,h)}{\Delta^1(\Phi_1^+,\widetilde{g})} = e^{\rho(\Phi_{r,1}^+)-\rho(\Phi_{r,2}^+)}(\gamma)e^{-\rho(\Phi_2^+)+\rho(\Phi_1^+)}(h)\frac{\Delta^1(\Phi_2^+,h)}{\Delta^1(\Phi_2^+,\widetilde{g})}.$$

Writing $h=\gamma h_0$, and since $\gamma^2=1$ we have

(6.7)(h)

$$\begin{aligned} e^{\rho(\Phi_{r,1}^+)-\rho(\Phi_{r,2}^+)}(\gamma)e^{-\rho(\Phi_2^+)+\rho(\Phi_1^+)}(h) \\ &= e^{\rho(\Phi_{r,1}^+)-\rho(\Phi_{r,2}^+)-\rho(\Phi_2^+)+\rho(\Phi_1^+)}(\gamma)e^{-\rho(\Phi_2^+)+\rho(\Phi_1^+)}(h_0) \\ &= e^{\rho(\Phi_1^+)-\rho(\Phi_{r,1}^+)}(\gamma)e^{\rho(\Phi_2^+)-\rho(\Phi_{r,2}^+)}(\gamma)e^{-\rho(\Phi_2^+)+\rho(\Phi_1^+)}(h_0) \\ &= e^{\rho(\Phi_1^+)-\rho(\Phi_2^+)}(h_0) \end{aligned}$$

since (by (5.1)(b))

$$\text{(6.7)(i)} \qquad e^{\rho(\Phi_1^+)-\rho(\Phi_{r,1}^+)}(\gamma) = \zeta_{cx}(\overline{G},\overline{H})(\gamma) = e^{\rho(\Phi_2^+)-\rho(\Phi_{r,2}^+)}(\gamma).$$

Therefore

$$\text{(6.7)(j)} \qquad \frac{\Delta^1(\Phi_1^+,h)}{\Delta^1(\Phi_1^+,\widetilde{g})} = e^{\rho(\Phi_1^+)-\rho(\Phi_2^+)}(h_0)\frac{\Delta^1(\Phi_2^+,h)}{\Delta^1(\Phi_2^+,\widetilde{g})}.$$

This completes the proof. □

Lemma 6.8 *Assume $\mu \in i\mathfrak{z}^*$ where μ is defined as in (**??**). With notation as in the Definition,*

$$\text{(6.9)} \qquad |\Delta_{\overline{G}}^{\widetilde{G}}(h,\widetilde{g})| = \frac{|\Delta_{\overline{G}}(h)|}{|\Delta_{\widetilde{G}}(\widetilde{g})|}.$$

Proof. Let $h\in\overline{H}'$, $\widetilde{g}\in\widetilde{H}'$ with $\phi(h)=p(\widetilde{g})$. Then

$$|\Delta_{\overline{G}}^{\widetilde{G}}(h,\widetilde{g})| = \frac{|\Delta_{\overline{G}}(h)|e^{\rho(\Phi^+)}(\widetilde{g})\widetilde{\chi}(\widetilde{g})|}{|\Delta_{\widetilde{G}}(\widetilde{g})||e^{\rho(\Phi^+)}(h)\chi(\Phi^+,h)|}.$$

Thus it suffices to prove that

$$|e^{\rho(\Phi^+)}(\widetilde{g})\widetilde{\chi}(\widetilde{g})| = |e^{\rho(\Phi^+)}(h)\chi(\Phi^+,h)|.$$

As in Lemma 5.40 we can write $h = z\overline{\exp}X$ where $z \in \overline{\Gamma}(\overline{H})Z(\overline{G})$ and $X \in \mathfrak{h}_d$ and $\widetilde{g} = \widetilde{z}\widetilde{\exp}2X$ where $\widetilde{z} \in Z(\widetilde{G})$ with $p(\widetilde{z}) = \phi(z)$. Now $|e^{\rho(\Phi^+)}(h)| = |e^{<\rho(\Phi^+),X>}|$ and $|e^{\rho(\Phi^+)}(\widetilde{g})| = |e^{<\rho(\Phi^+),2X>}|$ Thus by Lemma 5.40,

$$|e^{\rho(\Phi^+)}(\widetilde{g})\widetilde{\chi}(\widetilde{g})||e^{-\rho(\Phi^+)}(h)\chi^{-1}(\Phi^+,h)| = |\widetilde{\chi}_s(\widetilde{z})\chi_s^{-1}(z)|.$$

Write $z = t\overline{\exp}Z$ where $k \in \overline{\Gamma}(\overline{H})Z(\overline{G}) \cap \overline{T}$ and $Z \in \mathfrak{z} \cap \mathfrak{p}$. Then $\widetilde{z} = \widetilde{t}\widetilde{\exp}2Z$ where $\widetilde{t} \in \widetilde{T}$, so that $|\widetilde{\chi}_s(\widetilde{t})| = 1 = |\chi_s(t)|$. But

$$|\widetilde{\chi}_s(\widetilde{\exp}2Z)\chi_s^{-1}(\overline{\exp}Z)| = |e^{<2d\widetilde{\chi}_s - d\chi_s, Z>}| = |e^{-<\mu,Z>}| = 1$$

since $<\mu, Z> \in i\mathbb{R}$. □

The transfer factors satisfy the following invariance property with respect to the Weyl group.

Lemma 6.10 *Suppose* $\widetilde{g} \in \widetilde{H}', h \in \overline{H}'$ *satisfy* $p(\widetilde{g}) = \phi(h)$. *Fix* $\widetilde{x} \in \widetilde{G}$ *and let* $x = \overline{p}(p(\widetilde{x})) \in \overline{G}$. *Then*

$$\Delta_{\overline{G}}^{\widetilde{G}}(xhx^{-1}, \widetilde{x}\widetilde{g}\widetilde{x}^{-1}) = \Delta_{\overline{G}}^{\widetilde{G}}(h, \widetilde{g}). \tag{6.11}$$

Proof. Fix a real set of positive roots Φ^+ and $(\chi, \widetilde{\chi}) \in \mathcal{S}(H, \Phi^+)$. Let $\mathfrak{h}_1 = \mathrm{Ad}(x)\mathfrak{h}$, $H_1 = xHx^{-1}$, $\chi_1 = x\chi x^{-1}$, and $\widetilde{\chi}_1 = \widetilde{x}\widetilde{\chi}\widetilde{x}^{-1}$. We claim that $(\chi_1, \widetilde{\chi}_1) \in \mathcal{S}(H_1)$, and that $(\widetilde{\chi}_1, \chi_1)$ is compatible with $(\widetilde{\chi}_s, \chi_s)$ in the sense of Proposition 5.21.

Condition 5.3(b) is immediate. Further, for all $t \in \overline{H}_{1,d}^0$,

$$\chi_1(t) = \chi_1(x^{-1}tx) = \widetilde{\chi}^2 e^{\rho(\Phi^+)}(x^{-1}tx) = \widetilde{\chi}_1^2 e^{\rho(\Phi_1^+)}(t)$$

so (a) holds as well.

Assume $\gamma \in \Gamma(\overline{H})Z(\overline{G})$. By (5.23)(b)

$$\begin{aligned}\chi_1(\gamma) = \chi(x^{-1}\gamma x) &= (\widetilde{\chi}/\widetilde{\chi}_s)(\phi(x^{-1}\gamma x))\chi_s(yx^{-1}\gamma xy^{-1}) \\ &= (\widetilde{\chi}_1/\widetilde{\chi}_s)(\phi(\gamma))\chi_s((yx^{-1})\gamma(yx^{-1})^{-1}).\end{aligned} \tag{6.12}$$

This proves (5.23)(b) holds for $(\widetilde{\chi}_1, \chi_1)$.

Therefore we can use $(\widetilde{\chi}_1, \chi_1)$ and Φ_1^+ to define the transfer factor for $\overline{H}_1$. But

$$\widetilde{\chi}_1(\widetilde{x}\widetilde{g}\widetilde{x}^{-1}) = \widetilde{\chi}(\widetilde{g}), \quad \chi_1(xhx^{-1}) = \chi(h);$$

$$\Delta^1(\Phi_1^+, xhx^{-1}) = \Delta^1(\Phi^+, h), \quad \Delta^1(\Phi_1^+, \widetilde{x}\widetilde{g}\widetilde{x}^{-1}) = \Delta^1(\Phi^+, \widetilde{g}).$$

Therefore

$$\Delta_{\overline{G}}^{\widetilde{G}}(xhx^{-1}, \widetilde{x}\widetilde{g}\widetilde{x}^{-1}) = \Delta_{\overline{G}}^{\widetilde{G}}(h, \widetilde{g}).$$

□

We next show see how the transfer factor depends on the choice of $(\widetilde{\chi}_s, \chi_s)$. For now we include this data in the notation and write $\Delta_{\overline{G}}^{\widetilde{G}}(\widetilde{\chi}_s, \chi_s)$.

Lemma 6.13 *Suppose* $(\widetilde{\chi}_s, \chi_s)$, $(\widetilde{\chi}'_s, \chi'_s) \in \mathcal{S}(H_s)$. *Then there is a character* $\psi : \overline{G} \to \mathbb{C}^\times$ *so that for every Cartan subgroup* H *and* $h \in \overline{H}'$, $\widetilde{g} \in \widetilde{H}'$ *with* $\phi(h) = p(\widetilde{g})$ *we have*

$$\Delta_{\overline{G}}^{\widetilde{G}}(\widetilde{\chi}_s, \chi_s, h, \widetilde{g}) = \psi(h)\Delta_{\overline{G}}^{\widetilde{G}}(\widetilde{\chi}'_s, \chi'_s, h, \widetilde{g}). \tag{6.14}$$

Conversely, suppose that $\psi : \overline{G} \to \mathbb{C}^\times$ *is a character and* $(\widetilde{\chi}_s, \chi_s) \in \mathcal{S}(H_s)$. *Then there exists* $(\widetilde{\chi}'_s, \chi'_s) \in \mathcal{S}(H_s)$ *such that (6.14) is satisfied.*

Proof. This follows readily from Lemma 5.10. □

Proof. Since $\widetilde{\chi}_s, \widetilde{\chi}'_s$ are both genuine characters of $Z(\widetilde{H}_s)$, $\widetilde{\chi}_s\widetilde{\chi}'^{-1}_s$ factors to $Z_0(H_s)$. Thus we can define a character ψ_0 by

$$\psi_0(\gamma) = (\chi'/\chi)(\gamma)(\widetilde{\chi}/\widetilde{\chi}')(\phi(\gamma)) \quad (\gamma \in \Gamma(\overline{H}_s)Z(\overline{G})). \tag{6.15}$$

We claim ψ_0 is trivial on $\Gamma(\overline{H}_s)Z(\overline{G}) \cap \overline{G}^0_d$.

If $\gamma \in \overline{\Gamma}(\Phi_{R,s})$, then $\phi(\gamma) = 1$ by Lemma 3.11 and $\chi_s(\gamma) = \chi'_s(\gamma) = \zeta_{cpx}(\overline{G}, \overline{H}_s)(z)$, so $\psi_0(\gamma) = 1$. If $z \in \Gamma(\overline{H}_s)Z(\overline{G}) \cap \overline{H}^0_{s,d}$, then

$$\psi_0(z) = \widetilde{\chi}^2_{2,s}\widetilde{\chi}^{-2}_{1,s}e^{\rho(\Phi^+_{2,s})-\rho(\Phi^+_{1,s})}(z)\widetilde{\chi}_{1,s}\widetilde{\chi}^{-1}_{2,s}(\phi(z)) = e^{\rho(\Phi^+_{2,s})-\rho(\Phi^+_{1,s})}(z) = 1$$

by Lemma **??**. Thus $\psi_0(z) = 1$ for all $z \in \overline{\Gamma}(\Phi_{R,s})(\Gamma(\overline{H}_s)Z(\overline{G}) \cap \overline{H}^0_{s,d}) = \Gamma(\overline{H}_s)Z(\overline{G}) \cap \overline{G}^0_d$. Now since $\overline{G} = \Gamma(\overline{H}_s)Z(\overline{G})\overline{G}^0_d$, we can extend ψ_0 to $\overline{G}$ by $\psi(zh_0) = \psi_0(z), z \in \Gamma(\overline{H}_s)Z(\overline{G}), h_0 \in \overline{G}^0_d$.

Now let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be an arbitrary Cartan subalgebra of $\mathfrak{g}$. We may as well assume that $\mathfrak{a} \subset \mathfrak{a}_s$. Let $\widetilde{\chi}_i \in X_g(\widetilde{H})$ and let Φ^+ be a σ-stable choice of positive roots. Let χ_i be the character of $\overline{H}$ corresponding to $\widetilde{\chi}_i$ using Φ^+ and $(\widetilde{\chi}_{i,s}, \chi_{i,s})$.

Fix $h \in \overline{H}, \widetilde{g} \in Z(\widetilde{H})$ with $p(\widetilde{g}) = \phi(h)$. Then

$$\Delta_{\overline{G}}^{\widetilde{G}}(\widetilde{\chi}_{1,s}, \chi_{1,s}, h, \widetilde{g}) = c(h, \widetilde{g})\Delta_{\overline{G}}^{\widetilde{G}}(\widetilde{\chi}_{2,s}, \chi_{2,s}, h, \widetilde{g})$$

where

$$c(h,\widetilde{g})=\frac{\widetilde{\chi}_1\widetilde{\chi}_2^{-1}(\widetilde{g})}{\chi_1\chi_2^{-1}(h)}.$$

Write $h=z\overline{\exp}X, \widetilde{g}=\widetilde{z}\widetilde{\exp}2X$ as in Lemma 5.40 where $z\in\Gamma(\overline{H})Z(\overline{G}), X\in\mathfrak{h}_d, \widetilde{z}\in Z(\widetilde{G})$, with $p(\widetilde{z})=\phi(z)$. Now by Lemma 5.40

$$c(h,\widetilde{g})=\frac{\widetilde{\chi}_{1,s}\widetilde{\chi}_{2,s}^{-1}(\widetilde{z})}{\chi_{1,s}(z)\chi_{2,s}^{-1}(z)}=\frac{\widetilde{\chi}_{1,s}\widetilde{\chi}_{2,s}^{-1}(\phi(z))}{\chi_{1,s}(z)\chi_{2,s}^{-1}(z)}=\psi_0(z)=\psi(h).$$

Conversely, fix a character ψ of $\overline{G}$ and $(\widetilde{\chi}_{1,s},\chi_{1,s})\in X(\widetilde{H}_s,\overline{H}_s)$. Define $\widetilde{\chi}_{2,s}=\widetilde{\chi}_{1,s}$ and $\chi_{2,s}=\psi\chi_{1,s}$. By Lemma **??** $(\widetilde{\chi}_{2,s},\chi_{2,s})\in X(\widetilde{H}_s,\overline{H}_s)$, and it is easy to see that it satisfies (6.14). □

7 Lifting: Basic Propertires

Assume that $(\widetilde{G},G,\overline{G})$ is an admissible triple. In this section we define lifting from $\overline{G}$ to $\widetilde{G}$ and derive some of the basic properties of lifting. This depends on the choice of a set of transfer factors. Thus we fix a maximally split Cartan subgroup H_s of G and $(\widetilde{\chi}_s,\chi_s)\in\mathcal{S}(H_s)$, and obtain a set of transfer factors by Definition 6.2.

Recall (Section 3) $\mathcal{O}(G,g)$ is the conjugacy class of $g\in G$, and

$$\mathcal{O}_{st}(G,g)=\{xgx^{-1}\,|\,x\in G(\mathbb{C})\}\cap G \tag{7.1}$$

is the stable orbit. We also have $\mathcal{O}(\overline{G},g)$ and $\mathcal{O}_{st}(\overline{G},g)$, and the map (Lemma 3.4(3)) $\phi:\mathcal{O}_{st}(\overline{G})\to\mathcal{O}_{st}(G)$.

Definition 7.2 *Suppose $\widetilde{\mathcal{O}}\in\mathcal{O}(\widetilde{G})$ is a strongly regular semisimple orbit for $\widetilde{G}$, and and $\overline{\mathcal{O}}_{st}\in\mathcal{O}_{st}(\overline{G})$ is a strongly regular semisimple stable orbit for $\overline{G}$. Let $p(\widetilde{\mathcal{O}})_{st}$ be the stable orbit for G containing $p(\widetilde{\mathcal{O}})$.*

Define $\Delta_{\overline{G}}^{\widetilde{G}}(\overline{\mathcal{O}}_{st},\widetilde{\mathcal{O}})=0$ unless $\phi(\overline{\mathcal{O}}_{st})=p(\widetilde{\mathcal{O}})_{st}$.

Suppose $\phi(\overline{\mathcal{O}})=p(\widetilde{\mathcal{O}})_{st}$. Choose $h\in\overline{\mathcal{O}}$ and $\widetilde{g}\in\widetilde{\mathcal{O}}$ such that $\phi(h)=p(\widetilde{g})$. Define

$$\Delta_{\overline{G}}^{\widetilde{G}}(\overline{\mathcal{O}}_{st},\widetilde{\mathcal{O}})=\Delta_{\overline{G}}^{\widetilde{G}}(h,\widetilde{g}). \tag{7.3}$$

By Lemma 6.10 this is independent of the choices of h and $\widetilde{g}$.

Definition 7.4 *Suppose $\overline{\theta}$ is a stable virtual character of $\overline{G}$. Then $\theta(\overline{\mathcal{O}}_{st})$ is defined for any strongly regular semisimple orbit $\overline{\mathcal{O}}_{st}$. Suppose $\widetilde{\mathcal{O}}$ is a strongly regular semisimple orbit for $\widetilde{G}$. Define*

$$\text{(7.5)} \qquad Lift_{\overline{G}}^{\widetilde{G}}(\overline{\theta})(\widetilde{\mathcal{O}}) = \sum_{\overline{\mathcal{O}}_{st}\in\mathcal{O}_{st}(\overline{G})} \Delta_{\overline{G}}^{\widetilde{G}}(\overline{\mathcal{O}}_{st}, \widetilde{\mathcal{O}})\overline{\theta}(\overline{\mathcal{O}}_{st}).$$

This is a finite sum, over

$$\text{(7.6)} \qquad \{\overline{\mathcal{O}}_{st} \mid \phi(\overline{\mathcal{O}}_{st}) = p(\widetilde{\mathcal{O}})_{st}\}.$$

Thus $\text{Lift}_{\overline{G}}^{\widetilde{G}}\Theta$ is a genuine class function defined on the set of strongly regular semisimple orbits in $\widetilde{G}$.

Lemma 7.7 *Suppose $\widetilde{g}$ is a strongly regular semisimple element. Let $H = Cent_G(p(\widetilde{g}))$ and $\overline{H} = \overline{p}(H)$. Then*

$$\text{(7.8)} \qquad Lift_{\overline{G}}^{\widetilde{G}}(\overline{\theta})(\widetilde{g}) = \sum_{\{h\in\overline{H} \mid \phi(h)=p(\widetilde{g})\}} \Delta_{\overline{G}}^{\widetilde{G}}(h, \widetilde{g})\overline{\theta}(h).$$

In particular $Lift_{\overline{G}}^{\widetilde{G}}(\overline{\theta})(\widetilde{g}) = 0$ unless $p(\widetilde{g})$ is in the image of ϕ. Assume this holds, and choose h satisfying $\phi(h) = p(\widetilde{g})$. Then

$$\text{(7.9)} \qquad Lift_{\overline{G}}^{\widetilde{G}}(\overline{\theta})(\widetilde{g}) = \sum_{\{t\in\overline{H} \mid \phi(t)=1\}} \Delta_{\overline{G}}^{\widetilde{G}}(th, \widetilde{g})\overline{\theta}(th).$$

Proof. Let $g = p(\widetilde{g})$. The assertion is clear unless g is in the image of ϕ, so assume this holds.

We claim

$$\text{(7.10)} \qquad \{\overline{\mathcal{O}}_{\text{st}} \mid \phi(\overline{\mathcal{O}}_{\text{st}}) = p(\widetilde{\mathcal{O}})_{\text{st}}\} = \{\mathcal{O}_{\text{st}}(\overline{G}, h) \mid \phi(h) = g\}.$$

Suppose $\phi(\overline{\mathcal{O}}_{\text{st}}) = p(\widetilde{\mathcal{O}})_{\text{st}}$. Then $\phi(h') = g'$ for some $h' \in \overline{\mathcal{O}}_{\text{st}}$ and where g' is $G(\mathbb{C})$ conjugate to g. Since ϕ takes stable orbits to stable orbits, this implies $\phi(h) = g$ for some $h \in \overline{\mathcal{O}}_{\text{st}}$. Therefore the left hand side is contained in the right.

We just need to show that if $\phi(h) = \phi(h') = g$ then $\mathcal{O}_{\text{st}}(\overline{G}, h) \neq \mathcal{O}_{\text{st}}(\overline{G}, h')$. Suppose not, so there exists $x \in \overline{G}(\mathbb{C})$ such that $xhx^{-1} = h'$. Choose a preimage y of x in $G(\mathbb{C})$. Then $g = \phi(h') = \phi(xhx^{-1}) = y\phi(h)y^{-1} = ygy^{-1}$. Since g is strongly regular this implies $y \in H$, so $x \in \overline{H}$ and $h' = h$.

Since ϕ restricted to $\overline{H}$ is a homomorphism the final assertion is clear. □

We derive some elementary properties of lifting.

Lemma 7.11 $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\Theta)$ *is supported on* $Z(\widetilde{G})\widetilde{G}_d^0$.

Proof. Let $\overline{H}$ be a Cartan subgroup of $\overline{G}$. Then $\overline{H} = Z(\overline{G})\Gamma(\overline{H})\overline{H}_d^0$, so that $\phi(\overline{H}) \subset Z_0(G)H_d^0$. Thus $p^{-1}\phi(\overline{H}) \subset Z(\widetilde{G})\widetilde{G}_d^0$. □

Define $\widetilde{S} = p^{-1}\phi(Z(\overline{G}))$. Then

$$Z(\widetilde{G})^0 \subset \widetilde{S} \subset Z(\widetilde{G}). \tag{7.12}$$

Lemma 7.13 *Let* H *be a Cartan subgroup of* G. *Suppose* $\widetilde{g} \in \widetilde{H}', h \in \overline{H}'$ *satisfy* $\phi(h) = p(\widetilde{g})$. *Suppose we are also given* $\widetilde{z} \in \widetilde{S}, z \in Z(\overline{G})$ *satisfying* $\phi(z) = p(\widetilde{z})$. *Then*

$$\Delta_{\overline{G}}^{\widetilde{G}}(\widetilde{z}\widetilde{g}, zh) = \widetilde{\chi}_s(\widetilde{z})\chi_s(z^{-1})\Delta_{\overline{G}}^{\widetilde{G}}(\widetilde{g}, h). \tag{7.14}$$

Proof. Let Φ^+ be a real set of positive roots and choose $(\widetilde{\chi}, \chi) \in \mathcal{S}(H, \Phi^+)$. Since $z \in Z(\overline{G})$ and $\widetilde{z} \in \widetilde{S} \subset Z(\widetilde{G})$, we have

$$\Delta^1(\Phi^+, zh) = \Delta^1(\Phi^+, h),\ \Delta^1(\Phi^+, \widetilde{z}\widetilde{g}) = \Delta^1(\Phi^+, \widetilde{g}).$$

Furthermore

$$\frac{\widetilde{\chi}(\widetilde{z}\widetilde{g})}{\chi(zh)} = \frac{\widetilde{\chi}(\widetilde{z})}{\chi(z)}\frac{\widetilde{\chi}(\widetilde{g})}{\chi(h)} = \frac{\widetilde{\chi}_s(\widetilde{z})}{\chi_s(z)}\frac{\widetilde{\chi}(\widetilde{g})}{\chi(h)} \tag{7.15}$$

since $\frac{\widetilde{\chi}(\widetilde{z})}{\chi(z)} = \frac{\widetilde{\chi}_s(\widetilde{z})}{\chi_s(z)}$ by (5.23)(a). Inserting this into the defintion of $\Delta_{\overline{G}}^{\widetilde{G}}$ gives the result. □

Define $Z_1(\overline{G}) = \{z \in Z(\overline{G}) : \phi(z) = 1\}$.

Lemma 7.16 *Let* Θ *be a stable character of* $\overline{G}$ *with central character* ζ. *Then* $Lift_{\overline{G}}^{\widetilde{G}}\Theta \equiv 0$ *unless* $\zeta(z) = \chi_s(z)$ *for all* $z \in Z_1(\overline{G})$.

Proof. Fix a Cartan subgropu H, $\widetilde{g} \in \widetilde{H}'$, and $z \in Z_1(\overline{G})$. Then

$$\{h \in \overline{H} : \phi(h) = p(\widetilde{g})\} = \{zh : h \in \overline{H}, \phi(h) = p(\widetilde{g})\}.$$

Further, for all $h \in \overline{H}$ such that $\phi(h) = \widetilde{p}(\widetilde{g})$,

$$\Delta_{\overline{G}}^{\widetilde{G}}(\widetilde{g}, zh) = \chi_s(z^{-1})\Delta_{\overline{G}}^{\widetilde{G}}(\widetilde{g}, h)$$

by Lemma 7.13 applied to $h, \widetilde{g}, z$, and $\widetilde{z} = 1$. Thus

$$Lift_{\overline{G}}^{\widetilde{G}}(\Theta)(\widetilde{g}) = \sum_{\{h \in \overline{H} : \phi(h) = p(\widetilde{g})\}} \Delta_{\overline{G}}^{\widetilde{G}}(\widetilde{g}, zh)\Theta(zh) = \zeta(z)\chi_s(z^{-1})Lift_{\overline{G}}^{\widetilde{G}}(\Theta)(\widetilde{g}).$$

□

Fix a character ζ of $Z(\overline{G})$. Assume that $\zeta(z) = \chi_s(z)$ for all $z \in Z_1(\overline{G})$. Then $\zeta\chi_s^{-1}$ defines a character of $Z(\overline{G})/Z_1(\overline{G})$, and ϕ induces an isomorphism of this group and $\phi(Z(\overline{G})) = p(\widetilde{S})$. We can pull this character back to a character of $\widetilde{S}$. Tensoring with $\widetilde{\chi}_s$ we obtain a genuine character of $\widetilde{S}$, which we denote $\widetilde{\zeta}$.

In other words for $\widetilde{z} \in \widetilde{S}$ choose $z \in Z(\overline{G})$ so that $\phi(z) = p(\widetilde{z})$ and define

$$\widetilde{\zeta}(\widetilde{z}) = \widetilde{\chi}_s(\widetilde{z})(\zeta\chi_s^{-1})(z). \tag{7.17}$$

This is independent of the choice of z, and is a genuine character of $\widetilde{S}$.

Lemma 7.18 *Let Θ be a stable character of $\overline{G}$ with central character ζ such that $\zeta = \chi_s$ on $Z_1(\overline{G})$. Then for all $\widetilde{g} \in \widetilde{G}, \widetilde{z} \in \widetilde{S}$,*

$$Lift_{\overline{G}}^{\widetilde{G}}(\Theta)(\widetilde{z}\widetilde{g}) = \widetilde{\zeta}(\widetilde{z})Lift_{\overline{G}}^{\widetilde{G}}(\Theta)(\widetilde{g}).$$

Proof. Fix a Cartan subgroup H of G, $\widetilde{g} \in \widetilde{H}', \widetilde{z} \in \widetilde{S}$, and $z \in Z(\overline{G})$ such that $\phi(z) = p(\widetilde{z})$. Then

$$\{h \in \overline{H} : \phi(h) = p(\widetilde{z}\widetilde{g})\} = \{zh : h \in \overline{H}, \phi(h) = p(\widetilde{g})\}.$$

Thus by Lemma 7.13,

$$Lift_{\overline{G}}^{\widetilde{G}}(\Theta)(\widetilde{z}\widetilde{g}) = \sum_{\{h \in \overline{H} : \phi(h) = p(\widetilde{g})\}} \Delta_{\overline{G}}^{\widetilde{G}}(\widetilde{z}\widetilde{g}, zh)(\Theta)(zh)$$

$$= (\zeta\chi_s^{-1})(z)\widetilde{\chi}_s(\widetilde{z})Lift_{\overline{G}}^{\widetilde{G}}(\Theta)(\widetilde{g}) = \widetilde{\zeta}(\widetilde{z})Lift_{\overline{G}}^{\widetilde{G}}(\Theta)(\widetilde{g}).$$

□

8 Invariant Eigendistributions

In this section we review results of Hirai and Harish-Chandra on invariant eigendistributions. In the next section we apply these results to study lifting from G to $\widetilde{G}$.

Let G be a reductive Lie group in Harish-Chandra's class, and let H be a Cartan subgroup of G. Suppose $F : H' \to \mathbb{C}$ is differentiable. Corresponding to $X \in \mathfrak{h}$ is the differential operator

$$D_X F(h) = \frac{d}{dt}|_{t=0} F(h\exp(tX)) \quad (h \in H'). \tag{8.1}$$

Fix a set of positive roots Φ^+, with corresponding $\rho = \rho(\Phi^+)$. Then we also define

$$D^\rho_X F(h) = \frac{d}{dt}|_{t=0} e^{<\rho,tX>} F(h\exp(tX)) \quad (h \in H'). \tag{8.2}$$

The map $D_X \to D^\rho_X =< \rho, X > +D_X$ can be extended to an automorphism $D \to D^\rho$ of $S(\mathfrak{h}(\mathbb{C}))$.

Let ν be a character of the center $\mathcal{Z}$ of the universal enveloping algebra of $\mathfrak{g}(\mathbb{C})$. [Let λ be the corresponding $\mathfrak{h}(\mathbb{C})^*$ be the character of $I(\mathfrak{h}(\mathbb{C}))$ corresponding to ν by the Harish-Chandra homomorphism.]

We say $F : H' \to \mathbb{C}$ satisfies condition (C1, Φ^+, ν) if F is real analytic on H' and

$$D^\rho F(h) = \lambda(D) F(h) \quad (h \in H', D \in I(\mathfrak{h}(\mathbb{C}))). \tag{C1,Φ^+,ν}$$

Let $H'(R) = \{h \in H \mid e^\alpha(h) \neq 1, \alpha \in \Phi_R\}$. We say F satisfies condition (C2) if

$$F \text{ extends to a real analytic function on } H'(R). \tag{C2}$$

Now let Θ be a class function on G' and define

$$\Psi(H, \Phi^+, h) = \Delta^1(\Phi^+, h)\Theta(h) \quad (h \in H'). \tag{8.3}$$

Assume $\Psi(H, \Phi^+)$ satisfies (C1, Φ^+, ν) and (C2). Suppose α is a simple root for Φ^+_r. Let J be the Cartan subgroup obtained from H via a Cayley transform c defined by α. Let $\Phi_J^+ = c\Phi^+$ and $\rho_J = \rho(\Phi_J^+)$. Let $H(\alpha)$ be the set of $h \in H$ such that $e^\alpha(h) = 1$, but $e^\beta(h) \neq 1$ for all $\beta \neq \pm\alpha$.

Fix $h \in H(\alpha)$. Then $h\exp(t\check{\alpha}) \in H'$ for $0 \neq t$ sufficiently small, and there are well-defined one-sided limits

$$D^\rho_{\check{\alpha}}\Psi^\pm(H, \Phi^+, h) = \lim_{t\to 0^\pm} D^\rho_{\check{\alpha}}\Psi(H, \Phi^+, h\exp(t\check{\alpha})). \tag{8.4}$$

Let β be the imaginary root of J corresponding to α, so $i\beta^\vee \in \mathfrak{j}$ and $h \in J'(R)$. Thus $D_{\check{\beta}} = -iD_{i\check{\beta}}$ is defined as in (8.1), and $D^{\rho_J}_{\check{\beta}}\Psi(J, \Phi_J^+, h)$ is defined. We say $\Psi(H, \Phi^+)$ satisfies condition (C3) if

(C3) $D^\rho_{\check{\alpha}}\Psi^+(H, \Phi^+, h) - D^\rho_{\check{\alpha}}\Psi^-(H, \Phi^+, h) = 2D^{\rho_J}_{\check{\beta}}\Psi(J, \Phi_J^+, h) \quad (h \in H(\alpha)).$

The following theorem is a generalization of results of Hirai [8], [9] in the case that G is acceptable and satisfies an extra condition.

Theorem 8.5 *[7] Let Θ be a class function on G' and let ν be a character of $\mathcal{Z}$. Then Θ is an invariant eigendistribution with infinitesimal character ν if and only for every Cartan subgroup H of G and every real root β of H, there is a choice of positive roots Φ^+ such that β is simple for Φ_R^+ and the function $\Psi(H, \Phi^+)$ satisfies conditions (C1, Φ^+, ν), (C2) , and (C3).*

Moreover, suppose Θ is an invariant eigendistribution. Then $\Psi(H, \Phi^+)$ satisfies (C1, Φ^+, ν) and (C2) for every choice of Φ^+, and satisfies (C3) for every choice of Φ^+ such that β is simple for Φ_R^+.

Write $G = K\exp(\mathfrak{p})$ where K is a maximal compact subgroup and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition. Let $\|\cdot\|_G$ be a Euclidean norm on $\mathfrak{p}$ and define

$$\tau_G(k\exp X) = \|X\|, \; k \in K, X \in \mathfrak{p}. \tag{8.6}$$

Let Θ be an invariant eigendistribution on G. Then by §12 of [5], Θ is tempered if and only for each Cartan subgroup H of G there are numbers $C, r \geq 0$, so that

$$|\Delta_G(h)||\Theta(h)| \leq C(1 + \tau_G(h))^r, \; g \in H', \tag{8.7}$$

where $|\Delta_G(h)|$ is defined as in (6.1)(d).

Let $\mathfrak{z}$ be the center of $\mathfrak{g}$. As in [5] we can decompose $G = {}^0GZ_p$ where ${}^0G$ has Lie algebra $[\mathfrak{g},\mathfrak{g}] \oplus (\mathfrak{z} \cap \mathfrak{k})$ and $Z_p = \exp(\mathfrak{z} \cap \mathfrak{p})$. Let Θ be an invariant eigendistribution on G. We say Θ is relatively tempered (relatively supertempered) on G if its restriction to ${}^0G$ is tempered (supertempered) on ${}^0G$. By §4 of [6], Θ is supertempered on ${}^0G$ if and only if for every Cartan subgroup ${}^0H$ of ${}^0G$ and every $r \geq 0$,

$$\sup_{g \in {}^0H'} |\Delta_G(h)||\Theta(h)|(1+\tau_G(h))^r < \infty. \tag{8.8}$$

We will also use the following result from §4 of [6].

Theorem 8.9 *[6] Assume that G has a relatively compact Cartan subgroup B and let Θ be a relatively supertempered invariant eigendistribution such that $\Theta(b) = 0, b \in B'$. Then $\Theta = 0$.*

9 Lifting

We now return to the context of lifting. That is, we assume that $(\widetilde{G}, G, \overline{G})$ is an admissible triple and fix $(\widetilde{\chi}_s, \chi_s) \in X(\widetilde{H}_s, \overline{H}_s)$ where $\mathfrak{h}_s$ is a maximally split Cartan subalgebra of $\mathfrak{g}$. All three of the groups $\widetilde{G}, G$, and $\overline{G}$ are in the Harish-Chandra class.

Let Θ be a stable invariant eigendistribution on $\overline{G}$ with infinitesimal character ν. Define $\widetilde{\Theta} = Lift_{\overline{G}}^{\widetilde{G}}\Theta$. We know that $\widetilde{\Theta}$ is a class function on $\widetilde{G}'$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let Φ^+ be a choice of positive roots for $\mathfrak{h}$. Define

$$\Psi(\widetilde{H}, \Phi^+, \widetilde{g}) = \Delta^1(\Phi^+, \widetilde{g})\widetilde{\Theta}(\widetilde{g}), \ \widetilde{g} \in \widetilde{H}'; \tag{9.1}$$

$$\Psi(\overline{H}, \Phi^+, h) = \Delta^1(\Phi^+, h)\Theta(h), h \in \overline{H}'. \tag{9.2}$$

The following lemma follows from Theorem 8.5.

Lemma 9.3 $\Psi(\overline{H}, \Phi^+)$ *satisfies conditions (C1, Φ^+, ν) and (C2) for every choice of Φ^+. Let $\beta \in \Phi_R^+$. Then $\Psi(\overline{H}, \Phi^+)$ satisfies condition (C3) for any choice of Φ^+ such that β is a simple root for Φ_R^+.*

Fix $\widetilde{\chi} \in X_g(\widetilde{H})$ and a σ-stable choice of Φ^+. Write $\rho = \rho(\Phi^+)$. Let $\chi = \chi(\widetilde{\chi}, \Phi^+)$ and write $\Psi(\widetilde{H}) = \Psi(\widetilde{H}, \Phi^+)$ and $\Psi(\overline{H}) = \Psi(\overline{H}, \Phi^+)$.

Lemma 9.4 *For all* $\widetilde{g} \in \widetilde{H}, X \in \mathfrak{h}$ *such that* $\widetilde{g}\widetilde{\exp}X \in \widetilde{H}'$,

$$\Psi(\widetilde{H}, \widetilde{g}\widetilde{\exp}X) = \sum_{h \in X(\widetilde{g})} d(h, \widetilde{g}) e^{<-\rho-\mu, X/2>} \Psi(\overline{H}, h\overline{\exp}(X/2)),$$

where

$$X(\widetilde{g}) = \{h \in \overline{H} : \phi(h) = p(\widetilde{g})\},$$

and for $h \in X(\widetilde{g})$,

$$d(h, \widetilde{g}) = \widetilde{\chi}(\widetilde{g})\chi^{-1}(h).$$

Proof. Let $\widetilde{g} \in \widetilde{H}'$. ¿From the definition of lifting and transfer factor it is clear that

$$\Psi(\widetilde{H}, \widetilde{g}) = \sum_{h \in X(\widetilde{g})} \widetilde{\chi}(\widetilde{g})\chi^{-1}(h)\Psi(\overline{H}, h).$$

Fix $\widetilde{g} \in \widetilde{H}, X \in \mathfrak{h}$ such that $\widetilde{g}\widetilde{\exp}X \in \widetilde{H}'$. Then

$$X(\widetilde{g}\widetilde{\exp}X) = \{h\overline{\exp}(X/2) : h \in X(\widetilde{g})\}.$$

Thus $\Psi(\widetilde{H}, \widetilde{g}\widetilde{\exp}X) =$

$$\sum_{h \in X(\widetilde{g})} d(h, \widetilde{g})\widetilde{\chi}(\widetilde{\exp}X)\chi^{-1}(\overline{\exp}(X/2))\Psi(\overline{H}, h\overline{\exp}(X/2)).$$

But

$$\widetilde{\chi}(\widetilde{\exp}X)\chi(\overline{\exp}(X/2))^{-1} = e^{<2d\widetilde{\chi}-d\chi, X/2>} = e^{<-\rho-\mu, X/2>}.$$

□

Let ν be the infinitesimal character of Θ and let $\lambda \in \mathfrak{h}(\mathbb{C})^*$ correspond to ν by the Harish-Chandra homomorphism.

Lemma 9.5 $\Psi(\widetilde{H})$ *is real analytic on* H'. *Let* $\widetilde{g} \in \widetilde{H}'$. *Then for all* $X \in \mathfrak{h}_d$,

$$D^{\rho}_X \Psi(\widetilde{H}, \widetilde{g}) = (1/2) \sum_{h \in X(\widetilde{g})} d(h, \widetilde{g}) D^{\rho}_X \Psi(\overline{H}, h).$$

For all $D \in S(\mathfrak{z}(\mathbb{C}))$,

$$D^{\rho}\Psi(\widetilde{H}, \widetilde{g}) = ((\lambda - \mu)/2)(D)\Psi(\widetilde{H}, \widetilde{g}).$$

Proof. The fact that $\Psi(\widetilde{H})$ is real analytic on $\widetilde{H}'$ follows from Lemmas 9.3 and 9.4. Let $X \in \mathfrak{h}, t \in \mathbb{R}$. Then using Lemma 9.4 we have

$$e^{<\rho,tX>}\Psi(\widetilde{H},\widetilde{g\exp}tX) = sum_{h\in X(\widetilde{g})} d(h,\widetilde{g})e^{<\rho-\mu,tX/2>}\Psi(\overline{H},h\overline{\exp}(tX/2)).$$

Thus

$$D^\rho_X\Psi(\widetilde{H},\widetilde{g}) = \sum_{h\in X(\widetilde{g})} d(h,\widetilde{g})(D^\rho_{X/2}- <\mu,X/2>)\Psi(\overline{H},h).$$

Now if $X \in \mathfrak{h}_d$, then $<\mu,X>=0$. If $X = Z \in \mathfrak{z}$, then $D_{Z/2} \in I(\mathfrak{h}(\mathbb{C}))$, so that $D^\rho_{Z/2}\Psi(\overline{H},h) = \lambda(Z/2)\Psi(\overline{H},h)$ by Lemma 9.3. Thus

$$D^\rho_Z\Psi(\widetilde{H},\widetilde{g}) = \sum_{h\in X(\widetilde{g})} d(h,\widetilde{g}) <\lambda-\mu, Z/2> \Psi(\overline{H},h) = ((\lambda-\mu)/2)(D_Z)\Psi(\widetilde{H},\widetilde{g}).$$

By induction we see that $D^\rho\Psi(\widetilde{H},\widetilde{g}) = ((\lambda-\mu)/2)(D)\Psi(\widetilde{H},\widetilde{g})$ for all $D \in S(\mathfrak{z}(\mathbb{C}))$. □

Lemma 9.6 *For all $D \in I(\mathfrak{h}(\mathbb{C}))$ and $\widetilde{g} \in \widetilde{H}'$,*

$$D^\rho\Psi(\widetilde{H},\widetilde{g}) = ((\lambda-\mu)/2)(D)\Psi(\widetilde{H},\widetilde{g}).$$

Proof. We have $I(\mathfrak{h}(\mathbb{C})) = I(\mathfrak{h}_d(\mathbb{C}))S(\mathfrak{z}(\mathbb{C}))$. By Lemma 9.5 the result is true for $D \in S(\mathfrak{z}(\mathbb{C}))$.

Let $D \in I(\mathfrak{h}_d(\mathbb{C}))$ be homogeneous of degree $k \geq 0$. Since $\Psi(\overline{H})$ satisfies (C1,ν), using Lemma 9.5 and an inductive argument, for $\widetilde{g} \in \widetilde{H}'$,

$$D^\rho\Psi(\widetilde{H},\widetilde{g}) = 2^{-k}\sum_{h\in X(\widetilde{g})} d(h,\widetilde{g})D^\rho\Psi(\overline{H},h)$$

$$= 2^{-k}\lambda(D)\sum_{h\in X(\widetilde{g})} d(h,\widetilde{g})\Psi(\overline{H},h) = (\lambda/2)(D)\Psi(\widetilde{H},\widetilde{g}).$$

But $\mu(D) = 0, D \in I(\mathfrak{h}_d(\mathbb{C}))$, so the formula is also valid for $D \in I(\mathfrak{h}_d(\mathbb{C}))$. □

Since $(\nu-\mu)/2$ corresponds to $(\lambda-\mu)/2$ by the Harish-Chandra homomorphism, we have proven that $\Psi(\widetilde{H})$ satisfies condition (C1,$(\nu-\mu)/2$).

Define

(9.7) $$\overline{H}'(R) = \{h \in \overline{H} : e^\alpha(h) \neq 1,\ \alpha \in \Phi_R\};$$

(9.8) $$\widetilde{H}'(R) = \{\widetilde{g} \in \widetilde{H} : e^{\alpha}(\widetilde{g}) \neq 1,\ \alpha \in \Phi_R\}.$$

The next lemma shows that $\Psi(\widetilde{H})$ satisfies (C2).

Lemma 9.9 $\Psi(\widetilde{H})$ *extends to a real analytic function on* $\widetilde{H}'(R)$.

Proof. Since the restriction of $\widetilde{\Theta}$ to $\widetilde{H}$ is supported on $Z(\widetilde{G})\widetilde{H}^0$, $\Psi(\widetilde{H})$ is also supported on $Z(\widetilde{G})\widetilde{H}^0$. Fix $\widetilde{t} \in Z(\widetilde{G})\widetilde{T}^0$. Then connected components of $\widetilde{t}\widetilde{H}^0 \cap \widetilde{H}'(R)$ have the form $\widetilde{t}\widetilde{T}^0\widetilde{\exp}C(\Phi_R^+)$ where $C(\Phi_R^+) = \{X \in \mathfrak{a} : \alpha(X) > 0\ \forall \alpha \in \Phi_R^+\}$ and Φ_R^+ is a choice of positive roots for Φ_R.

By Lemma 9.4, for all $X \in \mathfrak{h}$ such that $\widetilde{t}\widetilde{\exp}X \in \widetilde{H}'$, $\Psi(\widetilde{H}, \widetilde{t}\widetilde{\exp}X) =$

$$\sum_{h \in X(\widetilde{t})} d(h, \widetilde{t}) e^{<-\rho-\mu, X/2>} \Psi(\overline{H}, h\overline{\exp}(X/2)).$$

Fix $h \in X(\widetilde{t})$. Then for all $\alpha \in \Phi_R$, $e^{2\alpha}(h) = e^{\alpha}(\widetilde{t}) = 1$, so that $e^{\alpha}(h) = \pm 1$. Define $\Phi_R(h) = \{\alpha \in \Phi_R : e^{\alpha}(h) = 1\}$. Then connected components of $h\overline{H}^0 \cap \overline{H}'(R)$ have the form $h\overline{T}^0\overline{\exp}C(\Phi_R^+(h))$ where $C(\Phi_R^+(h)) = \{X \in \mathfrak{a} : \alpha(X) > 0\ \forall \alpha \in \Phi_R^+(h)\}$ and $\Phi_R^+(h)$ is a choice of positive roots for $\Phi_R(h)$. Let Φ_R^+ be a choice of positive roots for Φ_R. Then $\Phi_R^+(h) = \Phi_R^+ \cap \Phi_R(h)$ is a choice of positive roots for $\Phi_R(h)$ and $X/2 \in \mathfrak{t} + C(\Phi_R^+(h))$ for all $X \in \mathfrak{t} + C(\Phi_R^+)$. Thus $X \to e^{<-\rho-\mu, X/2>}\Psi(\overline{H}, h\overline{\exp}(X/2))$ extends to a real analytic function on $\mathfrak{t} + C(\Phi_R^+)$. Thus

$$X \to \sum_{h \in X(\widetilde{t})} d(h, \widetilde{t}) e^{<-\rho-\mu, X/2>} \Psi(\overline{H}, h\overline{\exp}(X/2))$$

extends to a real analytic function on $\mathfrak{t} + C(\Phi_R^+)$. Thus there is an open neighborhood U of $\widetilde{t}$ in $\widetilde{t}\widetilde{T}^0$ such that $\Psi(\widetilde{H})$ extends to be real analytic on $U\widetilde{\exp}(C(\Phi_R^+))$. □

Fix $\beta \in \Phi_R$ and let $\mathfrak{j}$ be the Cartan subalgebra of $\mathfrak{g}$ obtained from $\mathfrak{h}$ by the Cayley transform c in the root β.

Lemma 9.10 *Assume that* Φ^+ *is* σ*-stable,* $\beta \in \Phi^+$*, and that* $\Phi_J^+ = c\Phi^+$ *is also* σ*-stable. Then* β *is a simple root for* Φ_R^+*.*

Proof. Let $\alpha \in \Phi_R^+, \alpha \neq \beta$. If $< \alpha, \check{\beta} >= 0$, then $s_\beta\alpha \in \Phi_R^+$. Suppose that $< \alpha, \check{\beta} >\neq 0$. Then $c\alpha \in \Phi_{J,CPX}^+$. Now since $\Phi_J^+ = c\Phi^+$ is also σ-stable, $(c\alpha)^\sigma = c(s_\beta\alpha) \in c\Phi^+$, so that $s_\beta\alpha \in \Phi_R^+$. Thus $s_\beta\alpha \in \Phi_R^+$ for all $\alpha \in \Phi_R^+, \alpha \neq \beta$, and hence β is simple. □

Lemma 9.11 *Let Φ^+ be a σ-stable choice of positive roots for Φ such that $\Phi_J^+ = c\Phi^+$ is also σ-stable. Let $\widetilde{g} \in \widetilde{H}(\beta)$. Then*

$$(D^{\rho}_{\check{\beta}}\Psi)^+(\widetilde{H}, \Phi^+, \widetilde{g}) - (D^{\rho}_{\check{\beta}}\Psi)^-(\widetilde{H}, \Phi^+, \widetilde{g}) = 2D^{\rho_J}_{c\check{\beta}}\Psi(\widetilde{J}, \Phi_J^+, \widetilde{g}).$$

Proof. Fix Φ^+ and $\Phi_J^+ = c\Phi^+$ as in the lemma and drop it from the notation in the various Ψ functions. Fix $\widetilde{\chi} \in X_g(\widetilde{H})$ and $\widetilde{\chi}_J \in X_g(\widetilde{J})$, and let $\chi = \chi(\widetilde{\chi}, \Phi^+)$ and $\chi_J = \chi(\widetilde{\chi}_J, \Phi_J^+)$ be the corresponding characters of $\overline{H}$ and $\overline{J}$.

Fix $\widetilde{g} \in \widetilde{H}(\beta)$ and $h \in X(\widetilde{g})$. For all $\alpha \in \Phi$ we have $e^{2\alpha}(h) = e^{\alpha}(\widetilde{g})$. Thus for $\alpha \neq \pm\beta$ we have $e^{\alpha}(h) \neq 1$, while $e^{\beta}(h) = \pm 1$. Thus h is either regular or semiregular with respect to β.

Suppose that $e^{\beta}(h) = -1$. Then $h \in \overline{H}'$ is regular so that

$$(D^{\rho}_{\check{\beta}}\Psi)^+(\overline{H}, h) = (D^{\rho}_{\check{\beta}}\Psi)^-(\overline{H}, h).$$

Thus

$$(D^{\rho}_{\check{\beta}}\Psi)^+(\widetilde{H}, \widetilde{g}) - (D^{\rho}_{\check{\beta}}\Psi)^-(\widetilde{H}, \widetilde{g})$$
$$= (1/2) \sum_{h \in X(\widetilde{g},\beta)} d(h, \widetilde{g})[(D^{\rho}_{\check{\beta}}\Psi)^+(\overline{H}, h) - (D^{\rho}_{\check{\beta}}\Psi)^-(\overline{H}, h)],$$

where

$$X(\widetilde{g}, \beta) = \{h \in X(\widetilde{g}) : e^{\beta}(h) = 1\}.$$

Let $h \in X(\widetilde{g}, \beta) = H(\beta) \cap X(\widetilde{g})$. Then since $\Psi(\overline{H})$ satisfies (C3),

$$(D^{\rho}_{\check{\beta}}\Psi)^+(\overline{H}, h) - (D^{\rho}_{\check{\beta}}\Psi)^-(\overline{H}, h) = 2D^{\rho_J}_{c\check{\beta}}\Psi(\overline{J}, h).$$

Thus

$$(D^{\rho}_{\check{\beta}}\Psi)^+(\widetilde{H}, \widetilde{g}) - (D^{\rho}_{\check{\beta}}\Psi)^-(\widetilde{H}, \widetilde{g}) = \sum_{h \in X(\widetilde{g},\beta)} d(h, \widetilde{g}) D^{\rho_J}_{c\check{\beta}}\Psi(\overline{J}, h).$$

Let $X_J(\widetilde{g}) = \{b \in \overline{J} : \phi(b) = p(\widetilde{g})\}$. Then $X_J(\widetilde{g}) \subset J'(R)$. Thus applying Lemma 9.5 to $\widetilde{J}$,

$$D^{\rho_J}_{c\check{\beta}}\Psi(\widetilde{J}, \widetilde{g}) = \sum_{b \in X_J(\widetilde{g})} d_J(b, \widetilde{g}) D^{\rho_J}_{c\check{\beta}}\Psi(\overline{J}, b),$$

where for $b \in X_J(\widetilde{g})$,

$$d_J(b, \widetilde{g}) = \widetilde{\chi}_J(\widetilde{g})\ \chi_J(b)^{-1}.$$

Now $X_J(\widetilde{g}, c\beta) = \{b \in X_J(\widetilde{g}) : e^{c\beta}(b) = 1\} = X(\widetilde{g}, \beta)$. Let $h \in X_J(\widetilde{g}, c\beta)$. Then by Lemma **??** we have

$$\widetilde{\chi}(\widetilde{g})\chi^{-1}(h) = \widetilde{\chi}_J(\widetilde{g})\chi_J^{-1}(h),$$

so that $d(h, \widetilde{g}) = d_J(h, \widetilde{g})$. Thus to complete the proof of the lemma we must show that

$$\sum_{b \in X_J(\widetilde{g}, c\beta)^c} d_J(b, \widetilde{g}) D^{\rho_J}_{c\check{\beta}} \Psi(\overline{J}, b) = 0$$

where $X_J(\widetilde{g}, c\beta)^c = \{b \in X_J(\widetilde{g}) : e^{c\beta}(b) = -1\}$.

Write $\beta' = c\beta$, and let w denote the reflection in β'. Since β' is an imaginary root and Θ is a stable class function on $\overline{G}'$, $\Theta(wb) = \Theta(b)$ and $\epsilon_R(\Phi_J^+, wb) = \epsilon_R(\Phi_J^+, b)$ for all $b \in \overline{J}'$. Further, since $\epsilon(w) = -1$,

$$\Delta^0(\Phi_J^+, wb) = -e^{w\rho_J - \rho_J}(b)\Delta^0(\Phi_J^+, b), \ b \in \overline{J}'.$$

Thus

$$\Psi(\overline{J}, wb) = -e^{\rho_J - w\rho_J}(b)\Psi(\overline{J}, b), \ b \in \overline{J}'.$$

Fix $b \in X_J(\widetilde{g}, \beta')^c$, so that $e^{\beta'}(b) = -1$. Then $wb = b\overline{\exp}(\pi i\check{\beta}')$, $\phi(wb) = \phi(b) = p(\widetilde{g})$, and $e^{\beta'}(wb) = e^{\beta'}(b) = -1$. Thus $wb \in X_J(\widetilde{g}, \beta')^c$, and since $wb\overline{\exp}(it\check{\beta}') = w(b\overline{\exp}(-it\check{\beta}'))$ for all $t \in \mathbb{R}$,

$$e^{<\rho_J, it\check{\beta}'>}\Psi(\overline{J}, (wb)\overline{\exp}(it\check{\beta}')) = -e^{\rho_J - w\rho_J}(b)e^{<\rho_J, -it\check{\beta}'>}\Psi(\overline{J}, b\overline{\exp}(-it\check{\beta}')).$$

Thus

$$D^{\rho_J}_{c\check{\beta}}\Psi(\overline{J}, wb) = e^{\rho_J - w\rho_J}(b)D^{\rho_J}_{c\check{\beta}}\Psi(\overline{J}, b).$$

Since $(wb)b^{-1} = \overline{\exp}(\pi i\check{\beta}')$, $e^{\rho_J - w\rho_J}(b) = (-1)^{<\rho_J, \check{\beta}'>}$ and $\chi_J((wb)b^{-1}) = (-1)^{<d\chi_J, \check{\beta}'>}$. Thus

$$d_J(wb, \widetilde{g})D^{\rho_J}_{c\check{\beta}}\Psi(\overline{J}, wb) = (-1)^{<d\chi_J + \rho_J, \check{\beta}'>}d_J(b, \widetilde{g})D^{\rho_J}_{c\check{\beta}}\Psi(\overline{J}, b).$$

But since β' is noncompact,

$$< d\chi_J + \rho_J, \check{\beta}' > = < 2d\widetilde{\chi}_J + 2\rho_J, \check{\beta}' > = \equiv < 2d\widetilde{\chi}_J, \check{\beta}' > \equiv 1$$

by Lemma **??**. Thus

$$d_J(wb, \widetilde{g})D^{\rho_J}_{c\check{\beta}}\Psi(\overline{J}, wb) = -d_J(b, \widetilde{g})D^{\rho_J}_{c\check{\beta}}\Psi(\overline{J}, b).$$

Thus if $\overline{\exp}(\pi i\check{\beta}') \neq 1$ so that $wb \neq b$, the terms for b and wb cancel. If $\overline{\exp}(\pi i\check{\beta}') = 1$ so that $wb = b$, this forces

$$D^{\rho_J}_{c\check{\beta}}\Psi(\overline{J}, b) = 0.$$

□

We have shown that the functions $\Psi(\widetilde{H}, \Phi^+)$ satisfy conditions (C1,$(\nu - \mu)/2$), (C2), and (C3). Thus by Theorem 8.5 we have the following.

Theorem 9.12 *Let Θ be a stable invariant eigendistribution on $\overline{G}$ with infinitesimal character ν. Then $\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}\Theta$ is an invariant eigendistribution on $\widetilde{G}$ with infinitesimal character $(\nu - \mu)/2$ where $\mu \in \mathfrak{z}(\mathbb{C})^*$ depends on the choice of $(\widetilde{\chi}_s, \chi_s)$ used to define the transfer factor and is defined as in (**??**).*

Theorem 9.13 *Let Θ be a tempered stable invariant eigendistribution on $\overline{G}$. If $(\widetilde{\chi}_s, \chi_s)$ is chosen so that $\mu \in i\mathfrak{z}^*$, then $\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}\Theta$ is tempered on $\widetilde{G}$.*

Proof. Since $\overline{G}$ and $\widetilde{G}$ have the same Lie algebra, we can use the same Euclidean norm $\|\cdot\|$ on $\mathfrak{p}$ to define both τ_H and $\tau_{\widetilde{G}}$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Fix $\widetilde{g} \in \widetilde{H}'$ and write $\widetilde{g} = \widetilde{t}\widetilde{\exp}X$ where $\widetilde{t} \in \widetilde{T}', X \in \mathfrak{a}$. Then $\tau_{\widetilde{G}}(\widetilde{g}) = \|X\|$. Further, for any $h \in \overline{H}$ such that $\phi(h) = p(\widetilde{g})$, we have $h = t\overline{X}/2$ where $t \in \overline{T}$ with $\phi(t) = p(\widetilde{t})$. Thus $\tau_H(h) = \|X/2\| = \tau_{\widetilde{G}}(\widetilde{g})/2$.

Now, using Lemma 6.8 and (8.8) applied to $\overline{G}$,

$$|\Delta_{\widetilde{G}}(\widetilde{g})||\widetilde{\Theta}(\widetilde{g})| \leq \sum_{\{h\in\overline{H}:\phi(h)=p(\widetilde{g})\}} |\Delta_{\widetilde{G}}(\widetilde{g})||\Delta^{\widetilde{G}}_{\overline{G}}(h,\widetilde{g})||\Theta(h)|$$

$$= \sum_{\{h\in\overline{H}:\phi(h)=p(\widetilde{g})\}} |\Delta_{\overline{G}}(h)||\Theta(h)| \leq [\overline{H}_1]C(1 + \tau_{\widetilde{G}}(\widetilde{g})/2)^r,$$

where $\overline{H}_1 = \{h \in \overline{H} : \phi(h) = 1\}$. Thus there is a constant C' such that

$$|\Delta_{\widetilde{G}}(\widetilde{g})||\widetilde{\Theta}(\widetilde{g})| \leq C'(1 + \tau_{\widetilde{G}}(\widetilde{g}))^r$$

for all $\widetilde{g} \in \widetilde{H}'$. Now $\widetilde{\Theta}$ is tempered using (8.8) applied to $\widetilde{G}$. □

Theorem 9.14 *Let Θ be a stable invariant eigendistribution on $\overline{G}$. If Θ is relatively tempered, then $\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}\Theta$ is relatively tempered. If Θ is relatively supertempered, then $\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}\Theta$ is relatively supertempered.*

Proof. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then ${}^0\widetilde{H} = \widetilde{H} \cap {}^0\widetilde{G} = \widetilde{T}\widetilde{\exp}\mathfrak{a}_d$ and ${}^0\overline{H} = \overline{H} \cap {}^0\overline{G} = \overline{T}\overline{\exp}\mathfrak{a}_d$. Fix $\widetilde{g} \in {}^0\widetilde{H}'$. Then $\widetilde{g} = \widetilde{t}\widetilde{\exp}X$ where $\widetilde{t} \in \widetilde{T}', X \in \mathfrak{a}_d$. Then for any $h \in \overline{H}$ such that $\phi(h) = p(\widetilde{g})$, we have $h = t\overline{X}/2$ where $t \in \overline{T}$ with $\phi(t) = p(\widetilde{t})$. Thus $h \in {}^0\overline{H}$. Further, as in the proof of Lemma 6.8, it is easy to see that

$$|\Delta_{\widetilde{G}}(\widetilde{g})||\Delta_{\overline{G}}^{\widetilde{G}}(h, \widetilde{g})| = |\Delta_{\overline{G}}(h)|$$

for any choice of $(\widetilde{\chi}_s, \chi_s)$. The first statement now follows easily by the same argument as that of Theorem 9.13. For the second statement we use as similar argument along with (8.8) applied to both $\overline{G}$ and $\widetilde{G}$. □

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