Character Theory

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Introduction

Representation Theory of Semisimple Lie Groups

A *representation* π of a group G is a complicated object, often infinite dimensional. The *character* of π, which determines π, is a function on G.

Unexpected and deep relationships between different groups can be seen by looking at the characters of their representations. It is often much harder to see the corresponding relationship between the representations.

"Characters Matter"

Motivation Two themes in number theory:

- Trace Formula (Langlands Program) - relate representations of *algebraic* (i.e. matrix) groups H, G.

  Basic problem: compute the multiplicity of a representation in the space of automorphic forms. This is very hard. "Endoscopy" compares the multiplicities for representations of G with those for other simpler groups H.

- Theta-Correspondence - relate representations of possibly *non-algebraic* groups G_1, G_2.

  Quantum mechanics: The quantum harmonic oscillator gives rise to the oscillator representation of the non-algebraic two-fold cover of $SL(2, \mathbb{R})$, more generally $Sp(2n, \mathbb{R})$. This gives rise to relationships between representations of subgroups (G_1, G_2) of $\widetilde{Sp}(2n, \mathbb{R})$.

Problem: Bring the Theta-Correspondence into the Langlands Program.

Approach: Character Theory - Relate the representation theory of each non-algebraic group to an algebraic group, using character theory.

Finite Groups

$G =$ finite group

$GL(n, \mathbb{C}) = \{n \times n$ complex matrices $g \,|\, det(g) \neq 0\}$

$\pi : G \to GL(n, \mathbb{C})$ a representation (group homomorphism)

Theorem: (Schur) *π is determined (up to isomorphism) by its* character, *the function*

$$\Theta_\pi(g) = \mathit{Trace}(\pi(g))$$

Corollary: (Pointwise Conjugacy $\Rightarrow$ Conjugacy) *Suppose ι_1 and ι_2 are two embeddings of G in $GL(n, \mathbb{C})$, satisfying:*

$$\iota_1(g) \sim \iota_2(g)$$

for all $g \in G$. Then there exists $x \in GL(n, \mathbb{C})$ such that for all $g \in G$,

$$x\iota_1(g)x^{-1} = \iota_2(g)$$

(In other words, if for all $g \in G$ there exists $x_g \in GL(n, \mathbb{C})$ such that $x_g\iota_1(g)x_g^{-1} = \iota_2(g)$, then x_g can be chosen independent of g.)

Proof: The assumption implies

$$\begin{aligned} &\Rightarrow \quad \Theta_{\iota_1} = \Theta_{\iota_2} \\ &\Rightarrow \quad \iota_1 \simeq \iota_2 \quad \text{(Theorem)} \\ &\Rightarrow \quad x\iota_1(g)x^{-1} = \iota_2(g) \quad (g \in G) \end{aligned}$$

by definition of isomorphism.

Remark: *This result is a special feature of $GL(n,\mathbb{C})$. It fails with $GL(n,\mathbb{C})$ replaced by other groups such as $PGL(n,\mathbb{C})$ ($n \geq 3$). This is related to the failure of multiplicity one for $SL(n)$ [Blasius 1994].*

The character Θ_π is conjugation invariant:

$$\Theta_\pi(xgx^{-1}) = \Theta_\pi(g)$$

Theorem: (Frobenius) *$\{\Theta_\pi \,|\, \pi \in G^\frown\}$ is a basis of $L^2(G)^G$, the conjugation invariant functions on G*

Note that $\{\chi_\mathcal{C} \,|\, \mathcal{C}$ a conjugacy class$\}$ is also a basis of $L^2(G)^G$. The matrix relating these two bases is the character table of G. For example:

$$\delta_1 = \sum_{\pi \in G^\frown} \mathrm{dim}(\pi)\Theta_\pi$$

Equivalently, as a representation of $G \times G$ (acting by left and right translation),

$$L^2(G) \simeq \sum_{\pi \in G^\frown} \pi \otimes \pi^*$$

Reductive Groups

G=reductive Lie group over $\mathbb{F}$ real or p-adic, for example $GL(n,\mathbb{F}), Sp(2n,\mathbb{F})$

G_0: the (strongly) regular semi-simple elements. These elements are diagonalizable over an algebraic closure, and G_0 is an open and dense subset of G.

Theorem: (HC 1965, 1978) *Let π be an irreducible unitary (more generally: admissible) representation of G.*

- *The character Θ_π of π is defined as a distribution: $\Theta_\pi(f) = Tr(\pi(f))$ for f smooth and compactly supported,*

- $\Theta_\pi(f) = \int_{G_0} f(g)\Theta_\pi(g)dg$ *where Θ_π is a function on G_0,*

- *The function Θ_π is independent of the choice of Haar measure dg, and Θ_π determines π up to isomorphism.*

All of the complicated structure of an infinite dimensional representation π is therefore encoded in a single function Θ_π. This function can be very complicated, and carries deep number theoretic information.

Suppose $\mathbb{F} = \mathbb{R}$. By analogy with finite groups, the "discrete" part of $L^2(G)$ may be written

$$L^2_d(G) = \sum \pi \otimes \pi^*$$

The (countable set of) π which appear are the "discrete series" of G. In spite of this natural definition, it is very difficult to construct π directly. Harish Chandra first classified the discrete series by studying their characters [HC 1965′].

The formulas for the characters of the discrete series are very complicated. They have been computed independently in [Herb 1983], [GKM 1997], and in some special cases in [Zuckerman 1976?], [Schmid 1974].

Exercise: *Write a computer program to compute these formulas. Compare them for G of type E_8.*

Example: (Kazhdan) *The character formula of certain discrete series representation of p-adic groups involve counting the number of points on an elliptic curve over a finite field.*

Although the general character of a representation (even a finite dimensional representation of a compact group) may be complicated, certain very special representations have remarkable characters.

Example: *Let $G = Spin(n)$, the two-fold cover of $SO(n)$. The $Spin$ representation of G, of dimension $2^{[\frac{n}{2}]}$, has character*

$$|\Theta_{Spin}(g)| = \frac{1}{\sqrt{2}^c} |det(1+g)|^{\frac{1}{2}}$$

($c = 0$ or 1 respectively if n is even or odd.) This is simpler than the character of a general finite-dimensional representation of G.

Remark: *The spin representation is the smallest genuine representation of $Spin(n)$ (genuine: does not factor to $SO(n)$).*

Example: *Let $G = \widetilde{Sp}(2n)$, the two-fold cover of $Sp(2n)$ over a real or p-adic field. Let*

$$\omega = \omega_+ \oplus \omega_-$$

be the oscillator representation, and

$$\Theta_\omega = \Theta_{\omega_+} \oplus \Theta_{\omega_-}$$

its character. Then [Howe]

$$|\Theta_{\omega_+ \pm \omega_-}(g)| = |det(1 \mp g)|^{-\frac{1}{2}}$$

The phase is easily determined up to a factor $\tau(g)$ with $\tau(g)^8 = 1$. The calculation of $\tau(g)$ involves some deep questions number theory.

Example: *Let G, ω, Θ_ω be as in the preceding example, with $\mathbb{F}$ now a finite field of order q. Then [Howe 1973]*

$$\Theta_\omega(g) = \tau(g) q^{\frac{1}{2}\dim(Ker(g-1))}$$

with $\tau(g)^4 = 1$

$$\tau(g) = \Gamma(Q_g, \psi).$$

Here Γ is the standard Gauss sum associated to a certain quadratic from Q_g on $Ker(g-1)$.

Remark: *The oscillator representation is the smallest non-trivial unitary representation of $\widetilde{Sp}(2n, \mathbb{R})$.*

Problem: *Compute the character of "small" unitary representations, for example the "minimal" representation. At least in some cases these should be simpler than the character of an arbitrary representation.*

Small representations often occur only for non-algebraic groups.

Basic Principle

Suppose the semisimple conjugacy classes of two Lie groups G, H are related in some simple way. Since the character of a representation is a function on conjugacy classes, the representation theory of G, H should be related, by carrying over the characters.

More precisely, suppose there is a map

$$\Psi : H_0/\sim \to G_0/\sim$$

Write $\Psi : \mathcal{C_H} \to \mathcal{C_G}$ for the map a conjugacy classes. Given a representation π_H of H, with character Θ_π, define:

$$\Theta_G(\mathcal{C_G}) = \sum_{\mathcal{C_H}, \Psi(\mathcal{C_H}) = \mathcal{C_G}} \Phi(\mathcal{C_H}, \mathcal{C_G}) \Theta_{\pi_{\mathcal{H}}}(\mathcal{C_H}) \tag{1}$$

Here $\Phi(\mathcal{C_H}, \mathcal{C_G})$ is a factor having to do with the difference between measures on H, G.

The question is then: is Θ_G the character of a representation π_G of G? If so, study the correspondence $\pi_H \to \pi_G$.

Example: *If $H \subset G$, Θ_G is the character of the induced representation $Ind_H^G(\pi)$.*

Non-algebraic groups

The Langlands program deals with an algebraic group G, i.e. G is a closed subgroup of a matrix group $GL(n)$.

The theta correspondence (and its generalizations) involves non-algebraic groups. The *existence* of non-algebraic covers of $Sp(2n)$ is implied by quantum mechanics. Even the existence of non-algebraic covers of other groups is a deep subject, intimately related to the theory of reciprocity laws in class field theory [Moore 1968].

If G is an algebraic group over a local field $\mathbb{F}$, $H^2(G)$ (in the appropriate category) parametrizes the central extensions of G. $H^2(G)$ is almost entirely independent of G: in most cases $H^2(G) \simeq \mu(\mathbb{F})$ (the roots of unity in $\mathbb{F}$). Brylinski and Deligne have recently given a very general construction of these central extensions [BD 1998].

The representation theory of these non-algebraic groups is somewhat mysterious. Flicker, Kazhdan and Patterson studied related the character theory of cover to $GL(n)$ to $GL(n)$ itself [Flicker 1980], [FK 1986], [KP 1984]. This is the first case of program outlined at the beginning of the talk, and of (1) for non-algebraic groups.

It is difficult to see the corresponding map on representations directly. For example, it takes the trivial representation of $GL(2)$ to the oscillator representation of the two-fold cover of $GL(2)$.

Example: *Let* $H = Spin(2n+1)$ *over* $\mathbb{R}$ *(the split spin group),* $G = \widetilde{Sp}(2n, \mathbb{R})$*. Define a correspondence between semisimple conjugacy classes of* $SO(2n+1)$ *and* $Sp(2n, \mathbb{R})$ *as follows:* $\tau(g) = g'$ *if* g, g' *have the same non-trivial eigenvalues. This is a bijection between* $SO(2n+1, \mathbb{C})$ *and* $Sp(2n, \mathbb{C})$ *orbits of (strongly) regular semisimple elements.*

Proposition: *There is a unique lifting* $\tilde{\tau}$ *of* τ *to a correspondence between* $Spin(2n+1)$ *and* $\widetilde{Sp}(2n, \mathbb{R})$*, such that if* $\tau(g) = g'$ *then*

$$\Theta_{Spin}(g)\Theta_{\omega_+ - \omega_-}(g') = \frac{1}{\sqrt{2}}$$

This is an aspect of Boson-Fermion duality: the spin representation is on the exterior algebra $\Lambda(V)$, and the oscillator representation is on the symmetric algebra $S(W) \simeq S(V)$. This is also related to Koszul duality.

$\widetilde{Sp}(2n)$-$SO(2n+1)$ **duality**

$\mathbb{F}$:local, of characteristic zero, i.e. $\mathbb{R}, \mathbb{C}$ or p-adic

$V, (,)$: symmetric bilinear form

$W, <,>$: skew-symmetric (symplectic) bilinear form

$\mathbb{W} = V \otimes W, <<,>>$, skew-symmetric bilinear form

$\mathbb{W} \simeq Hom_{\mathbb{F}}(V, W)$,

$G = SO(V), G' = Sp(W)$, $\mathfrak{g} = Lie(G), \mathfrak{g}' = Lie(G')$,

$$\begin{array}{rcl} \mathrm{Hom}(V,W) & \to & \mathrm{Hom}(W,V): \\ T & \to & T^* \\ < w, Tv > & = & (T^*w, v) \end{array}$$

Lemma:

$T^*T \in \Lambda^2(V) \simeq \mathfrak{g} \subset Hom(V,V)$,

$TT^* \in S^2(W) \simeq \mathfrak{g}' \subset Hom(W,W)$,

This defines the *orbit correspondence* between G-orbits on $\mathfrak{g}$ and G'-orbits on $\mathfrak{g}'$.

Remark: $(TT^*)^m = 0$ *implies* $(T^*T)^{m+1} = 0$*, so this defines a correspondence of nilpotent orbits.*

There are only finitely many nilpotent orbits, and they form a cone.

Problem: *Study the correspondence of nilpotent orbits given by the remark in more detail. It should have some interesting consequences in geometry and representation theory.*

For characters we are mainly interested in the semisimple orbits:

Remark: $TT^*v = \lambda v$ *implies* $(T^*T)(T^*v) = \lambda T^*v$*, so* TT^* *and* T^*T *have the same non-zero eigenvalues.*

In fact:

Proposition: *Fix W and $\delta \in \mathbb{F}^*/\mathbb{F}^{*2}$. Let*

$$\mathcal{V} = \{(V, (,)) \mid \dim(V) = \dim(W) + 1, disc(V) = \delta\}/\sim$$

Then the orbit correspondence is a bijection:

$$Sp(W)_0/\sim \leftrightarrow \bigcup_{V \in \mathcal{V}} SO(V)_0/\sim$$

This very strong relationship of conjugacy classes suggests a strong relationship between the representations.

However, there is *no* natural relationship between the representations of $Sp(W)$ and $SO(V)$. The reason: the oscillator representation.

Remark: The representations of $SO(V)$ may be obtained in a simple manner from the representations of the split form $SO(V_0) \simeq SO(n+1, n)$ (at least conjecturally). So for the application to representations it is enough to work with just the split form on the right hand side.

Let $p : \widetilde{Sp}(W) \to Sp(W)$ be the metaplectic group, with oscillator representation ω as before, and let

$$\Phi = \Theta_{\omega_+} - \Theta_{\omega_-},$$

as a function on $\widetilde{Sp}(2n)_0$.

Let V be the split orthogonal space with $\dim(V) = \dim(W) + 1$. Write

$$g \leftrightarrow g' \quad (g \in SO(V), g' \in Sp(W))$$

for the orbit correspondence, and for $g' \in \widetilde{Sp}(W)$, write $g \leftrightarrow g'$ if $g \leftrightarrow p(g')$.

For π a genuine representation of $SO(V)$, with character Θ_π, define:

$$\Theta'(g') = \Phi(g')\Theta_\pi(g) \qquad (g \leftrightarrow g')$$

This is a conjugation invariant function on $\widetilde{Sp}(W)$.

This is a special case of 1.

Conjecture: *Suppose* π *is a stable representation, i.e. its character is invariant by conjugation by* $SO(2n+1,\overline{\mathbb{F}})$*. Then*

(a) Θ' *is the character of a stable representation* π'*.*

(b) $\pi \to \pi'$ *is a bijection:*

$$\widetilde{Sp}(W)\widehat{\ }_{genuine,stable} \longleftrightarrow SO(V)\widehat{\ }_{stable}$$

This reduces the study of genuine representations of $\widetilde{Sp}(W)$ *to those of* $SO(V)$*.*

Theorem: (Adams 1998) *The conjecture is true for* $\mathbb{F} = \mathbb{R}$*.*

Theorem: (Schultz 1998) *The conjecture is true if* $\mathbb{F}$ *p-adic and* $n = 1$*.*

The case of $n = 1$ is related to work of Shintani, Waldspurger and others on modular forms of half-integral weight.

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