Strong real forms and the Kac classification

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October 9, 2005

This paper is expository. It is a mild generalization of the Kac classification of real forms of a simple Lie group to *strong* real forms. The basic reference for strong real forms in this language is [1]. For the Kac classification we follow [6]. There is also a treatment in [4], in slightly different terms.

1 Real forms and strong real forms

Let G be a reductive algebraic group. We will occasionally identify algebraic groups with their complex points. We have the standard exact sequence

$$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1 \tag{1.1}$$

where $\mathrm{Int}(G) \simeq G/Z(G)$ is the group of inner automorphisms of G, $\mathrm{Aut}(G)$ is the automorphims of G, and $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$.

Definition 1.2

1. *A real form of G is an equivalence class of involutions in $Aut(G)$, where equivalence is by conjugation by G, i.e. the action of $Int(G)$.*
2. *A traditional real form of G is an equivalence class of involutions, where equivalence is by the action of $Aut(G)$.*

The real form defined by θ has a maximal compact subgroup whose complexification is $K = G^\theta$.

We say two involutions $\theta, \theta' \in \mathrm{Aut}(G)$ are inner to each other, or in the same inner class, if they have the same image in $\mathrm{Out}(G)$. Such a class is determined by an involution $\gamma \in \mathrm{Out}(G)$, and we refer to the real forms of (G, γ).

We will work entirely in a fixed inner class, so fix an involution $\gamma \in \mathrm{Out}(G)$.

Fix a splitting data for the exact sequence (1.1). This is a set $(H, B, \{X_\alpha\})$ consisting of a Cartan subgroup H, a Borel subgroup B containing H, and a set of simple root vectors. This induces a splitting $\mathrm{Out}(G) \to \mathrm{Aut}(G)$ of (1.1), and we let θ be the image of γ in $\mathrm{Aut}(G)$. Thus θ is an involution of G, corresponding to the "most compact" real form in the given inner class. We let $K = G^\theta$.

Remark 1.3 Suppose G is simple and simply connected. It does not necessarily follow that K is simply connected; it is not simply connected if and only if the real form $G = G(\mathbb{R})$ of G corresponding to K has a non-linear cover. In fact K is simply connected unless $G = SL(2n+1)$, in which case $K = SO(2n+1)$ and $\pi_1(K) = \mathbb{Z}/2\mathbb{Z}$. This exception is due to the fact that Δ_θ (cf. Lemma 3.1) is not reduced in this case. See the table in Section 3.

Let
$$G^\Gamma = G \rtimes \langle \delta \rangle$$
where $\delta^2 = 1$ and $\delta g \delta^{-1} = \theta(g)$.

Definition 1.4 *A strong real form of (G, γ) is an equivalence class of elements $x \in G^\Gamma$, satisfying $x \notin G$, and $x^2 \in Z(G)$, where equivalence is by conjugation by G.*

The map $x \to \theta_x = int(x)$ is a surjection from the set of strong real forms to the set of real forms. Let
$$H^\Gamma = H \rtimes \langle \delta \rangle \subset G^\Gamma.$$
Let T be the identity component of H^θ, and A be the identity component of $H^{\theta^{-1}}$. Then $H = TA$. Let
$$T^\Gamma = T \times \langle \delta \rangle$$

Remark 1.5 We may write

$$H \simeq \mathbb{C}^{*a} \times \mathbb{C}^{*b} \times (\mathbb{C}^* \times \mathbb{C}^*)^c \qquad (1.6)$$

where θ acts trivially on the first a factors, by inverse on the next b ones, and $\theta(z, w) = (w, z)$ on each of the last c terms. Note that if $b \neq 0$ then T is a proper subset of H^θ. This happens, for example, in $SO(3, 1)$.

Lemma 1.7 *Suppose $x \in G\delta$ is a semi-simple element (i.e. $x = g\delta$ with $g \in G$ semisimple). Then x is G-conjugate to an element of $T\delta$.*

Proof. Write $x = g\delta$ and choose a Cartan subgroup H' containing g. Write $H' = T'A'$ as usual and $g = ta$ accordingly. Choose $h \in A'$ so that $h\theta(h) = h^2 = a$. Then $hxh^{-1} = t\delta$. Since T is a Cartan subgroup of K we may choose $k \in K$ so that $ktk^{-1} \in T$. Then $(kh)x(kh)^{-1} \in T\delta$. ∎

Lemma 1.8 *The strong real forms of (G, γ) are parametrized by elements x of $T\delta$ such that $x^2 \in Z$, modulo equivalence by conjugation by G.*

Remark 1.9 Since $h \in A$ acts on $T\delta$ by multiplication by a^2, and $A^2 = A$, we may replace T with $H/A \simeq T/T \cap A$. See the end of this section.

Let $W = \mathrm{Norm}_G(H)/H$. Then θ acts on W, and we let W^θ be its fixed points. Note that W^θ acts naturally on T, A and $T \cap A$.

It is not hard to see that if two elements of $T\delta$ are G-conjugate then they are conjugate by an element normalizing $T\delta$. It follows that G-conjugacy of elements of $T\delta$ is controlled by the group W_δ of the next definition.

Definition 1.10

$$W_\delta = Norm_G(T\delta)/Cent_G(T\delta) \tag{1.11}$$

Proposition 1.12

$$W_\delta \simeq W^\theta \ltimes (A \cap T) \tag{1.13}$$

The subgroup W^θ acts by its natural action on T (fixing δ), and $A \cap T$ acts by multiplication.

Proof.

It is easy to see that

$$\mathrm{Norm}_G(T\delta) = \{g \in \mathrm{Norm}_G(T) \mid g\delta g^{-1} \in T\} \tag{1.14}$$

It is well known that

$$\mathrm{Cent}_G(T) = H$$

$$\mathrm{Norm}_K(H)/T = \mathrm{Norm}_G(H)/H$$

From these it follows that

$$\begin{aligned}\mathrm{Norm}_G(T) &= \mathrm{Norm}_G(H)\\ &= \mathrm{Norm}_K(T)H\\ &= \mathrm{Norm}_K(T)A.\end{aligned}$$

If we let

$$A_0 = \{h \in A \,|\, h^2 \in T\}$$

from (1.14) we conclude

$$\mathrm{Norm}_G(T) = \mathrm{Norm}_K(T)A_0$$

On the other hand it is immediate that

$$\mathrm{Cent}_G(T\delta) = \mathrm{Cent}_G(T) \cap \mathrm{Cent}_G(\delta) = H^\theta.$$

From (1.6) we see

$$H^\theta = TA^\theta$$

and $A^\theta = (A_0)^\theta$ which gives

$$W_\delta = \mathrm{Norm}_K(T)A_0/T \simeq \mathrm{Norm}_K(T)/T \ltimes A_0/A^\theta$$

The first term is W^θ. For the second note that the map $a \to a^2$ takes A_0 onto $A \cap T$ and there is an exact sequence

$$1 \to A^\theta \to A_0 \to A \cap T \to 1 \tag{1.15}$$

Therefore $A_0/A^\theta \simeq A \cap T$.

Finally note that if $a \in A_0, t \in T$ then $a(t\delta)a^{-1} = a^2 t\delta$, so the second factor acts by multiplication by $a^2 \in A \cap T$. This completes the proof. ∎

Remark 1.16 With respect to the decomposition (1.6) we have

$$\begin{aligned}A_0 &\simeq (\mathbb{Z}/2\mathbb{Z})^b \times (\mathbb{Z}/4\mathbb{Z})^c\\ A^\theta &\simeq (\mathbb{Z}/2\mathbb{Z})^b \times (\mathbb{Z}/2\mathbb{Z})^c\\ A \cap T &\simeq 1 \times (\mathbb{Z}/2\mathbb{Z})^c\end{aligned}$$

where $\mathbb{Z}/4\mathbb{Z} = \{\pm(1,1), \pm(i,-i)\} \subset \mathbb{C}^* \times \mathbb{C}^*$.

Proposition 1.17 *The strong real forms of (G, γ) are are parametrized by elements x of $T\delta$ satisfying $x^2 \in Z$, modulo the action of W_δ.*

Let

$$\overline{T} = T/T \cap A. \tag{1.18}$$

As in Remark 1.9 we may use $\overline{T}$ in place of T. Note that W^θ acts on $\overline{T}$. Also every element of $T \cap A$ has order 2, so the condition $x^2 \in Z$ for $x \in \overline{T}$ is well defined. This gives:

Proposition 1.19 *The strong real forms of (G, γ) are are parametrized by elements x of $\overline{T}\delta$ satisfying $x^2 \in Z$, modulo the action of W^θ.*

For several reasons it is more convenient to use W^θ acting on $\overline{T}\delta$ instead of W_δ on $T\delta$. For one thing Z acts naturally on $\overline{T}$, by multiplication on $H/A \simeq \overline{T}$. Also the "translations" by $T \cap A$ acting on $T\delta$ naturally live in the lattice part of the affine Weyl group; see Remark 3.16.

To compute the orbits of W_δ on $\overline{T}\delta$ we pass to the tangent space, in which W_δ becomes an affine Weyl group. We begin with a discussion of the basics of affine root systems and Weyl groups.

2 Affine root systems and Weyl groups

Let V be a real vector space of dimension n and E an affine space with translations V. That is V acts simply transitively on E, written $v, e \to v + e$. A function If E, E' are affine spaces a function $f : E \to E'$ is said to be *affine* if there exists a linear function $df : V \to V'$ such that

$$f(v + e) = df(v) + f(e) \quad \text{for all } v \in V, e \in E. \tag{2.1}$$

In particular if E' is one dimensional we say f is an affine linear functional. In this case $df : V \to \mathbb{R}$, i.e. $df \in V^*$. We say df is the differential of f. The set $\text{Aff}(E)$ of all affine linear functionals is a vector space of dimension $n+1$. The map $f \to df$ is a linear map from $\text{Aff}(E)$ to V^*, and this gives an exact sequence

$$0 \to \mathbb{R} \to \text{Aff}(E) \to V^* \to 0. \tag{2.2}$$

The first inclusion takes $x \in \mathbb{R}$ to the constant function $f_x(e) = x$ for all $e \in E$; this satisfies $df = 0$.

Choose an element $e_0 \in E$. This gives an isomorphism $V \simeq E$ via $v \to v + e_0$. For $\lambda \in V^*$ let $s(\lambda)(v + e_0) = \lambda(v)$. This defines a splitting of (2.2):

Lemma 2.3 *Given e_0 we obtain an isomorphism*

$$Aff(E) \simeq V^* \oplus \mathbb{R} \tag{2.4)(a}$$

According to this decomposition we write $f \in Aff(E)$ as

$$f = (\lambda, c). \tag{2.4)(b}$$

We make the isomorphism (2.4)(a) explicit. In one direction $f \in \mathrm{Aff}(E)$ goes to $\lambda = df$ and $c = f(e_0)$. For the other direction (λ, c) goes to $f \in \mathrm{Aff}(E)$ defined by $f(v + e_0) = \lambda(v) + c$.

We now assume V is equipped with a positive definite non-degenerate symmetric form $(,)$, and identify V and V^*. In particular we may identify df with an element of V. Define $(,)$ on $\mathrm{Aff}(V)$ by

$$(f, g) = (df, dg)$$

and for $f \in \mathrm{Aff}(E)$ not a constant function let

$$f^\vee = \frac{2f}{(f, f)}.$$

The affine reflection $s_f : V \to V$ is

$$\begin{aligned} s_f(v) &= v - f^\vee(v) df \\ &= v - f(v)(df)^\vee \\ &= v - \frac{2f(v)}{(f, f)} df \end{aligned}$$

Definition 2.5 (Macdonald [5]) *An affine root system on E is a subset S of $Aff(E)$ satisfying*

1. *S spans $Aff(E)$, and the elements of S are non-constant functions,*

2. *$s_\alpha(\beta) \in S$ for all $\alpha, \beta \in S$,*

3. *$\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in S$,*

4. *The Weyl group $W = W(S)$ is the group generated by the reflections $\{s_\alpha \mid \alpha \in S\}$. We require that W acts properly on V.*

The Weyl group $W(S)$ is an *affine Weyl group*. The notions of simple roots $\Pi(S)$ and Dynkin diagram $D(S)$ are simlar to those for classical root systems. Also the dual $S^\vee$ of S defined in the obvious way is an affine root system, with Dynkin diagram $D(S^\vee) = D(S)^\vee$. Here the dual of a Dynkin diagram means the same diagram with arrows reversed, as usual.

Choose a base point e_0 in E and write elements of $\mathrm{Aff}(E)$ as (λ, c) as in Lemma 2.3.

Suppose $\Delta \subset V$ is a classical (not necessarily reduced) root system. If Δ is simply laced we say each root is long. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a set of simple roots. For each i let $\widetilde{\alpha}_i = (\alpha_i, 0)$, and let $\widetilde{\alpha}_0 = (-\beta, 1)$ where β is the highest root. Note that β is long. Then $\{\widetilde{\alpha}_0, \dots, \widetilde{\alpha}_n\}$ is a set of simple roots of an affine root system denoted $\widetilde{\Delta}$.

Let $D = D(\Delta)$ be the Dynkin diagram of Δ. Let $\widetilde{D}$ be the extended Dynkin diagram of D, i.e. obtained by adjoining $-\beta$ where β is the highest root. Then the Dynkin diagram of $\widetilde{\Delta}$ is the extended Dynkin diagram of Δ, i.e.

$$D(\widetilde{\Delta}) = \widetilde{D(\Delta)}.$$

We will use Δ (resp. S) to denote a typical classical (resp. affine) root system.

Suppose Δ is a classical root system with Dykin diagram $D = D(\Delta)$. Let and $S = \widetilde{\Delta}$, so $D(S) = \widetilde{D}$. Then $S^\vee = (\widetilde{\Delta})^\vee$ is also an affine root system, with Dynkin diagram $D(S^\vee) = (\widetilde{D})^\vee$. If Δ is not simply laced then it is *not* necessarily the case that $(\widetilde{\Delta})^\vee = \widetilde{(\Delta^\vee)}$ or $(\widetilde{D})^\vee = \widetilde{(D^\vee)}$. Note that $\widetilde{D}$ is obtained from D by adding a long root, so $(\widetilde{D})^\vee$ has an extra short root. On the other hand $\widetilde{(D^\vee)}$ is obtained from $D^\vee$ by adding an extra long root.

Theorem 2.6 (Macdonald [5]) *Every reduced, irreducible affine root system is equivalent to either $\widetilde{\Delta}$ or $(\widetilde{\Delta})^\vee$ where Δ is a classical (not necessarily reduced) root system.*

Remark 2.7 A remarkable fact is that every reduced, irreducible affine root system is also obtained from a classical root system and involution, as discussed in the next section.

3 Affine root system and Weyl group associated to (Δ, θ)

Let Δ be an irreducible root system, and θ an automorphism of Δ preserving a set of simple roots. Thus θ corresponds to an automorphism of the Dynkin diagram $D = D(\Delta)$ of Δ. Let $c \in \{1, 2, 3\}$ be the order of δ. Associated to (Δ, θ) is an affine root system, which we now describe.

The quotient Δ/θ is naturally a root system [7], which we denote Δ_θ. Here are the possibilities with $\theta \neq 1$. We list the finite root systems Δ, Δ_θ, the names of the affine root system according to [5] and [6], the simply connected group G with root system Δ, the real form of G corresponding to θ, and G^θ.

Δ	Δ_θ	Δ_{aff}	Δ_{aff}	G	$G(\mathbb{R})$	K
A_{2n}	BC_n	$\widetilde{BC_n}$	$A_{2n}^{(2)}$	$SL(2n+1)$	$SL(2n+1, \mathbb{R})$	$SO(2n+1)$
A_{2n-1}	C_n	$\widetilde{B}_n^\vee$	$A_{2n-1}^{(2)}$	$SL(2n)$	$SL(n, \mathbb{H})$	$Sp(n)$
D_n	B_n	$\widetilde{C}_n^\vee$	$D_n^{(2)}$	$Spin(2n)$	$Spin(2n-1, 1)$	$Spin(2n-1)$
$E_6 R$	F_4	$\widetilde{F_4}^\vee$	$E_6^{(2)}$	E_6	$E_6(F_4)$	F_4
$D_4, \quad \theta^3 = 1$	G_2	$\widetilde{G_2}^\vee$	$D_4^{(3)}$	$Spin(8)$		G_2

3.1 Affine root system

As in section 1 there is an algebraic group G, and splitting data $(H, B, \{X_\alpha\})$ so that $\Delta = \Delta(G, H)$, and θ may be viewed as an automorphism of G preserving the splitting data. (For these purposes we may as well take G simply connected.) Then $T = H^\theta$ acts on $\mathfrak{g}$, and the set of roots $\Delta(G, T) \subset \mathfrak{t}^*$ is a (possibly reduced) root system.

The following Lemma is more or less immediate.

Lemma 3.1 *Restriction from H to T defines isomorphisms*

$$\Delta(G, T) \simeq \Delta_\theta$$

and

$$W^\theta \simeq W(\Delta_\theta).$$

Also $\Delta(K,T)$ *is the reduced root system of* Δ_θ *(obtained by taking only the shorter of two roots* $\alpha, 2\alpha$*) and* $W(K,T) \simeq W(\Delta_\theta)$*. See Remark 1.3.*

Now T^Γ acts on the complex Lie algebra $\mathfrak{g}$ of G. Let $\Delta(G,T^\Gamma)$ be the set of roots, i.e. we have a root space decomposition

$$\mathfrak{g} = \sum_{\alpha\in\Delta(G,T^\Gamma)} \mathfrak{g}_\alpha.$$

Clearly restriction from T^Γ to T is a surjection $\Delta(G,T^\Gamma) \to \Delta(G,T)$.

If $c = 1$ this is simply $\Delta(G,T)$. For simplicity assume $c = 2$. Then $\Delta(G,T^\Gamma)$ may be thought of as a $\mathbb{Z}/2\mathbb{Z}$-graded root system. That is a character α of T^Γ is a pair (α_0, ϵ) with $\alpha_0 \in \Delta(G,T) \simeq \Delta_\theta$ and $\epsilon = \pm 1$, where $\alpha_0 = \alpha|_T$ and $\epsilon = \alpha(\delta)$. We can define the reflection associated to $\alpha \in \Delta(G,T^\Gamma)$ in the usual way, preserving $\Delta(G,T^\Gamma)$. To be precise, if $\alpha = (\alpha_0, \epsilon)$ and $\beta = (\beta_0, \delta)$ then

$$s_\alpha(\beta) = (s_{\alpha_0}(\beta_0), \epsilon\delta(-1)^{\langle\beta,\alpha^\vee\rangle}). \tag{3.2}$$

Let $\pi : E \to T\delta$ be the universal cover. Then E is an affine space with translations $\mathfrak{t} = \mathrm{Lie}(\mathfrak{t})$.

Suppose λ is a character of $T^\Gamma \to \mathbb{C}^*$. Note that λ is determined by its restriction to $T\delta$. By the property of covering spaces λ lifts to a family of functions $\widetilde{\lambda} : E \to \mathbb{C}$ satisfying

$$\lambda(\pi(X)) = e^{2\pi i\widetilde{\lambda}(X)}$$

i.e. $d\widetilde{\lambda} = d\lambda$, where the left hand side is in the sense of (2.1) and the right is the ordinary differential of λ. We say $\widetilde{\lambda}$ lies over λ. Any two such functions differ by constant.

Definition 3.3 *The affine root system* Δ_{aff} *associated to* (Δ,θ) *is the set of affine functions in* $\mathit{Aff}(E)$ *lying over* $\Delta(G,T^\Gamma)$*.*

Note that the underlying finite root system, i.e. the differentials of all affine roots is $\Delta(G,T) \simeq \Delta_\theta$, i.e.

$$d : \Delta_{\mathrm{aff}} \twoheadrightarrow \Delta_\theta$$

The following Lemma is an immediate consequence of the fact that $\Delta(G, T^\Gamma)$ is a root system in the sense of (3.2).

Lemma 3.4 *Δ_{aff} is an affine root system.*

To be explicit, choose $\widetilde{\delta} \in E$ with $\pi(\widetilde{\delta}) = \delta$. Suppose $\alpha \in \widehat{T^\Gamma}$. To avoid excessive notation we write α for the differential of α restricted to T, rather than $d\alpha$. Then in the decomposition of Lemma 2.3 we may write the set of $\widetilde{\alpha}$ lying over α as

$$\{(\alpha, c) \,|\, e^{2\pi i c} = \alpha(\delta)\}$$

In particular note that the set of roots lying over α is

$$\{(\alpha, c) \,|\, c \in \mathbb{Z}\} \quad \text{if } \alpha(\delta) = 1$$

or

$$\{(\alpha, c) \,|\, c \in \mathbb{Z} + \frac{1}{2}\} \quad \text{if } \alpha(\delta) = -1$$

Similarly if δ has order 3 then $c \in \mathbb{Z} + \frac{1}{3}$ or $\mathbb{Z} + \frac{2}{3}$.

For $\alpha \in \Delta_\theta$ let $c_\alpha = 1$ if α is long, or $\frac{1}{c}$ if α is short, where $c = \text{order}(\theta)$.

Proposition 3.5 *Let Δ_{aff} be the affine root system associated to (Δ, θ), and let $c = order(\theta) \in \{1, 2, 3\}$. Then*

$$\Delta_{aff} = \{(\alpha, x) \,|\, x \in c_\alpha \mathbb{Z}\}$$

Proposition 3.6 *Fix a set $\alpha_1, \ldots, \alpha_n$ of simple roots of Δ_θ. For each i let $\widetilde{\alpha}_i = (\alpha_i, 0)$. Let β be the highest (long) root of $\Delta = \Delta_\theta$ if $c = 1$ or the highest short root otherwise. Let*

$$\widetilde{\alpha}_0 = (-\beta, \frac{1}{c}).$$

Then $\{\widetilde{\alpha}_0, \widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n\}$ is a set of simple roots of Δ_{aff}.

3.2 Affine Weyl group

We now describe the affine Weyl group of Δ_{aff}.

Definition 3.7 *Let*

$$L(G) = X_*(T/T \cap A). \tag{3.8}$$

In particular we have

$$L(G)/X_*(T) \simeq T \cap A \tag{3.9}$$

Lemma 3.10

$$L(G) = \langle \frac{1}{c} \sum_{k=0}^{c-1} \theta^k(\gamma^\vee) \,|\, \gamma \in X_*(H) \rangle$$

The most important cases are $c = 1, 2$:

$$L = \{\frac{1}{2}(\alpha^\vee + \theta\alpha^\vee) \,|\, \alpha \in X_*(H\} \quad (c = 1, 2). \tag{3.11}$$

If G is understood we write $L = L(G)$. Let $L_{sc} = L(G_{sc})$, where G_{sc} is the simply connected cover of G, and similarly L_{ad}.

Lemma 3.12 *If $c = 1$ then $L_{sc} = R^\vee$. If $c = 2$ or 3 then*

$$L_{sc} = \langle R^\vee(\Delta_\theta) \cup \{\frac{1}{c}\alpha^\vee \,|\, \alpha \in \Delta_\theta, \alpha \text{ short}\}\rangle$$

Remark 3.13 By [2]

$$R^\vee(\Delta_\theta) = R^\vee(\Delta)^\theta$$

and this is the kernal of $\exp : \mathfrak{t} \to T$ if G is simply connected.

Proposition 3.14 *The lattice L_{sc} is the set of translations in W_{aff}. There is an exact sequence*

$$0 \to L_{sc} \to W_{aff} \to W^\theta \to 1 \tag{3.15)(a}$$

If we choose an element $\widetilde{\delta} \in E$ lying over δ we obtain a splitting of (3.14), *taking W^θ to the the stabilizer in Aff(E) of $\widetilde{\delta}$, i.e.*

$$W_{aff} \simeq W^\theta \ltimes L_{sc} \tag{3.15)(b}$$

Remark 3.16 As in Remark 1.9, and Propositions 1.17 and 1.19 we may use $T\delta$ and W_δ in place of $\overline{T}$ and W^θ. Then

(3.17) $$0 \to R^\vee \to W_{\text{aff}} \to W_\delta \to 1$$

This is not necessarily a split exact sequence.

We give a few details of the map $p : W_{\text{aff}} \to W_\delta$. Suppose $\alpha \in \Delta_\theta$ and $x \in \mathbb{Z}$. Then

$$p(s_{(\alpha,x)}) = s_\alpha.$$

Suppose $c = 2$, $\alpha \in \Delta_\theta$ is a short root and $x \in \mathbb{Z} + \frac{1}{2}$. Then $m_\alpha = \alpha^\vee(-1) \in T \cap A$ and

$$p(s_{(\alpha,x)}) = s_\alpha m_\alpha$$

and

$$p(t_{\frac{1}{2}\alpha^\vee}) = m_\alpha.$$

where $t_{\frac{1}{2}\alpha^\vee} \in W_{\text{aff}}$ is translation by $\frac{1}{2}\alpha^\vee$.

Definition 3.18 *Suppose B is a subgroup of $Aut(\overline{T}\delta)$. Let $\widetilde{B}$ be the lift of B to $Aff(E,E)$. That is*

$$\widetilde{B} = \{\phi \in Aff(E,E) \,|\, \phi \text{ factors to an element of } B\}.$$

With this notation W_{aff} lies over W^θ, i.e. $W_{\text{aff}} \subset \widetilde{W^\theta}$. In fact $\widetilde{W^\theta}$ has a structure similar to that of W_{aff}.

Lemma 3.19 *Setting $L = L(G)$ we have an exact sequence*

(3.20)(a) $$1 \to L \to \widetilde{W^\theta} \to W^\theta \to 1$$

Given a choice of $\widetilde{\delta}$ satisfying $p(\widetilde{\delta}) = \delta$ we obtain a splitting of (3.20)(a)*, so*

(3.20)(b) $$\widetilde{W^\theta} \simeq W^\theta \ltimes L.$$

If G is simply connected then (3.20)(a-b) reduce to (3.15)(a-b).

In general $\widetilde{W^\theta}$ is an "extended" affine Weyl group. It is not necessarily a Coxeter group, but can be realized as the semi-direct product of the Coxeter group W_{aff} by a finite group.

We need a choice of fundamental domain $\mathcal{D}$ for the action of W_{aff} on E. There is a standard natural choice for (the closure of) $\mathcal{D}$. Choose a set of simple roots $\widetilde{\alpha}_0, \ldots, \widetilde{\alpha}_n$ of Δ_{aff}, and let

$$\overline{\mathcal{D}} = \{e \in E \,|\, \widetilde{\alpha}_i(e) \geq 0, i = 0, \ldots, n\}.$$

If we choose $\widetilde{\delta}$ as usual then we may identify E with V, and write $\widetilde{\alpha}_i = (\alpha_i, 0)$ $(i = 1, \ldots, n)$ and $\widetilde{\alpha}_0 = (\alpha_0, c)$. Let $\beta = -\alpha_0$; recall β is the highest long (resp. short) root of Δ if $c = 1$ (respectively $c = 2$). Then

$$\overline{\mathcal{D}} = \{v \in V \mid \alpha_i(v) \geq 0 \, (i = 1, \ldots, n),\ \beta(v) \leq c\}.$$

Lemma 3.21 *We have an exact sequence*

$$1 \to W_{\mathit{aff}} \to \widetilde{W^\theta} \to L/L_{sc} \to 1 \tag{3.22}$$

Given $\widetilde{\delta}$ we obtain a splitting of (3.22)*, taking L/L_{sc} to the stabilizer of $\mathcal{D}$. Thus*

$$\widetilde{W^\theta} \simeq W_{\mathit{aff}} \rtimes L/L_{sc} \tag{3.23}$$

and L/L_{sc} acts as automorphisms of $\mathcal{D}$.

3.3 The group L/L_{sc}

Because of (3.23) we need to understand L/L_{sc}. From (3.11) we have

$$L/L_{sc} = \frac{\langle \{\frac{1}{2}(\gamma^\vee + \theta\gamma^\vee) \mid \gamma^\vee \in X_*(H)\}\rangle}{\langle \{\frac{1}{2}(\alpha^\vee + \theta\alpha^\vee) \mid \gamma^\vee \in R^\vee\}\rangle}$$

Let G_{sc} be the simply connected cover of G, with center $Z_{sc} = Z(G_{sc})$. We have an exact sequence

$$1 \to \pi_1(G) \to G_{sc} \to G \to 1$$

with $\pi_1(G) \subset Z_{sc}$. Write $H_{sc} = T_{sc}A_{sc}$ for the Cartan subgroup in G_{sc} with image H.

Lemma 3.24

(3.25)(a) $$L/L_{sc} \simeq \pi_1(G)/\pi_1(G) \cap A_{sc}$$

In particular

(3.25)(b) $$L_{ad}/L_{sc} \simeq Z_{sc}/Z_{sc} \cap A_{sc}$$

Proof. A standard fact is that $\pi_1(G) \simeq X_*(T)/R^\vee$. The map $\gamma^\vee \to \frac{1}{2}(\gamma^\vee + \theta\gamma^\vee)$ takes $X_*(T)$ to L and factors to a map from $\pi_1(G)$ to L/L_{sc}. It is not hard to see the kernel is $Z \cap A$. The main point is that if α is in the root lattice, then $\frac{1}{2}(\alpha^\vee + \theta\alpha^\vee)$ is equivalent to $\frac{1}{2}(\alpha^\vee - \theta\alpha^\vee)$ modulo $R^\vee$, and the second version shows this element is in A. ■

Remark 3.26 Note: $Z \cap A \subset Z^{\theta^{-1}}$, and the inclusion may be proper. Hence $Z/Z \cap A$ surjects onto $Z/Z^{\theta^{-1}}$, and this is not necessarily an isomorphism.

Definition 3.27 *Let*

$$\overline{\pi_1} = \overline{\pi_1}(G) = \pi_1(G)/\pi_1(G) \cap A_{sc} \tag{3.28}$$

Remark 3.29 I believe the next Proposition is correct, but it should be taken with a grain of salt. In any event I do not know how to make the isomorphism natural, and consequently I'm not sure if the subsequent definition is the right one. It seems like the right thing. . .

Write $K_{sc} = G^\theta_{sc}$, and let $\widetilde{K} \to K_{sc}$ be the simply connected cover of K. Recall (Remark 1.3) $\widetilde{K} = K_{sc}$ unless $G_{sc} = SL(2n+1)$, in which case $\widetilde{K} = Spin(2n+1) \to K = SO(2n+1)$ is a two-fold cover.

Proposition 3.30

$$L_{ad}/L_{sc} \simeq \pi_1(K)/\pi_1(K_{sc}) \tag{3.31}$$

In particular suppose G is adjoint. Then

$$L/L_{sc} \simeq Z(K_{sc}) \tag{3.32}$$

and this equals $Z(\widetilde{K})$ unless $G_{sc} = SL(2n+1)$.

Of course if $\gamma = 1$ these equations simplify considerably: $L/L_{sc} \simeq \pi_1(G)$, which equals $Z(G_{sc})$ if G is adjoint.

Definition 3.33 *Given (G, γ) let*

$$\pi_1'(K) = \pi_1(K)/\pi_1(K_{sc}) \tag{3.34}$$

Thus $\pi_1'(K) = \pi_1(K)$ unless $G_{sc} = SL(2n+1, \mathbb{C})$, and $\pi_1'(K) = \pi_1(G)$ if $\gamma = 1$. In particular if G is adjoint we have

$$\pi_1'(K) = Z(K_{sc}). \tag{3.35}$$

3.4 Action on the extended Dynkin diagram

A key ingredient of the computation of strong real forms is the action of $\overline{\pi_1}(G)$ on the extended Dynkin diagram.

Take G to be simply connected, so $Z = Z_{sc}$. First of all note that Z acts by left multiplication on $H\delta$ and therefore on $\overline{T}\delta$. Explicitly if $z \in Z$ write $Z = ta$ with $t \in T, a \in A$. Then for $t' \in T$,

$$z \cdot t'\delta = tt'\delta.$$

Note that although t, a are only defined up to $T \cap A$, this action is well defined on $\overline{T}\delta$. Clearly this action factors to $Z/Z \cap A$. This lifts to an action on E, and induces an action of $Z/Z \cap A$ on $\mathcal{D}$.

Now if z corresponds to $\gamma^\vee \in P^\vee$ via the isomorphism $Z \simeq P^\vee/R^\vee$, then $t = \frac{1}{2}(\gamma^\vee + \theta\gamma^\vee)$. It follows that under the isomorphism (3.25)(b) L_{ad}/L_{sc} acts by translation on E.

Now drop the assumption that G is simply connected. Then $\pi_1(G) \subset Z_{sc}$ acts on $\mathcal{D}$ and D_{Aff} by the preceding construction, and this action factors to an action of $\overline{\pi_1}(G)$.

Lemma 3.36 *The stabilizer of $\mathcal{D}$ in the Euclidean group of E is isomorphic to the automorphism group of D_{Aff}.*

Definition 3.37 *We write $\tau(m)$ for the action of $\overline{\pi_1}(G)$ on D_{Aff}.*

4 Affine Weyl group and strong real forms

We now return to the setting of Section 1. We relate the construction of the affine weyl group in Section 3 and the parametrization of real forms in Proposition 1.17.

Recall we are interested in computing the orbits of W_δ on $\overline{T}\delta$ (cf. Definition 3.14 and Proposition 1.17). To do this we pass to the tangent space E of $\overline{T}\delta$ at δ (cf. Section 3).

It is immediate that for any subgroup B of $\mathrm{Aut}(T\delta)$, $\pi : E \to T\delta$ factors to a bijection $E/\widetilde{B} \leftrightarrow T\delta/B$.

Lemma 4.1 *Strong real forms of G are parametrized by elements X of E satisying $\pi(X)^2 \in Z$ modulo the action of $\widetilde{W^\theta}$.*

Recall a choice of $\widetilde{\delta}$ gives

$$\begin{aligned} \widetilde{W^\theta} &\simeq W^\theta \ltimes L \quad (3.20)(b) \\ &\simeq W_{\text{aff}} \rtimes L/L_{sc} \quad (3.23) \\ &\simeq W_{\text{aff}} \rtimes \overline{\pi_1}(G) \quad \text{(Proposition 3.30 and Definition 3.27)} \end{aligned} \tag{4.2}$$

Lemma 4.3 *We may parametrize $\overline{D}$ as $\{(a_0, \ldots, a_n)\}$ where $a_i \geq 0$ and*

$$\sum_{i=0}^{n} n_i a_i = \frac{1}{c}. \tag{4.4}$$

Here $(a_0, \ldots, a_n)$ corresponds to the element X of $\mathcal{D}$ satisfying

$$\alpha_i(X) = a_i \quad (i = 1, \ldots, n)$$

Lemma 4.5 *Suppose $(a_1, \ldots, a_n)$ satisfies (4.4), and let $X \in \mathcal{D}$ be the corresponding element. Then $x = \pi(X) \in T\delta$ satisfies $x^m \in Z$ if and only if $ma_i \in \mathbb{Z}$ for all $i = 0, \ldots, n$.*

Example 4.6 Take $m = 1$. We must take $c = 1$ and each $a_i = 0$ or 1. We conclude from (4.4) that Z is in bijection with the nodes of $\widetilde{D}$ with label 1.

Theorem 4.7 *The strong real forms of (G, γ) are parametrized as follows. Let $c =$ order$(\gamma) = 1, 2$. Choose a set $S \subset \{0, \ldots, n\}$ satisfying*

$$\sum_{i \in S} n_i = \frac{2}{c}$$

Obviously $|S| \leq 2$ and $n_i \leq 2$ for all $i \in S$.

Two such subsets S, S' parametrize the same strong real form if and only if $\tau(m)S = S'$ for some $m \in \overline{\pi_1}(G)$.

4.1 Real forms and the Kac classification

The Kac classification of real forms of $\mathfrak{g}$ amounts to taking G to be the adjoint group. In this case $\overline{\pi_1}(G)$ is a quotient of $Z(G_{sc})$. Recall (3.28) acts by τ on D_{Aff} (Definition 3.37).

Theorem 4.8 *The real forms of* (G, γ) *are parametrized by subsets* S *as in Theorem 4.7, modulo the action of* $\overline{\pi_1}(G_{sc})$.

Recall (Definition 1.2) this definition of equivalence uses only $\mathrm{Int}(G)$ not $\mathrm{Aut}(G)$. The usual Kac classification is for what we refer to here as traditional real forms.

Recall we are given a finite root system Δ_θ. Let D_θ be its Dynkin diagram. Also Δ_θ is contained in an affine root system Δ_{aff}. Let $D_{\mathrm{Aff}} = D(\Delta_{\mathrm{aff}})$ be the Dynkin diagram of Δ_{aff}, so $D_\theta \subset D_{\mathrm{Aff}}$.

Lemma 4.9 *We have a split exact sequence*

$$1 \to \overline{\pi_1}(G) \to Aut(\mathcal{D}) \to Out(G) \to 1$$

or equivalently

$$1 \to \overline{\pi_1}(G) \to Aut(D_{Aff}) \to Aut(D_\theta) \to 1$$

Remark 4.10 If $\theta = 1$ and G is simply connected this becomes

$$1 \to Z \to \mathrm{Aut}(D_{\mathrm{Aff}}) \to \mathrm{Aut}(D)$$

See [6, Exercise 15, page 217]. For an explicit formula for the map $Z \to \mathrm{Aut}(D_{\mathrm{Aff}})$ see [3, Chapter VI, §2.3, Proposition 6].

Remark 4.11 Assuming Proposition 3.30 is correct we can replace $\overline{\pi_1}(G)$ with $Z(K_{sc})$, which is more natural:

$$1 \to Z(K_{sc}) \to \mathrm{Aut}(D_{\mathrm{Aff}}) \to \mathrm{Aut}(D_\theta) \to 1$$

In any event the usual Kac classification is stated as follows.

Theorem 4.12 *The traditional real forms of* G *are parametrized by subsets* S *as in Theorem 4.7, modulo the full automorphism group of* D_{Aff}.

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