ABSTRACT

Title of dissertation: LIFTING OF CHARACTERS OF $\widetilde{SL}_2(\mathbf{F})$ AND $SO_{1,2}(\mathbf{F})$ FOR $\mathbf{F}$ A NONARCHIMEDEAN LOCAL FIELD

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We make a definition of "stable" virtual character of the two fold cover of $SL_2(\mathbf{F})$ for $\mathbf{F}$ p–adic and establish a bijection between stable virtual characters $\widetilde{\pi}$ of $\widetilde{SL}_2(\mathbf{F})$ and irreducible representations π' of $SO_{1,2}(\mathbf{F})$. This is achieved through character theory; the characters $\Theta_{\widetilde{\pi}}$, $\Theta_{\pi'}$ of $\widetilde{\pi}$ and $\widetilde{\pi}'$ are related by

$$\Theta_{\widetilde{\pi}}(\widetilde{g}) = \Gamma_-(\widetilde{g})\Theta_{\pi'}(\tau(g)).$$

Here $\widetilde{g} \in \widetilde{SL}_2(\mathbf{F})$ and $\tau(g)$ is contained in the unique conjugacy class of elements of $SO_{1,2}(\mathbf{F})$ having the same nontrivial eigenvalues as $\widetilde{g}$. The transfer factor $\Gamma_-(\widetilde{g})$ is the character of the *difference* of the two halves of the oscillator representation of $\widetilde{SL}_2(\mathbf{F})$. Included is a complete development of character formulas for the oscillator representation of $\widetilde{SL}_2(\mathbf{F})$.

This work was motivated by the results of Adams in [3] where a very similar character formula affords a bijection of stable virtual characters of $\widetilde{Sp}(2n, \mathbf{R})$ and stable virtual characters of $SO_{n+1,n}(\mathbf{R})$ with good properties.

LIFTING OF CHARACTERS OF $\widetilde{SL}_2(\mathbf{F})$ AND $SO_{1,2}(\mathbf{F})$ FOR $\mathbf{F}$ A NONARCHIMEDEAN LOCAL FIELD

by

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Chapter 1

Introduction

Overview

The study of correspondences between representations of Lie groups is an important theme in the theory of representations and automorphic forms. Lifting, or transfer, of the representation theory of one group to another has been the subject of extensive study. Langlands functoriality, roughly stated, says that given *linear* groups G and H, a homomorphism ${}^L H \to {}^L G$ of their L–groups gives a transfer of representations of H to representations of G. On the other hand, the theta correspondence of Howe [16] provides a way to relate the representation theories of a reductive dual pair G and G' embedded in the metaplectic group. The groups G and G' are sometimes non–linear.

Shimura [34] defined a correspondence between the space of cusp forms of half integral weight $k/2$ and the space of cusp forms of even integral weight $k-1$. This correspondence preserves eigenvalues for the Hecke ring. Shimura's correspondence is based on L–functions and suggests a correspondence between automorphic forms on $\widetilde{SL}_2$ and PGL_2. Subsequently, Shintani [35] and Niwa [27] discovered this correspondence using the oscillator (or Weil) representation.

Interpreting the Shimura correspondence in representation theoretic terms then became an important problem. A systematic study of the dual pair correspondence for the reductive dual pair $(\widetilde{SL}_2, PGL_2)$ was made by Rallis and Schiffman [30] and was later parameterized by Manderscheid [24]. Gelbart initiated the study of the Shimura correspondence via group theory in [9]; this was continued with Piatetski–Shapiro in [10] and [11]. Finally, in [39], [40] and [41], Waldspurger gave a very deep study of the Shimura correspondence and automorphic forms on $\widetilde{SL}_2$. An excellent survey of this work was made by Piatetski–Shapiro in [28].

Using character theory, Flicker [6] has defined a correspondence between m–fold covering groups $\widetilde{GL}_2$ and GL_2. This correspondence has been generalized by Kazhdan and Patterson in [19] and [20] and by Flicker and Kazhdan in [7] in the case of $\widetilde{GL}_r$ and GL_r.

Our work is motivated by a correspondence of Adams in [3] using character theory. Suppose $\mathbf{F}$ is a local field of characteristic 0. Let $\widetilde{Sp}(2n, \mathbf{F})$ be the two-fold cover of $Sp(2n, \mathbf{F})$. Fix an additive character ψ of $\mathbf{F}$ and let $\omega(\psi) = \omega_e(\psi) \oplus \omega_o(\psi)$ be the oscillator representation of $\widetilde{Sp}(2n, \mathbf{F})$ attached to ψ. If π is a representation, let Θ_π be its character (see Chapter 2). We consider the character as a function on the regular semisimple elements, or equivalently, conjugacy classes. Let Γ_-^ψ be the *difference* of the characters of the two halves of the oscillator representation, i.e.,

$$\Gamma_-^\psi = \Theta_{\omega_e(\psi)} - \Theta_{\omega_o(\psi)}.$$

Now assume $\mathbf{F} = \mathbf{R}$. If $g \in Sp(2n, \mathbf{R})$, then we say that $h \in SO_{n+1,n}(\mathbf{R})$ and g correspond via the orbit correspondence if g and h have the same non–trivial eigenvalues (see Adams [3]). We write $t(g) = h$ for the orbit correspondence. If $g \in \widetilde{Sp}(2n, \mathbf{R})$ then write $t(g) = t(pr(g))$ (here pr is the projection $\widetilde{Sp}(2n, \mathbf{R}) \rightarrow$

$Sp(2n,\mathbf{R})$). Adams [3] (see also [1]) has given the definition of "stable" genuine invariant eigendistribution on $\widetilde{Sp}(2n,\mathbf{R})$ and has shown that the map t_* given by

$$t_*(\Theta)(g) = \Gamma_-^{\psi}(g)\Theta(t(g))$$

is a bijection between

$$\{\text{stable invariant eigendistributions of } SO_{n+1,n}(\mathbf{R})\}$$

and

$$\{\text{stable genuine invariant eigendistributions of } \widetilde{Sp}(2n,\mathbf{R})\}.$$

This bijection has good properties. For example, $t_*(\Theta)$ is a virtual character if and only if Θ is, $t_*(\Theta)$ is tempered if and only if Θ is, t_* takes stable discrete series representations of $SO_{n+1,n}(\mathbf{R})$ to the stable genuine discrete series representations of $\widetilde{Sp}(2n,\mathbf{R})$, t_* commutes with parabolic induction, and if Θ is a tempered virtual character, then $t_*(\Theta)$ is the normalized theta–lift of Θ. See Adams in [3] for the proof.

Essential to the proof of Adams' lifting are Hirai's matching conditions. These are special for the field $\mathbf{F}=\mathbf{R}$. Our problem considers the analogue of this correspondence in the case of $n=1$ with $\mathbf{F}$ p–adic. We now give a detailed summary of our methods and results.

Let π be an irreducible representation of $GL_2(\mathbf{F})$. If π has trivial central character, then π factors to a representation of $PGL_2(\mathbf{F}) \simeq SO_{1,2}(\mathbf{F})$. Let p be the map from $GL_2(\mathbf{F})$ to $SO_{1,2}(\mathbf{F})$. Then define π', a representation of $SO_{1,2}(\mathbf{F})$, by pulling back π via p, i.e., their characters satisfy

$$\Theta_\pi(g) = \Theta_{\pi'}(p(g)).$$

Assume $g \in SL_2(\mathbf{F})$. Let V be a vector space equipped with a quadratic form of signature $(1,2)$. For g regular semisimple let $\tau(g)$ be the unique conjugacy class of elements with the same nontrivial eigenvalues as g. Then

$$p(g) \in \tau(g^2).$$

(For details, see Section 2.3.) Our character identities relate character values at $\widetilde{g} \in \widetilde{SL}_2(\mathbf{F})$ satisfying $pr(\widetilde{g}) = g$ and character values at $\tau(g) \subseteq SO_{1,2}(\mathbf{F})$.

The key idea is then the following. Start with an irreducible representation π of $GL_2(\mathbf{F})$ whose central character χ_π satisfies $\chi_\pi(-I) = 1$ (here I is the 2×2 identity matrix) and lift it via Flicker's theory [6] to $\widetilde{\pi} = \mathrm{Lift}_{\mathrm{F}}(\pi)$ of $\widetilde{GL}_2(\mathbf{F})$. (The lifting is actually defined for arbitrary covers of $GL_2(\mathbf{F})$ but we only use the case of 2–fold covers. See Section 2.7 for a full discussion of Flicker's correspondence.) Then restrict $\widetilde{\pi}$ to $\widetilde{GL}_2(\mathbf{F})_+ = \{\widetilde{g} \in \widetilde{GL}_2(\mathbf{F}) : \det pr(\widetilde{g}) \in \mathbf{F}^{\times 2}\}$. Let $\widetilde{\sigma}$ be an irreducible summand of this restriction. By Clifford Theory (see Theorem 3.8), we have

$$\widetilde{\pi}|_{\widetilde{GL}_2(\mathbf{F})_+} \simeq \sum_{x \in \mathbf{F}^\times/\mathbf{F}^{\times 2}} \widetilde{\sigma}^x.$$

In particular, the summands are parameterized by $x \in \mathbf{F}^\times/\mathbf{F}^{\times 2}$. If G is any group, we let $Z(G)$ be its center. Since

$$\widetilde{GL}_2(\mathbf{F})_+ = \widetilde{SL}_2(\mathbf{F}) \cdot Z(\widetilde{GL}_2(\mathbf{F})_+),$$

there is no further reducibility when we restrict to $\widetilde{SL}_2(\mathbf{F})$. We are able to recover the character of $\widetilde{\sigma}^x$ by using an "inversion formula":

$$\Theta_{\widetilde{\sigma}^x}(\widetilde{g}) = \frac{1}{|\mathbf{F}^\times/\mathbf{F}^{\times 2}|} \sum_{\xi \in \mathbf{F}^\times/\mathbf{F}^{\times 2}} \chi_{\widetilde{\sigma}^x}(z_\xi)^{-1} \Theta_{\widetilde{\pi}}(z_\xi \widetilde{g}), \qquad \widetilde{g} \in \widetilde{GL}_2(\mathbf{F})_+.$$

This is given in Theorem 3.12. Note that this yields character formulas for all irreducible representations of $\widetilde{SL}_2(\mathbf{F})$, and that both Theorems 3.8 and 3.12 are valid for any genuine representation $\widetilde{\pi}$ of $\widetilde{GL}_2(\mathbf{F})$.

Flicker's correspondence is defined by a character identity, giving a relationship between the character of $\widetilde{\pi} = \mathrm{Lift}_{\mathrm{F}}(\pi)$ on $\widetilde{GL}_2(\mathbf{F})$ and the character of π on $GL_2(\mathbf{F})$. Flicker's formula is only nonzero on the "relevant elements", that is, elements $\widetilde{g}$ such that $pr(\widetilde{g})$ is a square. In particular, Flicker [6] points out that:

characters of genuine representations of $\widetilde{GL}_2(\mathbf{F})$ vanish off of the set $\{(g^2;\varepsilon)\}$.

Using Flicker's identity together with the inversion formula, we are able to relate the character of an individual $\widetilde{\sigma}^x$ with the character of π which is defined on *all* elements. The problem is then to relate the character of an individual $\widetilde{\sigma}^x$ with the character of a representation π' of $SO_{1,2}(\mathbf{F})$. It turns out that this can't be done because the resulting character formula has two terms. However, if we "stabilize" the formulas (see below) we are able to achieve this.

Here and throughout, if ν_0 is a character of $\mathbf{F}^\times$, then the letter ν will always be reserved for the one–dimensional representation $\nu = \nu_0 \circ \det$ of $GL_2(\mathbf{F})$.

Given a representation π on $GL_2(\mathbf{F})$ such that $\chi_\pi(-I) = 1$, there exists a character ν_0 of $\mathbf{F}^\times$ such that

$$\chi_\pi(xI) = \nu_0(x^2).$$

Let

$$\chi_{\nu_0}(xI;\varepsilon) = \nu_0(x)\gamma_{\mathbf{F}}(x,\psi)\varepsilon.$$

Note that this depends on the choice of the additive character ψ. The $\gamma_{\mathbf{F}}(x,\psi)$ are the gamma–factors introduced by Weil; see Section 2.5. Each genuine character of the center of $\widetilde{GL}_2(\mathbf{F})_+$ is of this form. In particular, there exists ν_0 such that $\chi_{\widetilde{\sigma}^x} = \chi_{\nu_0}$.

Definition 1.1. *Fix a character ν_0 of $\mathbf{F}^\times$ satisfying $\chi_\pi(xI) = \nu_0(x^2)$. Let $\widetilde{\rho}$ be* *the constituent of* $\widetilde{\pi}|_{\widetilde{GL_2}(\mathbf{F})_+}$ *with central character* χ_{ν_0}. *We define* $L(\pi,\nu_0)$ *to be the restriction of* $\widetilde{\rho}$ *to* $\widetilde{SL}_2(\mathbf{F})$.

This central character condition is consistent with Flicker's character condition in Definition 2.63.

Our first main result is stated in Theorem 5.30. We repeat it here.

Theorem 1.2. *Suppose that* π *is an irreducible representation of* $GL_2(\mathbf{F})$ *satisfying* $\chi_\pi(-I) = 1$. *Suppose that* ν_0 *is a character of* $\mathbf{F}^\times$ *satisfying* $\nu_0(x^2) = \chi_\pi(xI)$. *Let* $\widetilde{g}$ *be a regular semisimple element of* $\widetilde{SL}_2(\mathbf{F})$ *and let* $g = pr(\widetilde{g})$. *Then*

$$\boxed{\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \frac{1}{2}\left[\Gamma^\psi_-(\widetilde{g})\Theta_{(\pi\nu^{-1})'}(\tau(g)) + \nu_0(-1)\Gamma^\psi_+(\widetilde{g})\Theta_{(\pi\nu^{-1})'}(\tau(-g))\right].}$$

In the above statement, Γ^ψ_+ and Γ^ψ_- are respectively the sum and difference of the two halves of the oscillator representation attached to ψ. We remark that in Chapter 4, we have developed formulas for the characters of the even and odd halves of the oscillator representation and therefore obtain the character of their sum and difference. See Theorem 4.40 and Theorem 4.41.

If β_0 is a character of $\mathbf{F}^\times$ which satisfies $\beta_0(-1) = 1$, then

$$L(\pi,\nu_0) = L(\pi\beta,\nu_0\beta_0).$$

In fact, we have shown that $L(\pi,\nu_0) = L(\sigma,\lambda_0)$ if and only if there is a character β_0 satisfying $\sigma = \pi\beta$, $\lambda_0 = \nu_0\beta_0$, and $\beta_0(-1) = 1$.

Theorem 1.2 suggests that we have a map

$$L(\pi,\nu_0) \mapsto (\pi\nu^{-1})'.$$

However, the map is not one–to–one. Indeed, if β is such that $\beta_0(-1) = -1$, then $L(\pi,\nu_0)$ and $L(\pi\beta,\nu_0\beta_0)$ are generally non–zero, are inequivalent because they have different central characters, and map to the same representation $(\pi\nu^{-1})'$. To remedy this situation, we "stabilize" our formulas as follows.

Definition 1.3. *Let β_0 satisfy $\beta_0(-1) = -1$. Let π be an irreducible representation of $GL_2(\mathbf{F})$ satisfying $\chi_\pi(-I) = 1$. Let ν_0 be a character of $\mathbf{F}^\times$ such that $\chi_\pi(xI) = \nu_0(x^2)$. Define*

$$L_{st}(\pi, \nu_0) = L(\pi, \nu_0) + L(\pi\beta, \nu_0\beta_0).$$

This is independent of the choice of β.

Let $Gr_{st}(\widetilde{SL}_2(\mathbf{F}))$ be the linear span of all virtual characters $L_{st}(\pi, \nu_0)$ where π is an irreducible representation of $GL_2(\mathbf{F})$ satisfying $\chi_\pi(-I) = 1$ and ν_0 is a character of $\mathbf{F}^\times$ satisfying $\chi_\pi(xI) = \nu_0(x^2)$. If $\rho \in Gr_{st}(\widetilde{SL}_2(\mathbf{F}))$, then we say that ρ is a stable virtual representation of $\widetilde{SL}_2(\mathbf{F})$.

After making this definition, we obtain the following formula.

Theorem 1.4. *Continue with the notation in Theorem 1.2. Then*

$$\boxed{\Theta_{L_{st}(\pi,\nu_0)}(\widetilde{g}) = \Gamma_-^\psi(\widetilde{g})\Theta_{(\pi\nu^{-1})'}(\tau(g)).}$$

The formula defines a bijection

$$Gr_{st}(\widetilde{SL}_2(\mathbf{F})) \leftrightarrow Gr(SO_{1,2}(\mathbf{F}))$$
$$L_{st}(\pi, \nu_0) \leftrightarrow (\pi\nu^{-1})'.$$

[Here, $Gr(SO_{1,2}(\mathbf{F}))$ is the space of virtual representations of $SO_{1,2}(\mathbf{F})$.] The bijection satisfies the following properties. A precise statement appears in Theorem 6.27.

- principal series representations of $SO_{1,2}(\mathbf{F})$ are in bijection with genuine principal series representations of $\widetilde{SL}_2(\mathbf{F})$,

- the one dimensional representations of $SO_{1,2}(\mathbf{F})$ are in bijection with the finite set

$$\{\Gamma_{-}^{\psi} : \psi \in \widehat{\mathbf{F}}\},$$

- a special representation of $SO_{1,2}(\mathbf{F})$ corresponds with the sum of an odd oscillator representation and a special representation of $\widetilde{SL}_2(\mathbf{F})$,

- an irreducible supercuspidal representation of $SO_{1,2}(\mathbf{F})$ maps to the sum of two supercuspidal representations of $\widetilde{SL}_2(\mathbf{F})$.

Connection to Work of Waldspurger

Let $\widetilde{\sigma}$ be a discrete series representation of $\widetilde{SL}_2(\mathbf{F})$ and let ψ be an additive character of $\mathbf{F}$. In [41], Waldspurger has defined an involution $\widetilde{\sigma} \mapsto \widetilde{\sigma}^W$ as follows:

$$\widetilde{\sigma} \xrightarrow{\Theta(\psi)} \pi \xrightarrow{JL} \pi' \xrightarrow{\Theta'(\psi^{-1})} \widetilde{\sigma}^W.$$

In the above diagram, π is an irreducible admissible representation of $PGL_2(\mathbf{F})$ and JL denotes the Jacquet–Langlands correspondence (see [12]). The maps $\Theta(\psi)$, $\Theta'(\psi)$ are Waldspurger's correspondences (see [41]).

We have essentially defined a stable "packet"

$$L_{st}(\pi, \nu_0) = \{L(\pi, \nu_0), L(\pi\beta, \nu_0\beta_0)\}.$$

On the other hand, Waldspurger has proved the following theorem:

Theorem 1.5 ([41],[11]). *Let $\widetilde{\sigma}$ be a discrete series representation of $\widetilde{SL}_2(\mathbf{F})$ and suppose ψ is a nontrivial additive character of $\mathbf{F}$ such that $\widetilde{\sigma}$ has a ψ–Whittaker model (see [10]). Then:*

1. *There is an irreducible admissible representation* π *of* $PGL_2(\mathbf{F})$ *such that* $\Theta(\psi)^{-1}(\pi) = \widetilde{\sigma}$.

2. *The "near equivalence set"* $NE(\widetilde{\sigma})$ *defined by*

$$NE(\widetilde{\sigma}) = \{\Theta(\psi_\xi)^{-1}(\pi \otimes q_\xi) : \xi \in \mathbf{F}^\times / \mathbf{F}^{\times 2}\}$$

   has exactly two elements. (Here q_ξ *is the quadratic character associated to the extension* $\mathbf{F}(\sqrt{\xi})$ *of* $\mathbf{F}$*.)*

3. *Suppose that*

$$NE(\widetilde{\sigma}) = \{\Theta(\psi)^{-1}(\pi), \Theta(\psi_\xi)^{-1}(\pi \otimes q_\xi)\}.$$

   Then

$$NE(\widetilde{\sigma}) = \{\widetilde{\sigma}, \widetilde{\sigma}^W\}.$$

Our "packets" are the same as the "near equivalence sets" of Waldspurger. In particular, we claim that if $\widetilde{\sigma} = L(\pi, \nu_0)$ is a discrete series representation, then

$$\{L(\pi, \nu_0), L(\pi\beta, \nu_0\beta_0)\} = \{\widetilde{\sigma}, \widetilde{\sigma}^W\}.$$

Thus our stable packets are the character–theory analogue of near equivalence.

Chapter 2

Preliminaries

In this chapter we will explain various constructions and techniques that we will assume and use throughout.

In Section 1, we state the definition and properties of the quadratic Hilbert symbol. In Section 2, we explain how to embed the nonzero elements of a quadratic extension in $GL_2(\mathbf{F})$ and give information about squares in quadratic extensions. We define the group $SO_{1,2}(\mathbf{F})$ in Section 3. The basic theory of covering groups is given in Section 4 and the groups $\widetilde{SL}_2(\mathbf{F})$, $\widetilde{GL}_2(\mathbf{F})$ and $\widetilde{GL}_2(\mathbf{F})_+$ are defined. Section 5 is concerned with the definitions and properties of the gamma factors $\gamma_{\mathbf{F}}(x, \psi)$ introduced by Weil. We summarize the basic representation theory we will need, including a statement of the existence of global characters, in Section 6. Finally, in Section 7, we elucidate Flicker's correspondence in the special case $n = 2$. This is used heavily in the derivation of our character formulas. Flicker's character formula is given as it appears in Flicker–Kazhdan [7]. We have also included Flicker's character formulas in the cases of principal series representations and the oscillator representation.

2.1 The Hilbert Symbol

Let $\mathbf{F}$ be a local field. We recall briefly the definition and properties of the Hilbert symbol attached to $\mathbf{F}$. An elegant account of this material is found in Serre [32] and [33].

Definition 2.1. *Let a, b be elements of $\mathbf{F}$. We define*

$$(a,b)_{\mathbf{F}} = \begin{cases} 1 & \text{if } ax^2+by^2=z^2 \text{ has a nonzero solution } (x,y,z)\in\mathbf{F}^3 \\ -1 & \text{otherwise.} \end{cases}$$

The number $(a,b)_{\mathbf{F}}$ is called the Hilbert symbol of a and b relative to $\mathbf{F}$.

Proposition 2.2. *The following properties of the Hilbert symbol are true for all a and b in $\mathbf{F}^\times$.*

1. $(a,b)_{\mathbf{F}} = (b,a)_{\mathbf{F}}$,
2. $(a,b^2)_{\mathbf{F}} = 1$,
3. $(a,-a)_{\mathbf{F}} = 1$,
4. $(a,a)_{\mathbf{F}} = (-1,a)_{\mathbf{F}}$,
5. $(a,1-a)_{\mathbf{F}} = 1$,
6. $(a,b)_{\mathbf{F}} = (a+b,-ab)_{\mathbf{F}}$.

Proposition 2.3. *The Hilbert symbol is a nondegenerate, symmetric bilinear form on the $\mathbf{F}_2$–vector space $\mathbf{F}^\times/\mathbf{F}^{\times 2}\times\mathbf{F}^\times/\mathbf{F}^{\times 2}$. (Here $\mathbf{F}_2=\{0,1\}$.) In particular, the following properties hold.*

1. *If $(a,b)_{\mathbf{F}} = 1$ for all $a\in\mathbf{F}^\times/\mathbf{F}^{\times 2}$, then b is a square.*

2. *We have that* $(aa', b)_{\mathbf{F}} = (a, b)_{\mathbf{F}}(a', b)_{\mathbf{F}}$ *for all* a, a', $b \in \mathbf{F}^\times/\mathbf{F}^{\times 2}$.

Proof. See Serre [32]. □

2.2 Quadratic Extensions of F

Let $\mathbf{E}$ be the quadratic extension $\mathbf{F}(\delta)$ of $\mathbf{F}$ with $\delta^2 = \Delta \in \mathbf{F}$. We fix the ordered basis $\{1, \delta\}$ for $\mathbf{E}$ as a vector space over $\mathbf{F}$. If $z = x + y\delta \in \mathbf{E}$, the norm of z relative to this extension, written $N_{\mathbf{E}/\mathbf{F}}(z) = N(z)$, is defined to be $N(z) = x^2 - y^2\Delta$. The norm is a multiplicative map, i.e., $N(zw) = N(z)N(w)$. If $z = x + y\delta \in \mathbf{E}$, we write $\overline{z} = x - y\delta$ for the conjugate of z in $\mathbf{E}$. Note that $N(z) = z\overline{z}$. We have that $(a, b)_{\mathbf{F}} = 1$ if and only if a is a norm in $\mathbf{F}(\sqrt{b})$ and that $N(\mathbf{E}^\times)$ has index two in $\mathbf{F}^\times$. Let $\mathbf{E}^1$ denote the group of elements of norm one in $\mathbf{E}$. Let $\varphi(z) = z/\overline{z}$. Hilbert's Theorem 90 (see Jacobson, [17]) states that $z \in \mathbf{E}^1$ if and only if there exists $w \in \mathbf{E}^\times$ such that $z = \varphi(w) = w/\overline{w}$.

Let $z = x + y\delta \in \mathbf{E}$ and $a \in \mathbf{F}^\times$. The matrix of left multiplication of z on $\mathbf{E}$ in the ordered basis $\{1, a\delta\}$ is

$$\begin{pmatrix} x & y\Delta a \\ y/a & x \end{pmatrix}.$$

This gives an embedding $\iota_a : \mathbf{E}^\times \hookrightarrow GL_2(\mathbf{F})$. We write $\iota_a(\mathbf{E}^\times) = T_{a,\Delta}$, $\iota = \iota_1$ and $T_{1,\Delta} = T_\Delta$. Any embedding of $\mathbf{E}^\times$ in $GL_2(\mathbf{F})$ is conjugate to ι. Via ι, the determinant corresponds to the norm, so that $N = \det \circ \iota$. In other words,

$$N(x + y\sqrt{\Delta}) = \det \begin{pmatrix} x & y\Delta \\ y & x \end{pmatrix}.$$

In a similar way, we obtain an embedding

$$\iota_a(\mathbf{E}^1) \hookrightarrow SL_2(\mathbf{F})$$

and abuse notation by writing $\iota_a(\mathbf{E}^1) = T_{a,\Delta}$ when the context is clear. Any such embedding extends to an embedding ι of $\mathbf{E}^\times$ in $GL_2(\mathbf{F})$.

We use the following proposition repeatedly when we apply the inversion formula (see Theorem 3.12 below).

Proposition 2.4. *Suppose $z \in \mathbf{E}^1$. Then there are exactly two values of ξ in $\mathbf{F}^\times/\mathbf{F}^{\times 2}$ such that $\xi z \in \mathbf{E}^{\times 2}$.*

Proof. We claim that if $\xi \in \mathbf{E}^{\times 2} \cap \mathbf{F}^\times$, then $\xi \in \mathbf{F}^{\times 2} \cup \Delta\mathbf{F}^{\times 2}$. Indeed, write $\xi = (a + b\delta)^2$, so that $\xi = (a^2 + b^2\Delta) + 2ab\delta$. Now $\xi \in \mathbf{F}^\times$ implies that $2ab = 0$, i.e., $a = 0$ or $b = 0$, so that $\xi = b^2\Delta$ or a^2, respectively.

First assume that $z \in \mathbf{E}^1 \cap \mathbf{E}^{\times 2}$. If $\xi z \in \mathbf{E}^{\times 2}$ it follows that $\xi \in \mathbf{E}^{\times 2}$. Modulo squares in $\mathbf{F}^\times$, there are exactly two values of ξ, namely 1 and Δ.

Now assume that $z \notin \mathbf{E}^{\times 2}$. By Hilbert's Theorem 90 there exists $w \in \mathbf{E}^\times$ such that $z = w/\overline{w}$. Multiplying by w/w gives

$$z = \frac{1}{Nw} w^2.$$

Putting $\xi_0 = Nw$ we have that $\xi_0 z = w^2 \in \mathbf{E}^{\times 2}$. But $\xi z = \xi\xi_0^{-1} w^2 \in \mathbf{E}^{\times 2}$ implies that $\xi\xi_0^{-1}$ is a square in $\mathbf{E}^\times$. Modulo squares in $\mathbf{F}^\times$, we have that $\xi\xi_0^{-1} = 1$ or Δ and there are exactly two values of ξ, namely ξ_0 and $\xi_0\Delta$. □

2.3 $SO_{1,2}(\mathbf{F})$

Let $\mathbf{F}$ be a nonarchimedean local field, let $M_2(\mathbf{F})$ be the vector space of all two–by–two matrices with entries in $\mathbf{F}$, and let V be the subspace of $M_2(\mathbf{F})$ of trace–zero matrices. So

$$V = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbf{F} \right\}.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.5}$$

and define $\sigma : M_2(\mathbf{F}) \to M_2(\mathbf{F})$ by

$$\sigma(x) = AxA^{-1}. \tag{2.6}$$

Note that if $x \in GL_2(\mathbf{F})$,

$$(\sigma(x))^t = (\det x)x^{-1}.$$

This implies that for g invertible,

$$g^t\sigma(g) = \det(g)I.$$

In addition σ has the property

$$\sigma(xy) = \sigma(x)\sigma(y) \qquad \text{for all } x,\, y \in M_2(\mathbf{F}).$$

Now define a form $B(\cdot,\cdot)$ on V by

$$B(x, y) = \mathrm{trace}(\sigma(x)y^t). \tag{2.7}$$

We have a map $p : GL_2(\mathbf{F}) \to GL(V)$ defined by

$$p(g)x = gxg^{-1}.$$

This satisfies

$$\begin{aligned} B(p(g)x, p(g)y) &= \mathrm{trace}(\sigma(gxg^{-1})(gyg^{-1})^t) \\ &= \mathrm{trace}(\sigma(g)\sigma(x)\sigma(g^{-1})(g^{-1})^t y^t g^t) \\ &= \mathrm{trace}(g^t\sigma(g)\sigma(x)\sigma(g^{-1})(g^{-1})^t y^t) \\ &= \mathrm{trace}(\det(g)\sigma(x)\det(g^{-1})y^t) \\ &= \mathrm{trace}(\sigma(x)y^t) \\ &= B(x, y). \end{aligned}$$

So $p(g) \in O(V)$.

A basis of V consists of the elements

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We compute

$$\begin{aligned} B(X, X) &= 0 \\ B(X, Y) &= -1 \\ B(X, H) &= 0 \\ B(Y, Y) &= 0 \\ B(Y, H) &= 0 \\ B(H, H) &= -2. \end{aligned}$$

Therefore the matrix of $B(\cdot, \cdot)$ with respect to the ordered basis $\{X, Y, H\}$ is

$$[B] = \begin{pmatrix} & -1 & \\ -1 & & \\ & & -2 \end{pmatrix}.$$

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\begin{aligned} p(g)X &= \frac{1}{\det g}[(a^2)X + (-ac)H + (-c^2)Y] \\ p(g)Y &= \frac{1}{\det g}[(-b^2)X + (bd)H + (d^2)Y] \\ p(g)H &= \frac{1}{\det g}[(-2ab)X + (ad + bc)H + (2cd)Y]. \end{aligned}$$

Conseqently, with respect to the ordered basis $\{X, Y, H\}$, the map p is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{ad - bc} \begin{pmatrix} a^2 & -b^2 & -2ab \\ -c^2 & d^2 & 2cd \\ -ac & bd & ad + bc \end{pmatrix}.$$

In particular, the map p satisfies

$$diag(x, y) \mapsto diag(x/y, y/x, 1). \tag{2.8}$$

Also note that

$$[diag(x, 1/x, 1)]^t [B][diag(x, 1/x, 1)] = [B]$$

so that $diag(x, 1/x, 1) \in SO(V)$.

Now let $\mathbf{E}$ be the quadratic extension $\mathbf{F}(\delta)$ of $\mathbf{F}$ with $\delta^2 = \Delta \in \mathbf{F}$. Let ι be the corresponding embedding of $\mathbf{E}^\times$ in $GL_2(\mathbf{F})$. Another basis of V consists of the elements

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X' = \begin{pmatrix} 0 & -\Delta \\ 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad Y' = \begin{pmatrix} 0 & \Delta \\ 1 & 0 \end{pmatrix}.$$

We compute

$$\begin{aligned} B(X', X') &= 2\Delta \\ B(X', Y') &= 0 \\ B(X', H) &= 0 \\ B(Y', Y') &= -2\Delta \\ B(Y', H) &= 0 \\ B(H, H) &= -2. \end{aligned}$$

Therefore the matrix of $B(\cdot, \cdot)$ with respect to the ordered basis $\{H, X', Y'\}$ is

$$[B'] = diag(-2, 2\Delta, -2\Delta). \tag{2.9}$$

Letting $g = \begin{pmatrix} a & b\Delta \\ b & a \end{pmatrix}$ we have

$$\begin{aligned} p(g)H &= \frac{1}{\det g}[(a^2 + b^2\Delta)H + (2ab)X'] \\ p(g)X' &= \frac{1}{\det g}[(2ab\Delta)H + (a^2 + b^2\Delta)X'] \\ p(g)Y' &= Y'. \end{aligned} \tag{2.10}$$

We have shown that with respect to the ordered basis $\{H, X', Y'\}$, if $g \in \iota(\mathbf{E}^\times)$ then

$$g \mapsto \begin{pmatrix} g^2/(\det g) & \\ & 1 \end{pmatrix}.$$

Also note that

$$[diag(g,1)]^t[B'][diag(g,1)] = [B'].$$

In particular $diag(g,1)$ is conjugate to an element of $O(V)$. So if $\det g = 1$, $diag(g,1)$ is conjugate to a matrix in $SO(V)$.

Definition 2.11. *Let $g \in GL_2(\mathbf{F})$. We let $\tau(g) \subseteq SO_{1,2}(\mathbf{F})$ be the unique conjugacy class of elements having the same nontrivial eigenvalues as those of g.*

Proposition 2.12. *If $g \in SL_2(\mathbf{F})$ then $p(g) \in \tau(g^2)$.*

Remark. Fix a Cartan subgroup T of $SL_2(\mathbf{F})$. Then with respect to an appropriate basis of V,

$$\tau(g) = \begin{pmatrix} g & \\ & 1 \end{pmatrix}, \qquad g \in T.$$

This follows from the preceding computations.

In what follows, we do not distinguish between the conjugacy class $\tau(g)$ and elements of the conjugacy class. Thus if π is a representation with character Θ_π, we write $\Theta_\pi(\tau(g))$ for $\Theta_\pi(h)$ where h is any element of $\tau(g)$. Our main formulas, given in Chapter 5, relate characters of $\widetilde{SL}_2(\mathbf{F})$ on an element $(g;\varepsilon)$ in $\widetilde{SL}_2(\mathbf{F})$ to a character of $SO_{1,2}(\mathbf{F})$ on any element of $\tau(g)$.

Proposition 2.13. *Let $\mathbf{E} = \mathbf{F}(\sqrt{\Delta})$. Let ι be an embedding of $\mathbf{E}^\times$ in $GL_2(\mathbf{F})$ and $\varphi(z) = z/\overline{z}$. The map defined by $\iota'(z) = \tau(\iota(z))$, $z \in \mathbf{E}^1$, has image in $SO(V)$. Moreover, the diagram*

$$\begin{array}{ccc} \mathbf{E}^\times & \xrightarrow{\varphi} & \mathbf{E}^1 \\ \iota\downarrow & & \downarrow\iota' \\ GL(2) & \xrightarrow{p} & SO(V) \end{array}$$

commutes.

Proof. The first part follows from Proposition 2.12. By (2.10), if $z = x + y\delta \in \mathbf{E}^\times$ then

$$p\iota(z) = \begin{pmatrix} \iota(z^2)/Nz & \\ & 1 \end{pmatrix}$$

and

$$\iota'\varphi(z) = \iota'(z/\overline{z}) = \iota'(z^2/Nz) = \begin{pmatrix} \iota(z^2)/Nz & \\ & 1 \end{pmatrix}$$

as desired. □

2.4 Covering Groups

Let G be a locally compact topological group. We first explain how to define n–fold covers of G. See Moore, [25] and [26] for more details. Let μ_n denote the group of n–th roots of unity in $\mathbf{F}$. The group G acts trivially on μ_n.

Definition 2.14. *A Borel 2–cocycle c is a Borel measurable map*

$$c : G \times G \to \mu_n$$

which satisfies

$$c(g_1g_2, g_3)c(g_1, g_2) = c(g_1, g_2g_3)c(g_2, g_3)$$

and

$$c(1, g) = c(g, 1) = 1$$

for all g, g_i *in* G. *The cocycle* c *is said to be trivial if there is a map* $s : G \to \mu_n$ *which satisfies* $c(g, g')s(gg') = s(g)s(g')$. *Otherwise the cocycle is said to be nontrivial.*

Definition 2.15. *Let* c *be a Borel 2–cocycle. The* n*–fold covering group* $\widetilde{G}$ *of* G *with associated cocycle* c *is defined to be*

$$\widetilde{G} = \{(g; \varepsilon) : g \in G, \varepsilon \in \mu_n\}$$

with group law

$$(g; \varepsilon)(g'; \varepsilon') = (gg'; \varepsilon\varepsilon' c(g, g')).$$

The projection map $pr : \widetilde{G} \to G$ *is* $pr(g; \varepsilon) = g$. *The group* $\widetilde{G}$ *is a group extension of* G *in the sense that the sequence*

$$1 \longrightarrow \mu_n \longrightarrow \widetilde{G} \xrightarrow{pr} G \longrightarrow 1$$

is exact. This extension is central because μ_n *is contained in the center of* G *via the map* $\varepsilon \to (1; \varepsilon)$.

Definition 2.16. *Suppose that* G *is abelian. Let* g, h *be elements of* G *and* $\widetilde{g} \in pr^{-1}(g)$, $\widetilde{h} \in pr^{-1}(h)$. *We define the commutator of* g *and* h, *written* $\{g, h\}$, *to be*

$$\{g, h\} = \widetilde{g}\widetilde{h}\widetilde{g}^{-1}\widetilde{h}^{-1}. \tag{2.17}$$

It is clear that $\{g, h\}$ *is independent of the choices of* $\widetilde{g}$ *and* $\widetilde{h}$.

We now define the covering groups that we will use. We will construct two–fold covers of $SL_2(\mathbf{F})$ and $GL_2(\mathbf{F})$ by explicitly specifying the cocycle. We use the construction given in Gelbart [9], chapter 2.

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is any 2×2 matrix with entries in $\mathbf{F}$, define a map x by

$$x(g) = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{otherwise.} \end{cases} \tag{2.18}$$

Assume that $g_1, g_2 \in SL_2(\mathbf{F})$. Define a map c_α by

$$c_\alpha(g_1, g_2) = (x(g_1), x(g_2))_{\mathbf{F}}(-x(g_1)x(g_2), x(g_1g_2))_{\mathbf{F}}. \tag{2.19}$$

The following proposition is proved by Kubota [22].

Proposition 2.20. *The map c_α given in (2.19) is a nontrivial Borel 2–cocycle on $SL_2(\mathbf{F})$.*

Thus we have defined a two–fold covering group $\widetilde{SL}_2(\mathbf{F})$ of $SL_2(\mathbf{F})$ associated to the cocycle c_α. All nontrivial two–fold covers of $SL_2(\mathbf{F})$ are isomorphic to $\widetilde{SL}_2(\mathbf{F})$ (see Gelbart [9], pp. 14–15 for the proof).

We extend c_α to $GL_2(\mathbf{F})$, and, following Gelbart [9] define similarly a covering group $\widetilde{GL}_2(\mathbf{F})$ of $GL_2(\mathbf{F})$. This will describe a two–fold cover of $GL_2(\mathbf{F})$ which is a semidirect product of $\widetilde{SL}_2(\mathbf{F})$ and $\mathbf{F}^\times$.

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{F})$. Put

$$q(g) = \begin{pmatrix} a & b \\ c/\det g & d/\det g \end{pmatrix}. \tag{2.21}$$

We have that $q(g) \in SL_2(\mathbf{F})$. In addition, if $y \in \mathbf{F}^\times$, define

$$g^y = [diag(1,y)]^{-1} g [diag(1,y)] \tag{2.22}$$

and

$$v(y,g) = \begin{cases} 1 & \text{if } c \neq 0 \\ (y,d)_{\mathbf{F}} & \text{otherwise.} \end{cases} \tag{2.23}$$

A Borel cocycle c_β on $GL_2(\mathbf{F})$ is

$$c_\beta(g_1,g_2) = c_\alpha(q(g_1)^{\det g_2}, q(g_2)) v(\det(g_2), q(g_1)). \tag{2.24}$$

Note that c_β restricted to $SL_2(\mathbf{F})$ is equal to c_α.

Kazhdan and Patterson [19] have a convenient simplification of these formulas. In the same notation as above, the cocycle c_β on $GL_2(\mathbf{F})$ may be written

$$c_\beta(g_1,g_2) = \left(\frac{x(g_1g_2)}{x(g_1)}, \frac{x(g_1g_2)}{x(g_2)} \right)_{\mathbf{F}} \left(\det(g_1), \frac{x(g_1g_2)}{x(g_1)} \right)_{\mathbf{F}}. \tag{2.25}$$

Restricting this to $SL_2(\mathbf{F})$ we get the simple formula

$$c_\alpha(g_1,g_2) = \left(\frac{x(g_1g_2)}{x(g_1)}, \frac{x(g_1g_2)}{x(g_2)} \right)_{\mathbf{F}}. \tag{2.26}$$

The following proposition is mentioned in Gelbart [9] and is proved in Kubota [23].

Proposition 2.27. *Let $y \in \mathbf{F}^\times$. The map*

$$(g;\varepsilon) \mapsto (g^y; \varepsilon v(y,g))$$

is an automorphism of $\widetilde{SL}_2(\mathbf{F})$. It therefore determines a semidirect product of $\widetilde{SL}_2(\mathbf{F})$ and $\mathbf{F}^\times$. This semidirect product is isomorphic to the covering group $\widetilde{GL}_2(\mathbf{F})$ defined by (2.24).

Instead of c_β, it will be necessary to use a more convenient, but equivalent, cocycle c in defining $\widetilde{GL}_2(\mathbf{F})$. Define $s : GL_2(\mathbf{F}) \to \{\pm 1\}$ by

$$s(g) = \begin{cases} (c, d/\det g)_{\mathbf{F}} & \text{if } cd \neq 0 \text{ and } ord(c) \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases} \tag{2.28}$$

Then define c by

$$c(g_1, g_2) = c_\beta(g_1, g_2) s(g_1) s(g_2) s(g_1 g_2)^{-1}.$$

This is the cocycle we will use to do all of our calculations. (Recall that if $x \in \mathbf{F}$ and ϖ is a uniformizer then $x = \varpi^{ord(x)} u$ for some unit u.)

Definition 2.29. *If $g \in GL_2(\mathbf{F})$ then g^y is always given by (2.22). If $(g;\varepsilon) \in \widetilde{GL}_2(\mathbf{F})$, then $(g;\varepsilon)^y$ always denotes the element*

$$(g;\varepsilon)^y = (diag(1,y);1)^{-1}(g;\varepsilon)(diag(1,y);1). \tag{2.30}$$

Definition 2.31. *Let $\widetilde{GL}_2(\mathbf{F})_+$ be the subgroup of $\widetilde{GL}_2(\mathbf{F})$ defined by*

$$\widetilde{GL}_2(\mathbf{F})_+ = \{(g;\varepsilon) : \det g \in \mathbf{F}^{\times 2}\}. \tag{2.32}$$

If G is any group, we write $Z(G)$ for the center of G. The following proposition computes the centers of the various covering groups we have mentioned thus far. Let I denote the 2×2 identity matrix.

Proposition 2.33.

$$Z(\widetilde{SL}_2(\mathbf{F})) = \{(\pm I; \pm 1)\}$$
$$Z(\widetilde{GL}_2(\mathbf{F})) = \{(y^2 I; \pm 1) : y \in \mathbf{F}^\times\}$$
$$Z(\widetilde{GL}_2(\mathbf{F})_+) = \{(yI; \pm 1) : y \in \mathbf{F}^\times\}.$$

Proof. Write $\widetilde{G}$ for any of the three groups $\widetilde{SL}_2(\mathbf{F})$, $\widetilde{GL}_2(\mathbf{F})$, $\widetilde{GL}_2(\mathbf{F})_+$ and G for $pr(\widetilde{G})$. Clearly, if $\widetilde{z} = (z;\varepsilon) \in Z(\widetilde{G})$, then $z \in Z(G)$. So we may assume that $z = yI$.

If $g \in G$, then

$$c(z,g) = (y, x(g))_{\mathbf{F}} s(z)s(g)s(zg)^{-1},$$

while

$$c(g,z) = (y, x(g))_{\mathbf{F}} \cdot (y, \det g)_{\mathbf{F}} s(g)s(z)s(gz)^{-1}.$$

By definition, $s(z) = s(yI) = 1$ so that

$$s(z)s(g)s(zg)^{-1} = s(g)s(z)s(zg)^{-1} = s(g)s(z)s(gz)^{-1}.$$

Therefore, the commutator $\{z, g\}$ equals

$$\widetilde{z}\widetilde{g}\widetilde{z}^{-1}\widetilde{g}^{-1} = (y, \det g)_{\mathbf{F}}. \tag{2.34}$$

Hence $\widetilde{z} \in Z(\widetilde{G})$ if and only if $(y, \det g)_{\mathbf{F}} = 1$ for all $g \in G$.

If $\widetilde{G} = \widetilde{GL}_2(\mathbf{F})$, then $(y, \det g)_{\mathbf{F}} = 1$ for all $g \in G$ implies that y is a square, by the nondegeneracy of the Hilbert symbol. Therefore,

$$Z(\widetilde{GL}_2(\mathbf{F})) = \{(y^2 I; \pm 1) : y \in \mathbf{F}^\times\}.$$

If $\widetilde{G} = \widetilde{GL}_2(\mathbf{F})_+$ then $\det g$ is a square for all $g \in G$. The commutator is 1 for any y. Therefore,

$$Z(\widetilde{GL}_2(\mathbf{F})_+) = \{(yI; \pm 1) : y \in \mathbf{F}^\times\}.$$

Finally, if $\widetilde{G} = \widetilde{SL}_2(\mathbf{F})$, then $\det g = 1$ for all $g \in G$. Thus the commutator $\{z, g\} = 1$ for all g. The element $\widetilde{z} = (z, \varepsilon)$ will be in the center only if $\det z = y^2 = 1$, i.e. $y = \pm 1$. This shows that $Z(\widetilde{SL}_2(\mathbf{F})) = \{(\pm I; \pm 1)\}$. □

We end this section by doing some calculations which will simplify the formulas in Chapter 4.

Lemma 2.35. *If a, b, c, and d are in $\mathbf{F}^\times$, then*

$$c(diag(a,b), diag(c,d)) = (a,d)_{\mathbf{F}}. \tag{2.36}$$

Proof. By (2.25), we have

$$\begin{aligned} c_\beta(diag(a,b), diag(c,d)) &= (d,b)_{\mathbf{F}}(ab,d)_{\mathbf{F}} \\ &= (ab^2,d)_{\mathbf{F}} \\ &= (a,d)_{\mathbf{F}}. \end{aligned}$$

Since s is trivial on diagonal matrices, the result follows. □

Lemma 2.37. *For $\xi \in \mathbf{F}^\times$, $g \in GL_2(\mathbf{F})$,*

$$(-1,\xi)_{\mathbf{F}} s(\xi g) c(\xi I, g) = s(g) c_\beta(\xi I, -g). \tag{2.38}$$

Proof. We have that

$$\begin{aligned} (-1,\xi)_{\mathbf{F}} s(\xi g) c(\xi I, g) &= (-1,\xi)_{\mathbf{F}} s(\xi g) c_\beta(\xi I, g) s(\xi I) s(g) s(\xi g)^{-1} \\ &= s(g)(-1,\xi)_{\mathbf{F}} c_\beta(\xi I, g) \end{aligned}$$

since s is trivial on the diagonal matrix ξI. But

$$\begin{aligned} (-1,\xi)_{\mathbf{F}} c_\beta(\xi I, g) &= (-1,\xi)_{\mathbf{F}}(x(g),\xi)_{\mathbf{F}} \\ &= (\xi, -x(g))_{\mathbf{F}} \\ &= c_\beta(\xi I, -g) \end{aligned}$$

so the result follows. □

2.5 Weil's Gamma Factors

Let ψ be a nontrivial additive character of $\mathbf{F}$. If $a \in \mathbf{F}^\times$, the character ψ_a of $\mathbf{F}$ is given by

$$\psi_a(x) = \psi(ax). \tag{2.39}$$

Definition 2.40. *Let* $\gamma_{\mathbf{F}}(\psi)$ *be the Weil index of the map* $x \mapsto \psi(x^2)$ *(see [42]). Define* $\gamma_{\mathbf{F}}(x, \psi)$ *by*

$$\gamma_{\mathbf{F}}(x, \psi) = \frac{\gamma_{\mathbf{F}}(\psi_x)}{\gamma_{\mathbf{F}}(\psi)}. \tag{2.41}$$

From this definition, it immediately follows that $\gamma_{\mathbf{F}}(1, \psi) = 1$.

The next proposition gives the basic properties of the gamma–factors just defined. This is stated in Rao, [31].

Proposition 2.42. *The following formulas are valid for all* x *and* y *in* $\mathbf{F}^{\times}$.

1. $\gamma_{\mathbf{F}}(xy, \psi) = \gamma_{\mathbf{F}}(x, \psi)\gamma_{\mathbf{F}}(y, \psi)(x, y)_{\mathbf{F}}$.
2. $\gamma_{\mathbf{F}}(x^2 y, \psi) = \gamma_{\mathbf{F}}(y, \psi)$. *Hence* $\gamma_{\mathbf{F}}(x^2, \psi) = 1$ *and* $\gamma_{\mathbf{F}}(x^{-1}, \psi) = \gamma_{\mathbf{F}}(x, \psi)$.
3. $\gamma_{\mathbf{F}}(x, \psi_\xi) = (x, \xi)_{\mathbf{F}}\gamma_{\mathbf{F}}(x, \psi)$. *Hence* $\gamma_{\mathbf{F}}(x, \psi_{\xi^2}) = \gamma_{\mathbf{F}}(x, \psi)$.
4. $\{\gamma_{\mathbf{F}}(x, \psi)\}^2 = (-1, x)_{\mathbf{F}} = (x, x)_{\mathbf{F}}$. *Hence* $\gamma_{\mathbf{F}}(x, \psi)^{-1} = \gamma_{\mathbf{F}}(x, \psi)(-1, x)_{\mathbf{F}}$ *and* $\{\gamma_{\mathbf{F}}(x, \psi)\}^4 = 1$.

Proof. The formulas (1) and (2) are proved in Weil [42]. The other formulas follow from this. To prove (3), note that $\gamma_{\mathbf{F}}(x, \psi_\xi) = \gamma_{\mathbf{F}}(\psi_{x\xi})/\gamma_{\mathbf{F}}(\psi_\xi)$. Thus

$$\begin{aligned}
\gamma_{\mathbf{F}}(x, \psi_\xi)\gamma_{\mathbf{F}}(\xi, \psi) &= \left(\gamma_{\mathbf{F}}(\psi_{x\xi})/\gamma_{\mathbf{F}}(\psi_\xi)\right)\left(\gamma_{\mathbf{F}}(\psi_\xi)/\gamma_{\mathbf{F}}(\psi)\right) \\
&= \gamma_{\mathbf{F}}(x\xi, \psi) \\
&= \gamma_{\mathbf{F}}(x, \psi)\gamma_{\mathbf{F}}(\xi, \psi)(x, \xi)_{\mathbf{F}}.
\end{aligned}$$

Cancelling $\gamma_{\mathbf{F}}(\xi, \psi)$ gives (3). To prove (4), we use (1) and (2) to write

$$\gamma_{\mathbf{F}}(x, \psi)^2 = \gamma_{\mathbf{F}}(x^2, \psi)(x, x)_{\mathbf{F}} = (x, x)_{\mathbf{F}}.$$

□

2.6 Representation Theory

We state some basic definitions and theorems in the theory of representations of p–adic groups which we will need later. The basic references for this section are Silberger [36] and Gelbart [8]. Throughout this section, G is a locally compact p–adic group.

Definition 2.43. *Let Z be the center of G. If there is a character χ such that $\pi(z)v = \chi(z)v$ for all $z \in Z$ then we call π a χ–representation and say that χ is the central character of π.*

We remark that by Schur's Lemma, if π is irreducible then π has a central character. In general, if π has a central character then we denote it by χ_π.

Definition 2.44. *Let n be a positive integer. Let $\mathbf{F}$ be a field which contains μ_n, the n–th roots of unity. Then a character χ of $\mathbf{F}^\times$ is said to be n–even if $\chi(\mu_n) = \{1\}$. If $n = 2$ then we drop n from the notation and say that χ is even. If $n = 2$ and χ is not even then we say that χ is odd.*

Definition 2.45. *A χ–representation of G is called n–even if χ is an n–even character. If $n = 2$, we drop n from the notation and say simply that the representation is even.*

Definition 2.46. *Let H be a subgroup of G and let π be a representation of G on a complex vector space V. We write $\pi|_H$ for the restriction of the homomorphism π of G to H.*

Definition 2.47. *Let G be a group and H be a closed subgroup. Let δ_H denote the modular function for H. Let σ be a representation of H on a complex vector space V. Consider the space $F(G, V)$ of all locally constant functions $f : G \to V$*

satsifying $f(hg) = \delta_H(h)^{1/2}\sigma(h)f(g)$ *for all* $h \in H$, $g \in G$. *The representation* π *of* G *on the space* $F(G,V)$ *is called the representation induced by* σ *on* G *if*

$$\pi(g)(f)(x) = f(xg)$$

for all $x, g \in G$. *We write*

$$\pi = Ind_H^G(\sigma).$$

Definition 2.48. *Let* G *be a group and* $\widetilde{G}$ *be a two–fold cover of* G. *Let* $\widetilde{\pi}$ *be an irreducible representation of* $\widetilde{G}$ *on a complex vector space* V. *Then* $\widetilde{\pi}$ *is called genuine if* $\widetilde{\pi}$ *does not factor to* G.

Definition 2.49. *Let* G *be a group and* H *a closed subgroup of* G. *Let* π *be a representation of* H *on a complex vector space* V. *For any* $x \in G$, *the representation* π^x *is the representation of* $x^{-1}Hx$ *on* V *given by*

$$\pi^x(x^{-1}hx) = \pi(h), \qquad h \in H.$$

Definition 2.50. *Let* V *be a complex vector space. A representation* $\pi : G \to Aut(V)$ *is said to be admissible if*

1. *The stabilizer in* G *of each* v *in* V *is an open subgroup of* G, *and*

2. *For every compact open subgroup* K *of* G,

$$V^K = \{v \in V : \pi(k)v = v \text{ for all } k \in K\}$$

is finite–dimensional.

Definition 2.51 (Principal series, $GL_2(\mathbf{F})$**).** *Let* $G = GL_2(\mathbf{F})$. *Let* B *be the subgroup*

$$B = \left\{ \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} : x, y \in \mathbf{F}^\times, z \in \mathbf{F} \right\}.$$

Let η_1 and η_2 be two characters of $\mathbf{F}^\times$. Let σ be the character of B defined by

$$\sigma \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} = \eta_1(x)\eta_2(y).$$

Define the principal series representation $\pi(\eta_1, \eta_2)$ of $GL_2(\mathbf{F})$ to be

$$\pi(\eta_1, \eta_2) = Ind_B^G \sigma.$$

Proposition 2.52. *A principal series $\pi(\eta_1, \eta_2)$ is irreducible except when $\eta(x) = \eta_1\eta_2^{-1}(x) = |x|$ or $|x|^{-1}$. If $\eta(x) = |x|^{-1}$ then $\pi(\eta_1, \eta_2)$ contains a one–dimensional invariant subspace and the representation induced on the resulting factor space is irreducible. If $\eta(x) = |x|$, then $\pi(\eta_1, \eta_2)$ contains an irreducible invariant subspace of codimension one. Moreover, $\pi(\eta_1, \eta_2)$ and $\pi(\lambda_1, \lambda_2)$ are equivalent if and only if $(\eta_1, \eta_2) = (\lambda_1, \lambda_2)$ or (λ_2, λ_1).*

Proof. See Gelbart [8]. □

Definition 2.53 (Special representation). *If $\pi(\eta_1, \eta_2)$ is reducible, the resulting infinite dimensional subquotients of $\pi(\eta_1, \eta_2)$ are called special representations and are denoted $\sigma(\eta_1, \eta_2)$.*

Definition 2.54. *We call a principal series strongly even if both μ_1 and μ_2 are even characters.*

Definition 2.55 (Supercuspidal representation). *Suppose that π is an admissible representation of $G = GL_2(\mathbf{F})$ on V. Let N be the subgroup*

$$N = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbf{F} \right\}$$

of G. Then π is supercuspidal if for all $v \in V$,

$$\int_U \pi(n)v \, dn = 0$$

for some compact open subgroup U of N.

Proposition 2.56. *Suppose that π is an irreducible admissible representation of $GL_2(\mathbf{F})$. If π is not supercuspidal, then it is a subrepresentation of a principal series representation. In particular, if it is infinite dimensional then it is an irreducible principal series or a special representation. Conversely, if π is supercuspidal, then it is not equivalent to a subquotient of any principal series representation.*

Proof. See Gelbart [8]. □

We now give a brief explanation of principal series representations of $\widetilde{GL}_2(\mathbf{F})$. Let $A = \{diag(x, 1/x) : x \in \mathbf{F}^\times\}$. Hence $\widetilde{A} \simeq \widetilde{B}/N$. Let B_0 denote the subgroup of B whose diagonal entries have even v–adic order. (Recall that if $x \in \mathbf{F}$ and ϖ is a uniformizer then $x = \varpi^{ord_v(x)}u$ for some unit u.) Then $\widetilde{A}_0 = \widetilde{B}_0/N \simeq A_0 \times \mathbf{Z}/2\mathbf{Z}$ is a maximal abelian subgroup of finite index in $\widetilde{A}$. Moreover, $\widetilde{A}^2 \subseteq \widetilde{A}_0$. Let $\widetilde{\eta} = (\eta_1, \eta_2)$ be a genuine one–dimensional representation of $\widetilde{A}^2$ and extend this to $\widetilde{A}_0$ arbitrarily. Then extend $\widetilde{\eta}$ to $\widetilde{B}_0$ by requiring that $\widetilde{\eta}$ be trivial on N. Any genuine representation of $\widetilde{GL}_2(\mathbf{F})$ which is not supercuspidal is a subquotient of an induced representation $Ind_{\widetilde{B}_0}^{\widetilde{GL}_2(\mathbf{F})}(\widetilde{\tau})$ where $\widetilde{\tau} = Ind_{\widetilde{A}_0}^{\widetilde{A}}\widetilde{\eta}$. (For the proof, see Gelbart [9].) We write $\widetilde{\pi}(\eta_1, \eta_2) = Ind_{\widetilde{B}_0}^{\widetilde{GL}_2(\mathbf{F})}(\widetilde{\tau})$. Flicker [6] points out that this depends only on the restriction of the pair (η_1, η_2) to $\widetilde{A}^2$.

We end this section by recalling the theorem on the existence of a global character of an irreducible representation of a group G.

Definition 2.57. *Let G be a locally compact group and Z be the center of G. Let π be an irreducible admissible representation of G with central character χ. Let f be any function satisfying the property $f(zg) = \chi^{-1}(z)f(g)$ for all $z \in Z$,*

$g \in G$. Then define

$$\pi(f) = \int_{G/Z} \pi(g) f(g)\, dg.$$

Here dg is a Haar measure on G/Z.

Recall that the regular set (see Harish–Chandra [13]) is open and dense in G and its complement in G has measure zero.

The following theorem states that every admissible representation of a group G has a character defined on the regular set. The theorem holds for all of the groups that we will consider. It holds in particular for G any reductive algebraic group; it also holds for both of the covering groups $\widetilde{SL}_2(\mathbf{F})$ and $\widetilde{GL}_2(\mathbf{F})$. The proof is due to Harish–Chandra [14]. See also Flicker [6], p. 140 and Flicker–Kazhdan [7], p. 68.

Theorem 2.58. *The operator $\pi(f)$ is of finite rank. Let trace$(\pi(f))$ denote the trace of this operator. Then there is conjugation–invariant function Θ_π, defined on the regular set of G, which satisfies*

$$\mathit{trace}(\pi(f)) = \int_{G/Z} f(g)\Theta_\pi(g)\, dg.$$

The function Θ_π is called the character of π. Equivalent irreducible representations have the same character and conversely. If χ is the central character of π, then Θ_π transforms by χ on Z.

Note that if $\widetilde{\pi}$ is genuine, then $\Theta_{\widetilde{\pi}}(g;\varepsilon) = \varepsilon\Theta_{\widetilde{\pi}}(g;1)$.

2.7 Flicker's Correspondence

In this section, let $G = GL_2(\mathbf{F})$ and let $\widetilde{G}$ be an n–fold cover of G. In [6], Flicker has defined a correspondence between genuine irreducible admissible representations $\widetilde{\pi}$ of $\widetilde{G}$ and admissible representations π of G. We use Flicker's character formula extensively when we derive our character formulas.

Let g be regular with λ_1 and λ_2 the distinct eigenvalues of g. The map D is defined on regular g by

$$D(g) = \left| \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2} \right|^{1/2} \tag{2.59}$$

and satisfies

$$D(\zeta g) = D(g), \qquad\qquad \zeta \in \mathbf{F}^{\times}. \tag{2.60}$$

Flicker and Kazhdan [7], p. 93, have defined a constant b and showed that $b = n/(d|n^r/d|^{1/2})$. In their notation, we have taken $r = 2$ and $c = 0$, so that $d = gcd(n, r - 1 + 2rc) = 1$. Thus

$$b = n/|n|_p.$$

Kazhdan and Patterson have shown that

$$|\mathbf{F}^{\times}/\mathbf{F}^{\times n}| = n^2/|n|_p$$

(see Kazhdan and Patterson [19], Lemma 0.3.2 or Flicker and Kazhdan [7], Lemma 3.24.2). The constant b becomes

$$b = \frac{|\mathbf{F}^{\times}/\mathbf{F}^{\times n}|}{n}.$$

Flicker's correspondence for n–fold covers appears in Definition 5.0.1 of [6] and is repeated here. In our statement, we have corrected the character formula defining this correspondence to conform with the correspondence of Flicker and Kazhdan in [7].

Definition 2.61. *Suppose $\widetilde{\pi}$ is a genuine irreducible admissible representation of an n–fold cover of $GL_2(\mathbf{F})$ with central character $\widetilde{\chi}$ and that π is an admissible representation of $GL_2(\mathbf{F})$ with central character χ. Assume that χ and $\widetilde{\chi}$ satisfy the relationship*

$$\widetilde{\chi}(x^n I; 1) = \chi(xI)$$

for all $x \in \mathbf{F}^\times$. Then $\widetilde{\pi}$ and π correspond via Flicker's correspondence if whenever g^n is regular the following relationship holds:

$$D(g^n)\Theta_{\widetilde{\pi}}(g^n; s(g^n)^{-1}) = \begin{cases} b\, D(g)\Theta_\pi(g), & \text{if } g \text{ is elliptic} \\ b \displaystyle\sum_{\zeta \in \mu_n} D(g_\zeta)\Theta_\pi(g_\zeta), & \text{otherwise.} \end{cases}$$

In the above formula, $g_\zeta = diag(\zeta x, y)$ if g is conjugate to $diag(x, y)$.

Remark. For g hyperbolic, $s(g^n)^{-1} = 1$.

We are primarily interested in the case $n = 2$, so Flicker's correspondence is stated in this special case. When we quote Flicker's corresondence, we always mean the correspondence in this specific case. The constant b is

$$b = \frac{|\mathbf{F}^\times / \mathbf{F}^{\times 2}|}{2} = 2/|2|_p. \tag{2.62}$$

This is 2 if $p \neq 2$, and a power of 2 if $p = 2$.

Definition 2.63 (Flicker's Correspondence). *Let $\widetilde{\pi}$ be a genuine irreducible admissible representation of $\widetilde{GL}_2(\mathbf{F})$ with central character $\widetilde{\chi}$ and let π be an admissible representation of $GL_2(\mathbf{F})$ with central character χ. Suppose that χ and $\widetilde{\chi}$ satisfy the relationship*

$$\widetilde{\chi}(x^2 I; 1) = \chi(xI) \tag{2.64}$$

for all $x \in \mathbf{F}^{\times}$*. Then we say that* $\widetilde{\pi}$ *and* π *correspond via Flicker's correspondence if whenever* g^2 *is regular the following relationship holds:*

$$D(g^2)\Theta_{\widetilde{\pi}}(g^2; s(g^2)^{-1}) = \begin{cases} b\, D(g)\Theta_{\pi}(g), & \text{if } g \text{ is elliptic,} \\ b\,(D(g)\Theta_{\pi}(g) + D(g')\Theta_{\pi}(g')) & \text{if } g \text{ is hyperbolic.} \end{cases}$$

In the above formula, if g *is conjugate to the hyperbolic element* $diag(x, y)$ *then*

$$g' = diag(-x, y). \tag{2.65}$$

Suppose π and $\widetilde{\pi}$ correspond via Flicker's correspondence. Then we say that $\widetilde{\pi}$ is Flicker's lift of π and we write

$$\widetilde{\pi} = \mathrm{Lift}_{\mathrm{F}}(\pi). \tag{2.66}$$

Flicker has shown the following theorem about this correspondence (see Theorem 5.2 and Corollary 5.2.1 of Flicker [6]). Flicker's theorem is stated in the case of $n = 2$.

Theorem 2.67. *The correspondence* $\widetilde{\pi} = Lift_F(\pi)$ *is a bijection between irreducible admissible representations* π *of* $GL_2(\mathbf{F})$ *satisfying* $\chi_\pi(-I) = 1$ *and genuine irreducible admissible representations* $\widetilde{\pi}$ *of* $\widetilde{GL}_2(\mathbf{F})$*. If* π *is a supercuspidal representation of* $GL_2(\mathbf{F})$ *with even central character, then there is a supercuspidal representation* $\widetilde{\pi}$ *of* $\widetilde{GL}_2(\mathbf{F})$ *such that* $\widetilde{\pi} = Lift_F(\pi)$*. If* π *is a special representation* $\sigma(\mu_1, \mu_2)$ *of* $GL_2(\mathbf{F})$ *with both* μ_i *odd, then* $Lift_F(\pi)$ *is equivalent to an odd oscillator representation of* $\widetilde{GL}_2(\mathbf{F})$*. If* π *is an even one–dimensional representation of* $GL_2(\mathbf{F})$ *then* $Lift_F(\pi)$ *is an even oscillator representation of* $\widetilde{GL}_2(\mathbf{F})$*. (See Proposition 2.70 below.)*

Remark. An odd oscillator representation of $\widetilde{GL}_2(\mathbf{F})$ is supercuspidal. This is an example of a supercuspidal representation occuring in Flicker's lifting which does not come from a supercuspidal representation of $GL_2(\mathbf{F})$. This phenomenon is further illustrated by the following. Let $\pi = \pi(\mu_1, \mu_2)$ be a reducible principal series of $GL_2(\mathbf{F})$ such that μ_1 and μ_2 are both odd, i.e., π is even but not strongly even. Then Flicker's lifting of π is zero by Flicker's formula for principal series (see Proposition 2.68 below). On the other hand, π is the sum of a special representation and an odd one–dimensional representation. The special representation lifts to an odd oscillator representation and the odd one–dimensional lifts to minus that odd oscillator representation (Proposition 2.70 below). This gives a different way of seeing why the principal series π lifts to 0.

Flicker gives examples of special cases of this correspondence. He computes the correspondence for principal series and for the oscillator representations. We state these in the case $n = 2$. Flicker uses the full power of the trace formula to get the correspondence on supercuspidal representations.

Proposition 2.68. *Let η_1 and η_2 be even characters of $\mathbf{F}^\times$. Let μ_1 and μ_2 be two characters of $\mathbf{F}^{\times 2}$ which satisfy $\mu_i(x^2) = \eta_i(x)$ for all x and extend these to characters of $\mathbf{F}^\times$ arbitrarily. We have the character relation*

$$\Theta_{\widetilde{\pi}(\mu_1,\mu_2)}(g^2;1) = \begin{cases} b\,\dfrac{D(g)}{D(g^2)}\Theta_{\pi(\eta_1,\eta_2)}(g) & \textit{if } g \sim \textit{diag}(x,y) \\ 0 & \textit{otherwise.} \end{cases} \tag{2.69}$$

(In the above statement $g \sim h$ means that g is conjugate via $GL_2(\mathbf{F})$ to h.)

Proposition 2.70. *Let μ_0 be a character of $\mathbf{F}^\times$. Let $\omega(\mu)$ be the oscillator representation attached to μ on $\widetilde{GL}_2(\mathbf{F})$. Then*

$$\Theta_{\omega(\mu)}(g^2; s(g^2)^{-1}) = b\,\mu_0(-1)\mu(g)\frac{D(g)}{D(g^2)}$$

for g elliptic and

$$\Theta_{\omega(\mu)}\left(g^2;1\right) = b\left[\mu_0(-\det g)D(g) + \mu_0(\det g')D(g')\right]/D(g^2)$$

if $g = diag(x, y)$ is hyperbolic [with g' defined by (2.65)].

Corollary 2.71. *If μ is even, then*

$$D(g^2)\Theta_{\omega(\mu)}(g^2; s(g^2)^{-1}) = \begin{cases} b\,D(g)\mu(g) & \text{if } g \text{ is elliptic} \\ b\,(D(g)\mu(g) + D(g')\mu(g')) & \text{otherwise} \end{cases}$$

while if μ is odd,

$$D(g^2)\Theta_{\omega(\mu)}(g^2; s(g^2)^{-1}) = \begin{cases} -b\,D(g)\mu(g) & \text{if } g \text{ is elliptic} \\ -b\,(D(g)\mu(g) + D(g')\mu(g')) & \text{otherwise.} \end{cases}$$

$$= \begin{cases} b\,D(g)\Theta_\sigma(g) & \text{if } g \text{ is elliptic} \\ b\,(D(g)\Theta_\sigma(g) + D(g')\Theta_\sigma(g')) & \text{otherwise.} \end{cases}$$

Here g' is defined in (2.65) and σ is a special representation $\sigma(\mu|\cdot|^{-1/2}, \mu|\cdot|^{1/2})$ as in Definition 2.53.

The next proposition shows that characters of genuine admissible representations vanish off of elements of the form $(g^2; \varepsilon)$. This property, combined with Proposition 2.4 shows that when we apply the inversion formula (see Theorem 3.12 below) to Flicker's lifting, there will be exactly two nonzero terms which appear in the sum.

Proposition 2.72. *Let $\widetilde{\pi}$ be a genuine admissible representation of $\widetilde{GL}_2(\mathbf{F})$. Suppose that $y \in \widetilde{GL}_2(\mathbf{F})$ is regular and is not of the form $(g^2; \varepsilon)$. Then $\Theta_{\widetilde{\pi}}(y) = 0$.*

Proof. By Flicker [6], pp. 127–128, such y have the following property:

$$\text{there exists } h \in \widetilde{GL}_2(\mathbf{F}) \text{ such that } h^{-1}yh = y \cdot (1; -1).$$

Since $\widetilde{\pi}$ is genuine, we have

$$\Theta_{\widetilde{\pi}}(y) = \Theta_{\widetilde{\pi}}(h^{-1}yh) = \Theta_{\widetilde{\pi}}(y(1;-1)) = -\Theta_{\widetilde{\pi}}(y).$$

□

The commutator can be computed explicitly on an elliptic Cartan T_Δ.

Proposition 2.73. *Choose an embedding* $\iota : \mathbf{E}^\times = \mathbf{F}(\sqrt{\Delta}) \to T_\Delta$*. Let* $g = \iota(z)$ *and* $h = \iota(w)$*. Then*

$$\{g, h\} = (z, \overline{w})_{\mathbf{E}}.$$

Proof. See Flicker [6], pp. 127–128 and Blondel [4], p. 14. □

Remark. In the above proposition, $(\cdot,\cdot)_{\mathbf{E}}$ is the Hilbert symbol of the quadratic extension $\mathbf{E} = \mathbf{F}(\sqrt{\Delta})$ of $\mathbf{F}$. See Serre [33]. It satisfies the property

$$(\alpha, \beta)_{\mathbf{E}} = (N\alpha, \beta)_{\mathbf{F}}, \qquad \alpha \in \mathbf{E}^\times, \beta \in \mathbf{F}^\times.$$

Chapter 3

Representations of $\widetilde{GL}_2(\mathbf{F})_+$

We would like to derive a formula relating characters of $\widetilde{SL}_2(\mathbf{F})$ and characters of $SO_{1,2}(\mathbf{F})$. The first step is to restrict representations of $\widetilde{GL}_2(\mathbf{F})$ to $\widetilde{GL}_2(\mathbf{F})_+$; this is done in Theorem 3.8. The unexpected result says that the restriction is a sum of representations parameterized by $\mathbf{F}^\times/\mathbf{F}^{\times 2}$. We then derive an "inversion formula" which computes the character of an individual summand of this restriction if the character of the original representation is known. Finally, we explicitly parameterize summands appearing in the restriction of a representation to $\widetilde{GL}_2(\mathbf{F})_+$.

3.1 Restricting to $\widetilde{GL}_2(\mathbf{F})_+$

We address the problem of what happens when we restrict a genuine irreducible representation $\widetilde{\pi}$ of $\widetilde{GL}_2(\mathbf{F})$ to the subgroup $\widetilde{GL}_2(\mathbf{F})_+$.

Recall that $\widetilde{GL}_2(\mathbf{F})_+ = \{(h;\varepsilon) : \det h \in \mathbf{F}^{\times 2}\}$. Write

$$V = \mathbf{F}^\times/\mathbf{F}^{\times 2} = \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2 = \langle \alpha_1 \rangle \times \cdots \times \langle \alpha_n \rangle.$$

The Hilbert symbol $(\cdot,\cdot)_{\mathbf{F}}$ is a nondegenerate bilinear form on this $\mathbf{F}_2$ vector space (see Proposition 2.3). If W is a subspace of V, the orthogonal complement of W is

$$W^{\perp} = \{v \in V : (w,v)_{\mathbf{F}} = 1 \text{ for all } w \in W\ \}. \tag{3.1}$$

Write

$$\widetilde{H}_k = \widetilde{GL}_2(\mathbf{F})_{\alpha_1,\dots,\alpha_k} = \langle \left\{ (h;\varepsilon) : \det h \in \mathbf{F}^{\times 2} \cup \bigcup_{i=1}^{k} \alpha_i \mathbf{F}^{\times 2} \right\} \rangle \tag{3.2}$$

for $k = 0,\dots,n$. Note that $\widetilde{H}_0 = \widetilde{GL}_2(\mathbf{F})_+$ and $\widetilde{H}_n = \widetilde{GL}_2(\mathbf{F})$. We have the chain of inclusions

$$\widetilde{H}_0 \subseteq \widetilde{H}_1 \subseteq \cdots \subseteq \widetilde{H}_k \subseteq \cdots \subseteq \widetilde{H}_n$$

with $\widetilde{H}_k/\widetilde{H}_{k-1}$ of order 2 for all $k = 1,\dots,n$.

Proposition 3.3. *Let $\widetilde{H}$ be a subgroup of $\widetilde{GL}_2(\mathbf{F})$. Then $Z(\widetilde{H})$, the center of $\widetilde{H}$, equals*

$$Z(\widetilde{H}) = \{(h;\varepsilon) : h \in Z(H) \text{ and } (h, \det g)_{\mathbf{F}} = 1 \text{ for all } g \in H\}. \tag{3.4}$$

In particular, we have

$$\begin{aligned} Z(\widetilde{H}_k) &= \{(diag(y,y);\pm 1) : (\alpha_i, y)_{\mathbf{F}} = 1 \text{ for all } i = 1,\dots,k\} \\ &= \{(diag(y,y);\pm 1) : y \in \langle \alpha_1,\dots,\alpha_k\rangle^{\perp}\} \end{aligned}$$

for all $k = 0,\dots,n$.

Proof. The proof of this proposition is very similar to that of Proposition 2.33.

It is clear that if $(h;\varepsilon) \in Z(\widetilde{H}_k)$ then $h = diag(y,y)$ for some $y \in \mathbf{F}^{\times}$. The condition $(y, \det g)_{\mathbf{F}} = 1$ [see (2.34)] for all $g \in H_k$ means that $(y,\alpha_i)_{\mathbf{F}} = 1$ for all $i = 1,\dots,k$. This is true if and only if $y \in \langle \alpha_i \rangle^{\perp}$ for all $i = 1,\dots,k$. □

Proposition 3.5. *For some $\beta \in \langle \alpha_1, \ldots, \alpha_{k-1} \rangle^\perp$, we have that $(\alpha_k, \beta)_{\mathbf{F}} = -1$.*

Proof. Assume not. This implies that if $\beta \in \langle \alpha_1, \ldots, \alpha_{k-1} \rangle^\perp$, then $(\beta, \alpha_k)_{\mathbf{F}} = 1$, that is, $\beta \in \langle \alpha_k \rangle^\perp$. So $\langle \alpha_1, \ldots, \alpha_{k-1} \rangle^\perp \subseteq \langle \alpha_k \rangle^\perp$. Taking $^\perp$ of both sides, we get that $\langle \alpha_k \rangle \subseteq \langle \alpha_1, \ldots, \alpha_{k-1} \rangle$. This contradicts the fact that $\{\alpha_1, \ldots, \alpha_k\}$ is a linearly independent set. □

Let I be the 2×2 identity matrix.

Proposition 3.6. *Let $\widetilde{H}$ be a subgroup of $\widetilde{GL}_2(\mathbf{F})$ such that $\widetilde{SL}_2(\mathbf{F}) \subseteq \widetilde{H}$ and $Z(\widetilde{H}) \subseteq \{(xI; \varepsilon)\}$. Let $\widetilde{\sigma}$ be a genuine irreducible representation of $\widetilde{H}$ with central character $\chi_{\widetilde{\sigma}}$. Then*

$$\chi_{\widetilde{\sigma}^y}(zI; \varepsilon) = (y, z)_{\mathbf{F}} \cdot \chi_{\widetilde{\sigma}}(zI; \varepsilon).$$

Proof. This follows from the computation

$$(zI; \varepsilon)^y = (y, z)_{\mathbf{F}}(zI; \varepsilon)$$

(see Definition 2.29) and the fact that $\chi_{\widetilde{\sigma}}$ is genuine. □

Remark. Recall that $Z(\widetilde{H}_{k-1}) = \{(diag(y, y), \pm 1) : y \in \langle \alpha_1, \ldots, \alpha_{k-1} \rangle^\perp\}$. Proposition 3.6 says that $(diag(1, \alpha_k), 1)$ acts nontrivially on $Z(\widetilde{H}_{k-1})$.

Proposition 3.7 (Restriction principle). *Let $H \subseteq G$ be a subgroup of G having index two. Let y be a representative for the nontrivial coset of G/H. Let ω be the nontrivial character of G/H (so $\omega(h) = 1$, $\omega(yh) = -1$ for $h \in H$). Let π be an irreducible representation of G.*

1. Suppose $\pi \otimes \omega \neq \pi$. Then $\sigma = \pi|_H$ is irreducible. Furthermore $\sigma^y = \sigma$ and $Ind_H^G(\sigma) = \pi \oplus (\pi \otimes \omega)$.

2. Suppose $\pi \otimes \omega = \pi$. Then $\pi|_H = \sigma_1 \oplus \sigma_2$ is reducible. Here each σ_i is irreducible. Furthermore $\sigma_1 \neq \sigma_1^y \simeq \sigma_2$ and $Ind_H^G(\sigma_1) = Ind_H^G(\sigma_2)$ are irreducible.

Proof. See Bröcker and tom Dieck, [5]. $\square$

The following theorem achieves the goal of this section. It states that the restriction of a genuine irreducible representation of $\widetilde{GL}_2(\mathbf{F})$ is a sum of representations parameterized by $\mathbf{F}^\times/\mathbf{F}^{\times 2}$.

Theorem 3.8. *Let $\widetilde{\pi}$ be a genuine irreducible representation of $\widetilde{GL}_2(\mathbf{F})$. Let $\widetilde{\sigma}$ be an irreducible constituent of $\widetilde{\pi}|_{\widetilde{GL_2(\mathbf{F})_+}}$. Then*

$$\widetilde{\pi}|_{\widetilde{GL_2(\mathbf{F})_+}} = \sum_{\alpha \in \mathbf{F}^\times/\mathbf{F}^{\times 2}} \widetilde{\sigma}^\alpha.$$

Hence the irreducible constituents of $\widetilde{\pi}|_{\widetilde{GL_2(\mathbf{F})_+}}$ are parameterized by $\alpha \in \mathbf{F}^\times/\mathbf{F}^{\times 2}$.

Proof. Fix $k \geq 1$. Let $\widetilde{\rho}$ be an irreducible representation of $\widetilde{H}_k$. Let $\widetilde{\sigma}$ be an irreducible constituent of $\widetilde{\rho}_{\widetilde{H}_{k-1}}$. We may assume by Proposition 3.5 (and the remark after Proposition 3.6) that there exists β having the properties $(\alpha_k, \beta)_{\mathbf{F}} = -1$ and $(diag(\beta,\beta);\varepsilon) \in Z(\widetilde{H}_{k-1})$. By Proposition 3.6, the central characters $\chi_{\widetilde{\sigma}^{\alpha_k}}$ and $\chi_{\widetilde{\sigma}}$ are of opposite sign on $(diag(\beta,\beta);\varepsilon)$ in $Z(\widetilde{H}_{k-1})$. Therefore $\widetilde{\sigma}^{\alpha_k} \neq \widetilde{\sigma}$. Since $(diag(1,\alpha_k);\varepsilon)$ is in $\widetilde{H}_k$ but not in $\widetilde{H}_{k-1}$, the restriction principle gives that

$$\widetilde{\rho}|_{\widetilde{H}_{k-1}} = \widetilde{\sigma}_1 \oplus \widetilde{\sigma}_2$$

with each $\widetilde{\sigma}_i$ irreducible. The proof now follows by restriction in stages. $\square$

Remark. A statement of this result appears in Gelbart and Piatetski–Shapiro [11] on p. 101.

3.2 Inversion via the Center

In light of Theorem 3.8, we now consider the problem of finding a formula for the character of the individual constituents $\widetilde{\sigma}^\alpha$ appearing in the restriction of an irreducible genuine representation $\widetilde{\pi}$ of $\widetilde{GL}_2(\mathbf{F})$ to $\widetilde{GL}_2(\mathbf{F})_+$. We will show that $\Theta_{\widetilde{\sigma}^\alpha}$ can be written in terms of $\chi_{\widetilde{\sigma}^\alpha}$ and $\Theta_{\widetilde{\pi}}$.

To begin, we let $\widetilde{\pi}$ be a genuine irreducible representation of $\widetilde{GL}_2(\mathbf{F})$. By Theorem 3.8 we may write $\widetilde{\pi}|_{\widetilde{GL}_2(\mathbf{F})_+} = \sum_\alpha \widetilde{\sigma}^\alpha$, where each $\widetilde{\sigma}^\alpha$ is a genuine irreducible representation of $\widetilde{GL}_2(\mathbf{F})_+$. The group $\mathbf{F}^\times/\mathbf{F}^{\times 2}$ acts transitively on the set $\{\widetilde{\sigma}^\alpha\}$.

Fix a set of coset representatives $\{\alpha\}$ for $\mathbf{F}^\times/\mathbf{F}^{\times 2}$. For each α we have that

$$\chi_{\widetilde{\pi}}(x^2 I;\varepsilon) = \chi_{\widetilde{\sigma}^\alpha}(x^2 I;\varepsilon). \tag{3.9}$$

By Proposition 3.6 we also have

$$\chi_{(\widetilde{\sigma}^\alpha)^a}(xI;\varepsilon) = (a,x)_{\mathbf{F}}\chi_{\widetilde{\sigma}^\alpha}(xI;1) \tag{3.10}$$

for every α.

Let z_α be any element satisfying

$$pr(z_\alpha) = \alpha I. \tag{3.11}$$

Each z_α is in the center of $\widetilde{GL}_2(\mathbf{F})_+$.

Theorem 3.12 (Inversion Formula). *Let $\widetilde{\pi}$ be a genuine, irreducible representation of $\widetilde{GL}_2(\mathbf{F})$ and write*

$$\widetilde{\pi}|_{\widetilde{GL}_2(\mathbf{F})_+} = \sum_{\alpha\in\mathbf{F}^\times/\mathbf{F}^{\times 2}} \widetilde{\sigma}^\alpha.$$

Then for any $\alpha \in \mathbf{F}^\times/\mathbf{F}^{\times 2}$,

$$\Theta_{\widetilde{\sigma}^\alpha}(\widetilde{g}) = \frac{1}{|\mathbf{F}^\times/\mathbf{F}^{\times 2}|}\sum_{\xi\in\mathbf{F}^\times/\mathbf{F}^{\times 2}} \chi_{\widetilde{\sigma}^\alpha}(z_\xi)^{-1}\Theta_{\widetilde{\pi}}(z_\xi\widetilde{g}). \tag{3.13}$$

Here we abuse notation and write ξ for the coset $\xi\mathbf{F}^{\times 2}$ as well as an element of that coset.

Proof. First we show that the inversion formula is well defined. In particular, we claim that the formula is independent of the choice of coset representatives for $\mathbf{F}^\times/\mathbf{F}^{\times 2}$. Fix ξ and let $\xi' = y^2\xi$, so that $\xi\mathbf{F}^{\times 2} = \xi'\mathbf{F}^{\times 2}$. By (2.35), $(y^2\xi I; 1) = (y^2 I; 1)(\xi I; 1)$. It follows that

$$\begin{aligned}\chi_{\widetilde{\sigma}^\alpha}(z_{\xi'})^{-1}\Theta_{\widetilde{\pi}}(z_{\xi'}\widetilde{g}) &= \chi_{\widetilde{\sigma}^\alpha}(y^2I;1)^{-1}\chi_{\widetilde{\sigma}^\alpha}(z_\xi)^{-1}\Theta_{\widetilde{\pi}}((y^2I;1)z_\xi\widetilde{g}) \\ &= \chi_{\widetilde{\sigma}^\alpha}(y^2I;1)^{-1}\chi_{\widetilde{\sigma}^\alpha}(z_\xi)^{-1}\chi_{\widetilde{\pi}}(y^2I;1)\Theta_{\widetilde{\pi}}(z_\xi\widetilde{g}).\end{aligned}$$

From (3.9), we see

$$\begin{aligned}\chi_{\widetilde{\sigma}^\alpha}(z_{\xi'})^{-1}\Theta_{\widetilde{\pi}}(z_{\xi'}\widetilde{g}) &= \chi_{\widetilde{\sigma}^\alpha}(y^2I;1)^{-1}\chi_{\widetilde{\sigma}^\alpha}(z_\xi)^{-1}\chi_{\widetilde{\sigma}^\alpha}(y^2I;1)\Theta_{\widetilde{\pi}}(z_\xi\widetilde{g}) \\ &= \chi_{\widetilde{\sigma}^\alpha}(z_\xi)^{-1}\Theta_{\widetilde{\pi}}(z_\xi\widetilde{g}).\end{aligned}$$

This proves the claim.

Now we derive the inversion formula. Let z be in the center of $\widetilde{GL}_2(\mathbf{F})_+$. Then

$$\Theta_{\widetilde{\pi}}(z\widetilde{g}) = \sum_\alpha \chi_{\widetilde{\sigma}^\alpha}(z)\Theta_{\widetilde{\sigma}^\alpha}(\widetilde{g}).$$

For any constants c_ξ, we have

$$\begin{aligned}\sum_\xi c_\xi\Theta_{\widetilde{\pi}}(z_\xi\widetilde{g}) &= \sum_\xi c_\xi \sum_\alpha \chi_{\widetilde{\sigma}^\alpha}(z_\xi)\Theta_{\widetilde{\sigma}^\alpha}(\widetilde{g}) \\ &= \sum_\alpha \Theta_{\widetilde{\sigma}^\alpha}(\widetilde{g}) \sum_\xi c_\xi\chi_{\widetilde{\sigma}^\alpha}(z_\xi).\end{aligned}$$

Fix α_0 and let $c_\xi = \chi_{\widetilde{\sigma}^{\alpha_0}}(z_\xi)^{-1}$. Substitute in to get

$$\sum_\xi \chi_{\widetilde{\sigma}^{\alpha_0}}(z_\xi)^{-1}\Theta_{\widetilde{\pi}}(z_\xi\widetilde{g}) = \sum_\alpha \Theta_{\widetilde{\sigma}^\alpha}(\widetilde{g}) \sum_\xi \chi_{\widetilde{\sigma}^{\alpha_0}}(z_\xi)^{-1}\chi_{\widetilde{\sigma}^\alpha}(z_\xi).$$

By transitivity of the action of $\mathbf{F}^\times/\mathbf{F}^{\times 2}$ on the $\widetilde{\sigma}^\alpha$'s, we may pick a so that $(\widetilde{\sigma}^{\alpha_0})^a = \widetilde{\sigma}^\alpha$. (This a depends on α.) By (3.6),

$$\sum_\xi \chi_{\widetilde{\sigma}^{\alpha_0}}(z_\xi)^{-1}\Theta_{\widetilde{\pi}}(z_\xi\widetilde{g}) = \sum_\alpha \Theta_{\widetilde{\sigma}^\alpha}(\widetilde{g}) \sum_\xi (a,\xi)_{\mathbf{F}}.$$

But

$$\sum_{\xi}(a,\xi)_{\mathbf{F}} = \begin{cases} |\mathbf{F}^{\times}/\mathbf{F}^{\times 2}| & \text{if } a \text{ is a square} \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\Theta_{\widetilde{\sigma}^{\alpha_0}}(\widetilde{g}) = \frac{1}{|\mathbf{F}^{\times}/\mathbf{F}^{\times 2}|} \sum_{\xi} \chi_{\widetilde{\sigma}^{\alpha_0}}(z_{\xi})^{-1} \Theta_{\widetilde{\pi}}(z_{\xi}\widetilde{g}).$$

This completes the proof, since α_0 was arbitrary. □

3.3 Parameterization

Our goal in this section is to parameterize the representations which appear in the restriction to $\widetilde{GL}_2(\mathbf{F})_+$ of a Flicker lifting. In order to do this, some preparation and notation is required.

Fix, once and for all, a nontrivial character ψ of $\mathbf{F}$.

A two–fold cover of $\mathbf{F}^{\times}$ is given by the cocycle $c(x,y) = (x,y)_{\mathbf{F}}$.

Proposition 3.14. *Let $\gamma_{\mathbf{F}}(x,\psi)$ be Weil's gamma factor as defined in Definition 2.40. The map*

$$\widetilde{\lambda}(x;\varepsilon) = \gamma_{\mathbf{F}}(x,\psi)\varepsilon$$

is a genuine character of the two–fold cover of $\mathbf{F}^{\times}$.

Proof. We have

$$\begin{aligned} \widetilde{\lambda}((x;\varepsilon)(x';\varepsilon')) &= \widetilde{\lambda}(xx';\varepsilon\varepsilon'(x,x')_{\mathbf{F}}) \\ &= \gamma_{\mathbf{F}}(xx',\psi)\varepsilon\varepsilon'(x,x')_{\mathbf{F}} \\ &= \gamma_{\mathbf{F}}(x,\psi)\gamma_{\mathbf{F}}(x',\psi)(x,x')_{\mathbf{F}}\varepsilon\varepsilon'(x,x')_{\mathbf{F}} \\ &= \gamma_{\mathbf{F}}(x,\psi)\varepsilon\gamma_{\mathbf{F}}(x',\psi)\varepsilon' \\ &= \widetilde{\lambda}(x;\varepsilon)\widetilde{\lambda}(x';\varepsilon'); \end{aligned}$$

this establishes the proposition. □

Definition 3.15. *Suppose that ν_0 is any character of $\mathbf{F}^\times$. Define a character of $Z(\widetilde{GL}_2(\mathbf{F})_+)$ by the formula*

$$\chi_{\nu_0}(xI;\varepsilon) = \nu_0(x)\gamma_{\mathbf{F}}(x,\psi)\varepsilon. \tag{3.16}$$

It is clear that χ_{ν_0} is a genuine character of $Z(\widetilde{GL}_2(\mathbf{F})_+)$. Since the quotient of two genuine characters is not genuine, i.e., a character of $\mathbf{F}^\times$, it is also obvious that each genuine character of $Z(\widetilde{GL}_2(\mathbf{F})_+)$ is of the form χ_{ν_0} for some character ν_0 of $\mathbf{F}^\times$.

Now let π be an irreducible admissible representation of $GL_2(\mathbf{F})$ and let $\widetilde{\pi} = \mathrm{Lift}_{\mathrm{F}}(\pi)$. Recall that the central characters of $\widetilde{\pi}$ and π are related by (2.64):

$$\chi_{\widetilde{\pi}}(x^2I;1) = \chi_\pi(xI).$$

In particular, $\chi_\pi(-I) = 1$. The central character of any representation appearing in the restriction of $\widetilde{\pi}$ to $\widetilde{GL}_2(\mathbf{F})_+$ also satisfies (2.64), see (3.9).

Definition 3.17. *Let $\widetilde{\pi} = Lift_F(\pi)$. Let ν_0 be a character of $\mathbf{F}^\times$ which satisfies*

$$\chi_\pi(xI) = \nu_0(x^2). \tag{3.18}$$

If π and ν_0 satisfy (3.18), we define $L(\pi,\nu_0)$ to be the constituent of $\widetilde{\pi}|_{\widetilde{GL}_2(\mathbf{F})_+}$ with central character χ_{ν_0}.

Remark. Since $L(\pi,\nu_0)$ has central character χ_{ν_0}, by (3.9) it follows that

$$\chi_{\widetilde{\pi}}(x^2;1) = \nu_0(x^2)\gamma_{\mathbf{F}}(x^2,\psi) = \nu_0(x^2).$$

By (2.64) this is $\chi_\pi(xI)$. This explains condition (3.18).

Proposition 3.19. *Fix ν_0. Let $x \in \mathbf{F}^\times$. Put $q_x = (x, \cdot)_\mathbf{F}$. Then*

$$L(\pi, \nu_0)^x = L(\pi, \nu_0 q_x).$$

(See Definition 2.49.) Hence

$$\widetilde{\pi}|_{\widetilde{GL}_2(\mathbf{F})_+} = \sum_{x \in \mathbf{F}^\times / \mathbf{F}^{\times 2}} L(\pi, \nu_0 q_x).$$

Proof. This follows from the way we are parameterizing summands in the restriction and the "central character trick" [see (3.10)]:

$$\begin{aligned}\chi_{\nu_0}^x(yI; \varepsilon) &= (x, y)_\mathbf{F} \chi_{\nu_0}(yI; \varepsilon) \\ &= q_x(y)\nu_0(y)\gamma_\mathbf{F}(y, \psi)\varepsilon \\ &= \nu_0(y)\gamma_\mathbf{F}(y, \psi_x)\varepsilon \\ &= \chi_{\nu_0 q_x}(yI; \varepsilon).\end{aligned}$$

□

Remark. We have that $\widetilde{GL}_2(\mathbf{F})_+ \simeq \widetilde{SL}_2(\mathbf{F}) \cdot Z(\widetilde{GL}_2(\mathbf{F})_+)$. Hence there is no further reducibility of the restriction of $\widetilde{\pi}$ to $\widetilde{SL}_2(\mathbf{F})$, and the irreducible constituents of $\widetilde{\pi}|_{\widetilde{SL}_2(\mathbf{F})}$ are parameterized by $\mathbf{F}^\times/\mathbf{F}^{\times 2}$. This result is surprising and its analogue for the linear group $GL_2(\mathbf{F})$ is false (see Gelbart and Knapp, [21] and Tadic, [37]). In general, understanding the representation theory of $SL_2(\mathbf{F})$ by restricting representations of $GL_2(\mathbf{F})$ is quite difficult; see Labesse–Langlands [18].

We will restrict $L(\pi, \nu_0)$ to $\widetilde{SL}_2(\mathbf{F})$ in Chapter 5. The condition on (π, ν_0) and (σ, λ_0) needeed to guarantee $L(\pi, \nu_0) = L(\sigma, \lambda_0)$ as representations of $\widetilde{SL}_2(\mathbf{F})$ is nontrivial and is given in Lemma 5.34.

Chapter 4

The Character of the Oscillator Representation

Let $\omega(\psi) = \omega_e(\psi) \oplus \omega_o(\psi)$ be the oscillator representation of $\widetilde{SL}_2(\mathbf{F})$ attached to ψ. (See Gelbart and Piatetski–Shapiro in [10], p. 150, for a clear exposition of the definition of $\omega(\psi)$.) In this chapter, we explicitly compute the characters of $\omega_e(\psi)$ and $\omega_0(\psi)$ and get formulas for the sums and differences of these. The derivation uses the inversion formula and Flicker's formula for the character of the oscillator representation (see Proposition 2.70). The idea motivates the derivation of the general character identities in the next chapter. In fact, we could have proved these general identities first and obtained the formulas in this chapter as a consequence. In the end, we felt it would be useful to have both computations and have included the explicit derivation of the character of the oscillator representation as a firm starting point. The fact that these identities are a special case of the general identities is given in Corollary 5.19.

For a general reference for the character of the oscillator representation, see Howe [15]. Additional information in the real case is given in Torasso [38] and Adams [2]. For the p–adic case see Prasad [29]. We remark that Prasad's formula is given in terms of the norm one element $z = w/\overline{w}$ corresponding to an element of $SL_2(\mathbf{F})$.

We require some notation. Let $m = |\mathbf{F}^\times/\mathbf{F}^{\times 2}|$. Let D_{SO} and D_{SL} be the Weyl denominators for $SO_{1,2}(\mathbf{F})$ and $SL_2(\mathbf{F})$ respectively. So if $x, y \in \mathbf{F}^\times$,

$$D_{SO}(diag(x, 1/x, 1)) = |(1-x)(1-1/x)|^{1/2},$$

and

$$D_{SL}(diag(x, 1/x)) = |(1-x^2)(1-1/x^2)|^{1/2}.$$

We have the following lemmas.

Lemma 4.1. *Suppose that $g \in SL_2(\mathbf{F})$. Then*

$$\frac{D_{SO}(\tau(\pm g))}{D_{SL}(g)} = |\det(1 \pm g)|^{-1/2}.$$

Proof. Let $x \in \mathbf{F}^\times$ and put $g = diag(x, 1/x)$. We have

$$\frac{D_{SO}(\tau(\pm g))}{D_{SL}(g)} = \frac{|(1 \mp x)(1 \mp 1/x)|^{1/2}}{|(1-x^2)(1-1/x^2)|^{1/2}} = |(1 \pm x)(1 \pm 1/x)|^{-1/2} = |\det(1 \pm g)|^{-1/2}.$$

Let $\mathbf{E}$ be a the quadratic extension $\mathbf{F}(\sqrt{\Delta})$. Let ι be an embedding of $\mathbf{E}^\times$ in $GL_2(\mathbf{F})$ and write $\iota(\mathbf{E}^1) = S_\Delta \subset SL_2(\mathbf{F})$.

Now suppose that $g \in S_\Delta$. Write $g = \iota(z)$ with $z \in \mathbf{E}^1$. Then

$$\begin{aligned} \frac{D_{SO}(\tau(\pm g))}{D_{SL}(g)} &= \frac{|(1 \mp z/\overline{z})(1 \mp \overline{z}/z)|^{1/2}}{|(1-z^2/\overline{z}^2)(1-\overline{z}^2/z^2)|^{1/2}} \\ &= |(1 \pm z/\overline{z})(1 \pm \overline{z}/z)|^{-1/2} \\ &= |\det(1 \pm g)|^{-1/2}. \end{aligned}$$

Since any element of $SL_2(\mathbf{F})$ is conjugate to an element of one of the above two forms, the result follows.

□

Definition 4.2. *For $a \in \mathbf{F}^\times$, define*

$$\Phi_a(g) = |\det(1 + g/a)|^{-1/2} \tag{4.3}$$

We abuse notation and write $\Phi_\pm(g)$ in the case that $a = \pm 1$.

Notation 4.4. *If ν_0 is a character of $\mathbf{F}^\times$, we always reserve the letter ν to signify the one–dimensional representation of $GL_2(\mathbf{F})$ given by $\nu = \nu_0 \circ \det$.*

Let μ_0 be a character of $\mathbf{F}^\times$ and suppose that $\omega(\mu)$ is an oscillator representation of $\widetilde{GL}_2(\mathbf{F})$. (See Gelbart and Piatetski–Shapiro in [10], p. 151.) Let ν_0 be another character of $\mathbf{F}^\times$ and let $\omega(\mu)(\nu_0)$ be the summand of $\omega(\mu)|_{\widetilde{GL}_2(\mathbf{F})_+}$ which has central character

$$\chi_{\nu_0}(xI;\varepsilon) = \nu_0(x)\gamma_{\mathbf{F}}(x,\psi)\varepsilon.$$

4.1 Hyperbolic Set

We begin this section with some identities involving the map D defined by (2.59). We have that

$$\frac{D(diag(x,y))}{D(diag(x^2,y^2))} = |(1+y/x)(1+x/y)|^{-1/2}$$

and

$$\frac{D(diag(-x,y))}{D(diag(x^2,y^2))} = |(1-y/x)(1-x/y)|^{-1/2}.$$

Taking $y = 1$ and $g = diag(x, 1/x)$ we obtain

$$\frac{D(diag(x,1))}{D(diag(x^2,1))} = |(1+x)(1+1/x)|^{-1/2} = |\det(1+g)|^{-1/2} = \Phi_+(g) \qquad (4.5)$$

and

$$\frac{D(diag(-x,1))}{D(diag(x^2,y^2))} = |(1-x)(1-1/x)|^{-1/2} = |\det(1-g)|^{-1/2} = \Phi_-(g). \qquad (4.6)$$

Let $g = diag(x, 1/x)$ and let $\widetilde{g} = (g;\varepsilon)$. By the inversion formula, we have that

$$\Theta_{\omega(\mu)(\nu_0)}(\widetilde{g}) = \frac{1}{m}\sum_{\alpha\in\mathbf{F}^\times/\mathbf{F}^{\times 2}} \chi_{\nu_0}(z_\alpha)^{-1}\Theta_{\omega(\mu)}\left(z_\alpha\widetilde{g}\right).$$

By Proposition 2.72, all but one of the terms in the sum is zero, i.e.,

$$\Theta_{\omega(\mu)(\nu_0)}(\widetilde{g}) = \frac{\varepsilon}{m}(\xi, x)_{\mathbf{F}}\chi_{\nu_0}(\xi I; 1)^{-1}\Theta_{\omega(\mu)}(diag(\xi x, \xi/x); 1) \qquad (4.7)$$

for ξ such that $\xi \equiv x$ mod squares. This is a ξ which makes $diag(\xi x, \xi/x)$ a square; without loss of generality we take $\xi = x$.

We now make the following observations. By definition of χ_{ν_0} and properties of gamma factors (see Proposition 2.42) we have that (recall $\xi = x$)

$$\begin{aligned}(\xi, x)_{\mathbf{F}}\chi_{\nu_0}(\xi I; 1)^{-1} &= (-1, x)_{\mathbf{F}}\chi_{\nu_0}(x^{-1}I; (x,x)_{\mathbf{F}}) \\ &= \gamma_{\mathbf{F}}(x^{-1}, \psi)\nu_0(x^{-1}) \\ &= \gamma_{\mathbf{F}}(x, \psi)\nu_0(x)^{-1}.\end{aligned}$$

By Flicker's formula for the character of the oscillator representation (see Proposition 2.70), we have that

$$\begin{aligned}\Theta_{\omega(\mu)}\left(diag(\xi x, \xi/x); 1\right) &= \Theta_{\omega(\mu)}\left(diag(x^2, 1); 1\right) \\ &= b\left[\mu_0(-x)\frac{D(diag(x,1))}{D(diag(x^2,1))} + \mu_0(x)\frac{D(diag(-x,1))}{D(diag(x^2,1))}\right].\end{aligned}$$

By (4.5) and (4.6), this becomes

$$\Theta_{\omega(\mu)}\left(diag(\xi x, \xi/x); 1\right) = b\mu_0(x)\left[\frac{\mu_0(-1)}{|\det(1+g)|^{1/2}} + \frac{1}{|\det(1-g)|^{1/2}}\right].$$

Incorporating these observations in (4.7) we obtain the following proposition.

Proposition 4.8. *Let $g = diag(x, 1/x)$ and $\widetilde{g} = (g; \varepsilon)$. Suppose that μ_0 and ν_0 are characters of $\mathbf{F}^\times$. Then*

$$\Theta_{\omega(\mu)(\nu_0)}(\widetilde{g}) = \frac{b\varepsilon\mu_0(x)\nu_0(x)^{-1}\gamma_{\mathbf{F}}(x,\psi)}{m}\left[\frac{\mu_0(-1)}{|\det(1+g)|^{1/2}} + \frac{1}{|\det(1-g)|^{1/2}}\right]. \qquad (4.9)$$

Taking $\mu_0 = \nu_0$ in (4.9), and hence $\mu = \nu$, we get

$$\Theta_{\omega(\nu)(\nu_0)}(\widetilde{g}) = \frac{b\varepsilon\gamma_{\mathbf{F}}(x,\psi)}{m}\left[\frac{\nu_0(-1)}{|\det(1+g)|^{1/2}} + \frac{1}{|\det(1-g)|^{1/2}}\right]. \tag{4.10}$$

We apply (4.10) twice, once to a character ν satisfying $\nu_0(-1) = 1$ and once to a character ν' satisfying $\nu_0'(-1) = -1$. Adding and subtracting the results we obtain

$$\Theta_{\omega(\nu)(\nu_0)+\omega(\nu')(\nu_0')}(\widetilde{g}) = \frac{2b\varepsilon\gamma_{\mathbf{F}}(x,\psi)}{m|\det(1-g)|^{1/2}}$$

and

$$\Theta_{\omega(\nu)(\nu_0)-\omega(\nu')(\nu_0')}(\widetilde{g}) = \frac{2b\varepsilon\gamma_{\mathbf{F}}(x,\psi)}{m|\det(1+g)|^{1/2}}.$$

But by (2.62), $b = |\mathbf{F}^\times/\mathbf{F}^{\times 2}|/2$, so that $2b/m = 1$. We have shown the following proposition.

Proposition 4.11. *Let $g = diag(x, 1/x)$ and $\widetilde{g} = (g;\varepsilon)$. Suppose that ν_0 and ν_0' are characters of $\mathbf{F}^\times$ that satisfy $\nu_0(-1) = 1$ and $\nu_0'(-1) = -1$. Then*

$$\Theta_{\omega(\nu)(\nu_0)\pm\omega(\nu')(\nu_0')}(\widetilde{g}) = \frac{\gamma_{\mathbf{F}}(x,\psi)\varepsilon}{|\det(1\mp g)|^{1/2}}.$$

This is consistent with the calculation of the character of the oscillator representation given by Howe [15].

4.2 Elliptic Set

Let $\mathbf{E} = \mathbf{F}(\delta)$ be a quadratic extension of $\mathbf{F}$, $\delta^2 = \Delta \in \mathbf{F}$. Fix an embedding ι of $\mathbf{E}^\times$ in $GL_2(\mathbf{F})$ and write $S_\Delta = \iota(\mathbf{E}^1)$ (see Section 2.2). Set $\delta' = \iota(\delta)$.

Suppose g is an elliptic element of $SL_2(\mathbf{F})$. Without loss of generality, $g \in S_\Delta$ for some Δ. By the inversion formula, we have that

$$\Theta_{\omega(\mu)(\nu_0)}(\widetilde{g}) = \frac{1}{m}\sum_{\alpha\in\mathbf{F}^\times/\mathbf{F}^{\times 2}} \chi_{\nu_0}(z_\alpha)^{-1}\Theta_{\omega(\mu)}(z_\alpha\widetilde{g}).$$

The analysis is divided into two cases, the case of g a square and g not a square.

4.2.1 Elliptic Set, Case I: $g = h^2$

Suppose that g is the square of an element h in $GL_2(\mathbf{F})$. Write $\iota(z) = g$, $\iota(w) = h$ and $\widetilde{g} = (g; \varepsilon)$. Since $\det g = 1$, it follows that $\det h = Nw = \pm 1$.

We will need the following calculations involving the function D from (2.59).

Lemma 4.12. *Suppose that $g = h^2$. Then*

$$\frac{D(h)}{D(g)} = |\det(1 + g/\det h)|^{-1/2} = \Phi_{\det h}(g). \tag{4.13}$$

Proof. Note that

$$\begin{aligned}
\frac{D(h)}{D(g)} &= \left| \frac{(w - \overline{w})^2 / w\overline{w}}{(z - \overline{z})^2 / z\overline{z}} \right|^{1/2} \\
&= \left| \frac{(1 - w/\overline{w})(1 - \overline{w}/w)}{(1 - z/\overline{z})(1 - \overline{z}/z)} \right|^{1/2} \\
&= |(1 + w/\overline{w})(1 + \overline{w}/w)|^{-1/2} \\
&= |\det(1 + \iota(w^2/Nw))|^{-1/2} \ = |\det(1 + g/\det h)|^{-1/2} \\
&= \Phi_{\det h}(g).
\end{aligned}$$

□

Lemma 4.14. *Suppose that $g = h^2$. Then*

$$\frac{D(\delta' h)}{D(g)} = |\det(1 - g/\det h)|^{-1/2} = \Phi_{-\det h}(g). \tag{4.15}$$

Proof. We have

$$\begin{aligned}
\frac{D(\delta' h)}{D(g)} &= \left| \frac{(\delta w - \overline{\delta w})^2 / \delta w \overline{\delta w}}{(z - \overline{z})^2 / z\overline{z}} \right|^{1/2} \\
&= \left| \frac{(1 - \delta w/\overline{\delta w})(1 - \overline{\delta w}/\delta w)}{(1 - z/\overline{z})(1 - \overline{z}/z)} \right|^{1/2} \\
&= \left| \frac{(1 + w/\overline{w})(1 + \overline{w}/w)}{(1 - z/\overline{z})(1 - \overline{z}/z)} \right|^{1/2} \\
&= |(1 - w/\overline{w})(1 - \overline{w}/w)|^{-1/2} \\
&= |\det(1 - \iota(w^2/Nw))|^{-1/2}
\end{aligned}$$

so that

$$\frac{D(\delta' h)}{D(g)} = |\det(1 - g/\det h)|^{-1/2} = \Phi_{-\det h}(g).$$

□

By Proposition 2.4, ξg is a square if and only if $\xi = 1$ or Δ. [In particular, $\Delta g = (\delta' h)^2$.]

The inversion formula becomes

$$\Theta_{\omega(\mu)(\nu_0)}(\widetilde{g}) = \frac{1}{m}\left[\Theta_{\omega(\mu)}(\widetilde{g}) + \chi_{\nu_0}(\Delta I; 1)^{-1}\Theta_{\omega(\mu)}((\Delta I; 1)\widetilde{g})\right] \tag{4.16}$$

by Proposition 2.72. By Flicker's formula for the character of the oscillator representation, we have that

$$\begin{aligned}\Theta_{\omega(\mu)}(\widetilde{g}) &= \varepsilon s(g)\Theta_{\omega(\mu)}(g; s(g)^{-1}) \\ &= b\varepsilon s(g)\mu_0(-1)\mu(h)\frac{D(h)}{D(g)} \\ &= b\varepsilon s(g)\mu_0(-\det h)\frac{D(h)}{D(g)}.\end{aligned}$$

By (4.13) this becomes

$$\Theta_{\omega(\mu)}(\widetilde{g}) = \frac{b\varepsilon s(g)\mu_0(-\det h)}{|\det(1 + g/\det h)|^{1/2}}. \tag{4.17}$$

We now consider the term $\chi_{\nu_0}(\Delta I; 1)^{-1}\Theta_{\omega(\mu)}((\Delta I; 1)\widetilde{g})$. By definition, we have

$$\chi_{\nu_0}(\Delta I; 1)^{-1} = \nu_0(\Delta)^{-1}\gamma_{\mathbf{F}}(\Delta, \psi)(-1, \Delta)_{\mathbf{F}}. \tag{4.18}$$

Flicker's formula for the character of the oscillator representation yields

$$\begin{aligned}\Theta_{\omega(\mu)}((\Delta I; 1)\widetilde{g}) &= \varepsilon s(\Delta g)c(\Delta I, g)\Theta_{\omega(\mu)}(\Delta g; s(\Delta g)^{-1}) \\ &= b\varepsilon s(\Delta g)c(\Delta I, g)\mu_0(-1)\mu(\delta' h)\frac{D(\delta' h)}{D(\Delta g)} \\ &= b\varepsilon s(\Delta g)c(\Delta I, g)\mu_0(\Delta \det h)\frac{D(\delta' h)}{D(\Delta g)}.\end{aligned}$$

Equation (2.60) implies that the right–hand side equals

$$b\varepsilon s(\Delta g)c(\Delta I, g)\mu_0(\Delta \det h)\frac{D(\delta' h)}{D(g)}.$$

Hence, by (4.15),

$$\Theta_{\omega(\mu)}((\Delta I;1)\widetilde{g}) = \frac{b\varepsilon s(\Delta g)c(\Delta I, g)\mu_0(\Delta \det h)}{|\det(1-g/\det h)|^{1/2}}. \tag{4.19}$$

Putting (4.18) and (4.19) together we obtain

$$\chi_{\nu_0}(\Delta I;1)^{-1}\Theta_{\omega(\mu)}((\Delta I;1)\widetilde{g}) = \frac{b\varepsilon(-1,\Delta)_{\mathbf{F}} s(\Delta g)c(\Delta I, g)\mu_0(\Delta \det h)\nu_0(\Delta)^{-1}\gamma_{\mathbf{F}}(\Delta,\psi)}{|\det(1-g/\det h)|^{1/2}}.$$

We condense the result using (2.38) and obtain

$$\chi_{\nu_0}(\Delta I;1)^{-1}\Theta_{\omega(\mu)}((\Delta I;1)\widetilde{g}) = \frac{b\varepsilon s(g)c_\beta(\Delta I, -g)\mu_0(\Delta \det h)\nu_0(\Delta)^{-1}\gamma_{\mathbf{F}}(\Delta,\psi)}{|\det(1-g/\det h)|^{1/2}}. \tag{4.20}$$

Plugging (4.17) and (4.20) into (4.16), the following proposition follows.

Proposition 4.21. *Suppose $g = h^2 \in S_\Delta$ is regular elliptic and $\widetilde{g} = (g;\varepsilon)$. Suppose that μ_0 and ν_0 are characters of $\mathbf{F}^\times$. Then*

$$\Theta_{\omega(\mu)(\nu_0)}(\widetilde{g}) = \frac{b\varepsilon s(g)}{m}\left[\frac{\mu_0(-\det h)}{|\det(1+g/\det h)|^{1/2}} + \frac{c_\beta(\Delta I, -g)\gamma_{\mathbf{F}}(\Delta,\psi)\mu_0(\Delta \det h)\nu_0(\Delta)^{-1}}{|\det(1-g/\det h)|^{1/2}}\right]. \tag{4.22}$$

Taking $\mu_0 = \nu_0$ in (4.22), and hence $\mu = \nu$, we get

$$\Theta_{\omega(\nu)(\nu_0)}(\widetilde{g}) = \frac{b\varepsilon s(g)}{m}\left[\frac{\nu_0(-\det h)}{|\det(1+g/\det h)|^{1/2}} + \frac{c_\beta(\Delta I, -g)\gamma_{\mathbf{F}}(\Delta,\psi)\nu_0(\det h)}{|\det(1-g/\det h)|^{1/2}}\right].$$

Hence we have that

$$\Theta_{\omega(\nu)(\nu_0)}(\widetilde{g}) = \begin{cases} \dfrac{b\varepsilon s(g)}{m}\left[\dfrac{\nu_0(-1)}{|\det(1+g)|^{1/2}} + \dfrac{c_\beta(\Delta I, -g)\gamma_{\mathbf{F}}(\Delta,\psi)}{|\det(1-g)|^{1/2}}\right] \\ \\ \dfrac{b\varepsilon s(g)}{m}\left[\dfrac{1}{|\det(1-g)|^{1/2}} + \dfrac{\nu_0(-1)c_\beta(\Delta I, -g)\gamma_{\mathbf{F}}(\Delta,\psi)}{|\det(1+g)|^{1/2}}\right] \end{cases} \tag{4.23}$$

according as $\det h = \pm 1$.

We apply this twice, once to a character ν satisfying $\nu_0(-1) = 1$ and once to a character ν' satisfying $\nu_0'(-1) = -1$. Adding and subtracting the results and noticing that $2b/m = 1$, the following proposition follows.

Proposition 4.24. *Suppose that $g = h^2 \in S_\Delta$ is regular elliptic and $\widetilde{g} = (g;\varepsilon)$. Suppose that ν_0 and ν_0' are characters of $\mathbf{F}^\times$ that satisfy $\nu_0(-1) = 1$ and $\nu_0'(-1) = -1$. Then*

$$\Theta_{\omega(\nu)(\nu_0)+\omega(\nu')(\nu_0')}(\widetilde{g}) = \begin{cases} \dfrac{\varepsilon s(g)\gamma_{\mathbf{F}}(\Delta,\psi)c_\beta(\Delta I, -g)}{|\det(1-g)|^{1/2}}, & \det h = 1 \\ \dfrac{\varepsilon s(g)}{|\det(1-g)|^{1/2}}, & \det h = -1. \end{cases}$$

and

$$\Theta_{\omega(\nu)(\nu_0)-\omega(\nu')(\nu_0')}(\widetilde{g}) = \begin{cases} \dfrac{\varepsilon s(g)}{|\det(1+g)|^{1/2}}, & \det h = 1 \\ \dfrac{\varepsilon s(g)\gamma_{\mathbf{F}}(\Delta,\psi)c_\beta(\Delta I, -g)}{|\det(1+g)|^{1/2}}, & \det h = -1. \end{cases}$$

4.2.2 Elliptic Set, Case II: $g \neq h^2$

Assume now that g is not the square of an element in $GL_2(\mathbf{F})$.

Put $\iota(z) = g$. By Proposition 2.4, we may write $\xi_0 z = w^2$ with $\xi_0 = Nw$. Let $h = \iota(w)$ and $\widetilde{g} = (g;\varepsilon)$. Note that $\xi_0\Delta z = (\delta w)^2$ and if ξz is a square, then $\xi = \xi_0$ or $\xi_0\Delta$.

We begin again with some computations involving D.

Lemma 4.25. *Suppose that $\xi_0 g = h^2$ with $\xi_0 = \det h$. Then*

$$\frac{D(h)}{D(\xi_0 g)} = |\det(1+g)|^{-1/2} = \Phi_+(g). \tag{4.26}$$

Proof. Note that

$$\begin{aligned}
\frac{D(h)}{D(\xi_0 g)} &= \frac{D(h)}{D(g)} \\
&= \left|\frac{(w-\overline{w})^2/w\overline{w}}{(z-\overline{z})^2/z\overline{z}}\right|^{1/2} \\
&= \left|\frac{(1-w/\overline{w})(1-\overline{w}/w)}{(1-z/\overline{z})(1-\overline{z}/z)}\right|^{1/2} \\
&= \left|\frac{(1-w/\overline{w})(1-\overline{w}/w)}{(1-\xi_0 z/\overline{\xi_0 z})(1-\overline{\xi_0 z}/\xi_0 z)}\right|^{1/2} \\
&= |(1+w/\overline{w})(1+\overline{w}/w)|^{-1/2} \\
&= |\det(1+\iota(w^2/Nw))|^{-1/2}.
\end{aligned}$$

Since $\iota(w^2/Nw) = h^2/\det h = g$, we get

$$\frac{D(h)}{D(\xi_0 g)} = |\det(1+g)|^{-1/2} = \Phi_+(g).$$

□

Lemma 4.27. *Suppose that $\xi_0 g = h^2$ with $\xi_0 = \det h$. Then*

$$\frac{D(\delta' h)}{D(\xi_0 \Delta g)} = |\det(1-g)|^{-1/2} = \Phi_-(g). \tag{4.28}$$

Proof. We have

$$\frac{D(\delta' h)}{D(\xi_0 \Delta g)} = \frac{D(\delta' h)}{D(g)}.$$

Hence,

$$\begin{aligned}
\frac{D(\delta' h)}{D(\xi_0 \Delta g)} &= \left| \frac{(\delta w - \overline{\delta w})^2 / \delta w \overline{\delta w}}{(z - \overline{z})^2 / z\overline{z}} \right|^{1/2} \\
&= \left| \frac{(1 - \delta w/\overline{\delta w})(1 - \overline{\delta w}/\delta w)}{(1 - z/\overline{z})(1 - \overline{z}/z)} \right|^{1/2} \\
&= \left| \frac{(1 + w/\overline{w})(1 + \overline{w}/w)}{(1 - z/\overline{z})(1 - \overline{z}/z)} \right|^{1/2} \\
&= \left| \frac{(1 + w/\overline{w})(1 + \overline{w}/w)}{(1 - \xi_0 z/\overline{\xi_0 z})(1 - \overline{\xi_0 z}/\xi_0 z)} \right|^{1/2} \\
&= |(1 - w/\overline{w})(1 - \overline{w}/w)|^{-1/2} \\
&= |\det(1 - \iota(w^2/Nw))|^{-1/2}.
\end{aligned}$$

Since $\iota(w^2/Nw) = h^2/\det h = g$, we get

$$\frac{D(\delta' h)}{D(\xi_0 \Delta g)} = |\det(1 - g)|^{-1/2} = \Phi_-(g).$$

□

By Proposition 2.72, the inversion formula becomes

$$\begin{aligned}
\Theta_{\omega(\mu)(\nu_0)}(\widetilde{g}) = \frac{1}{m} \big[&\chi_{\nu_0}(\xi_0 I; 1)^{-1} \Theta_{\omega(\mu)}((\xi_0 I; 1)\widetilde{g}) \\
&+ \chi_{\nu_0}(\xi_0 \Delta I; 1)^{-1} \Theta_{\omega(\mu)}((\xi_0 \Delta I; 1)\widetilde{g}) \big]
\end{aligned} \tag{4.29}$$

First we calculate

$$\chi_{\nu_0}(\xi_0 I; 1)^{-1} \Theta_{\omega(\mu)}((\xi_0 I; 1)\widetilde{g}).$$

By definition, we have that

$$\chi_{\nu_0}(\xi_0 I; 1)^{-1} = \nu_0(\xi_0)^{-1} \gamma_{\mathbf{F}}(\xi_0, \psi)(-1, \xi_0)_{\mathbf{F}}. \tag{4.30}$$

By Flicker's formula for the character of the oscillator representation, we have that

$$\begin{aligned}
\Theta_{\omega(\mu)}((\xi_0 I; 1)\widetilde{g}) &= \varepsilon s(\xi_0 g) c(\xi_0 I, g) \Theta_{\omega(\mu)}(\xi_0 g; s(\xi_0 g)^{-1}) \\
&= b\varepsilon s(\xi_0 g) c(\xi_0 I, g) \mu_0(-1) \mu(h) \frac{D(h)}{D(\xi_0 g)} \\
&= b\varepsilon s(\xi_0 g) c(\xi_0 I, g) \mu_0(-\xi_0) \frac{D(h)}{D(\xi_0 g)}.
\end{aligned}$$

By (4.26) this becomes

$$\Theta_{\omega(\mu)}((\xi_0 I;1)\widetilde{g}) = \frac{b\varepsilon s(\xi_0 g)c(\xi_0 I,g)\mu_0(-\xi_0)}{|\det(1+g)|^{1/2}}. \tag{4.31}$$

Patching (4.30) and (4.31) together, we get

$$\chi_{\nu_0}(\xi_0 I;1)^{-1}\Theta_{\omega(\mu)}((\xi_0 I;1)\widetilde{g}) = \frac{b\varepsilon\gamma_{\mathbf{F}}(\xi_0,\psi)(-1,\xi_0)_{\mathbf{F}} s(\xi_0 g)c(\xi_0 I,g)\mu_0(-\xi_0)\nu_0(\xi_0)^{-1}}{|\det(1+g)|^{1/2}}.$$

We condense this using (2.38) and obtain

$$\chi_{\nu_0}(\xi_0 I;1)^{-1}\Theta_{\omega(\mu)}((\xi_0 I;1)\widetilde{g}) = \frac{b\varepsilon\gamma_{\mathbf{F}}(\xi_0,\psi)s(g)c_\beta(\xi_0 I,-g)\mu_0(-\xi_0)\nu_0(\xi_0)^{-1}}{|\det(1+g)|^{1/2}}. \tag{4.32}$$

Next we calculate

$$\chi_{\nu_0}(\xi_0\Delta I;1)^{-1}\Theta_{\omega(\mu)}((\xi_0\Delta I;1)\widetilde{g}).$$

By definition, we have that

$$\chi_{\nu_0}(\xi_0\Delta I;1)^{-1} = \nu_0(\xi_0\Delta)^{-1}\gamma_{\mathbf{F}}(\xi_0\Delta,\psi)(-1,\xi_0\Delta)_{\mathbf{F}}. \tag{4.33}$$

By Flicker's formula for the character of the oscillator representation, we have that

$$\begin{aligned}\Theta_{\omega(\mu)}((\xi_0\Delta I;1)\widetilde{g}) &= \varepsilon s(\xi_0\Delta g)c(\xi_0\Delta I,g)\Theta_{\omega(\mu)}(\xi_0\Delta g; s(\xi_0\Delta g)^{-1}) \\ &= b\varepsilon s(\xi_0\Delta g)c(\xi_0\Delta I,g)\mu_0(-1)\mu(\xi_0\delta' h)\frac{D(\delta' h)}{D(\xi_0\Delta g)} \\ &= b\varepsilon s(\xi_0\Delta g)c(\xi_0\Delta I,g)\mu_0(\xi_0\Delta)\frac{D(\delta' h)}{D(\xi_0\Delta g)}.\end{aligned}$$

By (4.28) this becomes

$$\Theta_{\omega(\mu)}((\xi_0\Delta I;1)\widetilde{g}) = \frac{b\varepsilon s(\xi_0\Delta g)c(\xi_0\Delta I,g)\mu_0(\xi_0\Delta)}{|\det(1-g)|^{1/2}}. \tag{4.34}$$

Patching (4.33) and (4.34) together, we get

$$\chi_{\nu_0}(\xi_0\Delta I;1)^{-1}\Theta_{\omega(\mu)}((\xi_0\Delta I;1)\widetilde{g}) = \frac{b\varepsilon\gamma_{\mathbf{F}}(\xi_0\Delta,\psi)(-1,\xi_0\Delta)_{\mathbf{F}}s(\xi_0\Delta g)c(\xi_0\Delta I,g)\mu_0(\xi_0\Delta)\nu_0(\xi_0\Delta)^{-1}}{|\det(1-g)|^{1/2}}.$$

We condense this using (2.38) and obtain

$$\chi_{\nu_0}(\xi_0\Delta I;1)^{-1}\Theta_{\omega(\mu)}((\xi_0\Delta I;1)\widetilde{g}) = \frac{b\varepsilon\gamma_{\mathbf{F}}(\xi_0\Delta,\psi)s(g)c_\beta(\xi_0\Delta I,-g)\mu_0(\xi_0\Delta)\nu_0(\xi_0\Delta)^{-1}}{|\det(1-g)|^{1/2}}. \tag{4.35}$$

Rewriting (4.29) using (4.32) and (4.35) results in the following proposition.

Proposition 4.36. *Suppose that $g \in S_\Delta$ is regular elliptic and is not a square. Let $\widetilde{g} = (g;\varepsilon)$. Write $\xi_0 g = h^2$ with $\xi_0 = \det h$. Suppose that μ_0 and ν_0 are characters of $\mathbf{F}^\times$. Then*

$$\Theta_{\omega(\mu)(\nu_0)}(\widetilde{g}) = \frac{b\varepsilon s(g)}{m}\left[\frac{\gamma_{\mathbf{F}}(\xi_0,\psi)c_\beta(\xi_0 I,-g)\mu_0(-\xi_0)\nu_0(\xi_0)^{-1}}{|\det(1+g)|^{1/2}} + \frac{\gamma_{\mathbf{F}}(\xi_0\Delta,\psi)c_\beta(\xi_0\Delta I,-g)\mu_0(\xi_0\Delta)\nu_0(\xi_0\Delta)^{-1}}{|\det(1-g)|^{1/2}}\right]. \tag{4.37}$$

Taking $\mu_0 = \nu_0$ in (4.37), and hence $\mu = \nu$, we get

$$\Theta_{\omega(\nu)(\nu_0)}(\widetilde{g}) = \frac{b\varepsilon s(g)}{m}\left[\frac{\gamma_{\mathbf{F}}(\xi_0,\psi)c_\beta(\xi_0 I,-g)\nu_0(-1)}{|\det(1+g)|^{1/2}} + \frac{\gamma_{\mathbf{F}}(\xi_0\Delta,\psi)c_\beta(\xi_0\Delta I,-g)}{|\det(1-g)|^{1/2}}\right]. \tag{4.38}$$

We apply this twice, once to a character ν satisfying $\nu_0(-1) = 1$ and once to a character ν' satisfying $\nu_0'(-1) = -1$. Adding and subtracting the results and noticing that $2b/m = 1$, the following proposition follows.

Proposition 4.39. *Suppose that $g \in S_\Delta$ is regular elliptic and is not a square. Write $\xi_0 g = h^2$ with $\xi_0 = \det h$. Suppose that ν_0 and ν_0' are characters of $\mathbf{F}^\times$ that satisfy $\nu_0(-1) = 1$ and $\nu_0'(-1) = -1$. Then*

$$\Theta_{\omega(\nu)(\nu_0)+\omega(\nu')(\nu_0')}(\widetilde{g}) = \frac{\varepsilon s(g)c_\beta(\xi_0\Delta I,-g)\gamma_{\mathbf{F}}(\xi_0\Delta,\psi)}{|\det(1-g)|^{1/2}}$$

and

$$\Theta_{\omega(\nu)(\nu_0)-\omega(\nu')(\nu_0')}(\widetilde{g}) = \frac{\varepsilon s(g) c_\beta(\xi_0 I, -g)\gamma_{\mathbf{F}}(\xi_0, \psi)}{|\det(1+g)|^{1/2}}.$$

4.3 Recapitulation

We collect the results of this chapter in two theorems.

Theorem 4.40. *Let $\omega(\mu)(\nu_0)$ be the oscillator representation of $\widetilde{GL}_2(\mathbf{F})_+$ with central character χ_{ν_0} where*

$$\chi_{\nu_0}(xI; \varepsilon) = \nu_0(x)\gamma_{\mathbf{F}}(x, \psi)\varepsilon.$$

1. *Let $g = diag(x, 1/x)$ and $\widetilde{g} = (g; \varepsilon)$. Then*

$$\Theta_{\omega(\mu)(\nu_0)}(\widetilde{g}) = \frac{b\varepsilon\mu_0(x)\nu_0(x)^{-1}\gamma_{\mathbf{F}}(x, \psi)}{m}\left[\frac{\mu_0(-1)}{|\det(1+g)|^{1/2}} + \frac{1}{|\det(1-g)|^{1/2}}\right].$$

2. *Suppose $g = h^2 \in S_\Delta$ is regular elliptic and $\widetilde{g} = (g; \varepsilon)$. Then*

$$\Theta_{\omega(\mu)(\nu_0)}(\widetilde{g}) = \frac{b\varepsilon s(g)}{m}\left[\frac{\mu_0(-\det h)}{|\det(1+g/\det h)|^{1/2}} + \frac{c_\beta(\Delta I, -g)\gamma_{\mathbf{F}}(\Delta, \psi)\mu_0(\Delta \det h)\nu_0(\Delta)^{-1}}{|\det(1-g/\det h)|^{1/2}}\right].$$

3. *Suppose that $g \in S_\Delta$ is regular elliptic and is not a square. Let $\widetilde{g} = (g; \varepsilon)$. Write $\xi_0 g = h^2$ with $\xi_0 = \det h$. Then*

$$\Theta_{\omega(\mu)(\nu_0)}(\widetilde{g}) = \frac{b\varepsilon s(g)}{m}\left[\frac{\gamma_{\mathbf{F}}(\xi_0, \psi)c_\beta(\xi_0 I, -g)\mu_0(-\xi_0)\nu_0(\xi_0)^{-1}}{|\det(1+g)|^{1/2}} + \frac{\gamma_{\mathbf{F}}(\xi_0\Delta, \psi)c_\beta(\xi_0\Delta I, -g)\mu_0(\xi_0\Delta)\nu_0(\xi_0\Delta)^{-1}}{|\det(1-g)|^{1/2}}\right].$$

Suppose that ν_0 and ν_0' are characters of $\mathbf{F}^\times$ that satisfy $\nu_0(-1) = 1$ and $\nu_0'(-1) = -1$. Put $\widetilde{g} = (g; \varepsilon)$. Define

$$\Gamma_\pm(\widetilde{g}) = \Theta_{\omega(\nu)(\nu_0)\pm\omega(\nu')(\nu_0')}(\widetilde{g}).$$

Theorem 4.41.

$$\Gamma_+(\widetilde{g}) = \begin{cases} \dfrac{\gamma_{\mathbf{F}}(x,\psi)\varepsilon}{|\det(1-g)|^{1/2}}, & \textit{if } g = diag(x, 1/x) \\[2em] \dfrac{\varepsilon s(g)\gamma_{\mathbf{F}}(\Delta,\psi)c_\beta(\Delta I, -g)}{|\det(1-g)|^{1/2}}, & \textit{if } g = h^2 \in S_\Delta \textit{ and } \det h = 1 \\[2em] \dfrac{\varepsilon s(g)}{|\det(1-g)|^{1/2}}, & \textit{if } g = h^2 \in S_\Delta \textit{ and } \det h = -1 \\[2em] \dfrac{\varepsilon s(g)\gamma_{\mathbf{F}}(\xi_0\Delta,\psi)c_\beta(\xi_0\Delta I, -g)}{|\det(1-g)|^{1/2}}, & \textit{if } g \in S_\Delta,\ \xi_0 g = h^2 \textit{ and } \det h = \xi_0 \end{cases}$$

and

$$\Gamma_-(\widetilde{g}) = \begin{cases} \dfrac{\gamma_{\mathbf{F}}(x,\psi)\varepsilon}{|\det(1+g)|^{1/2}}, & \textit{if } g = diag(x, 1/x) \\[2em] \dfrac{\varepsilon s(g)}{|\det(1+g)|^{1/2}}, & \textit{if } g = h^2 \in S_\Delta \textit{ and } \det h = +1 \\[2em] \dfrac{\varepsilon s(g)\gamma_{\mathbf{F}}(\Delta,\psi)c_\beta(\Delta I, -g)}{|\det(1+g)|^{1/2}}, & \textit{if } g = h^2 \in S_\Delta \textit{ and } \det h = -1 \\[2em] \dfrac{\varepsilon s(g)\gamma_{\mathbf{F}}(\xi_0,\psi)c_\beta(\xi_0 I, -g)}{|\det(1+g)|^{1/2}}, & \textit{if } g \in S_\Delta,\ \xi_0 g = h^2 \textit{ and } \det h = \xi_0. \end{cases}$$

Remark. We have proved in Lemma 4.1 that for all $g \in SL_2(\mathbf{F})$,

$$|\det(1 \pm g)|^{1/2} = \frac{D_{SO}(\tau(\pm g))}{D_{SL}(g)}.$$

Chapter 5

The Character Formula

Let π be an irreducible representation of $GL_2(\mathbf{F})$ satisfying $\chi_\pi(-I) = 1$ and $\widetilde{\pi} = \mathrm{Lift}_{\mathrm{F}}(\pi)$ be Flicker's lifting of π to $\widetilde{GL}_2(\mathbf{F})$. In Chapter 3 we let $L(\pi, \nu_0)$ be the constituent of $\widetilde{\pi}|_{\widetilde{GL}_2(\mathbf{F})_+}$ having central character $\chi_{\nu_0}(xI; \varepsilon) = \nu_0(x)\gamma_{\mathbf{F}}(x, \psi)\varepsilon$.

We restrict $L(\pi, \nu_0)$ to $\widetilde{SL}_2(\mathbf{F})$ and also write $L(\pi, \nu_0)$ for this restriction. Throughout the rest of this chapter and beyond, whenever we write $L(\pi, \nu_0)$ we intend the representation of $\widetilde{SL}_2(\mathbf{F})$. The representation π and the character ν_0 are related by the equation $\chi_\pi(xI) = \nu_0(x^2)$, see (3.18). The representation $\pi\nu^{-1}$ has trivial central character and factors to a representation $(\pi\nu^{-1})'$ of $SO_{1,2}(\mathbf{F})$.

This chapter contains the main results of this thesis. We derive a character formula relating the representations $L(\pi, \nu_0)$ of $\widetilde{SL}_2(\mathbf{F})$ and $(\pi\nu^{-1})'$ of $SO_{1,2}(\mathbf{F})$. This character formula suggests a map $L(\pi, \nu_0) \mapsto (\pi\nu^{-1})'$ from irreducible representations of $\widetilde{SL}_2(\mathbf{F})$ to irreducible representations of $SO_{1,2}(\mathbf{F})$. The map is surjective but not injective.

In order to obtain a bijection, we "stabilize" the formulas; we get a character formula relating "stable" representations $L_{st}(\pi, \nu_0)$ of $\widetilde{SL}_2(\mathbf{F})$ and representations

$(\pi\nu^{-1})'$ of $SO_{1,2}(\mathbf{F})$ through the following remarkably simple formula:

$$\Theta_{L_{st}(\pi,\nu_0)}(\widetilde{g}) = \Gamma_-(\widetilde{g})\Theta_{(\pi\nu^{-1})'}(\tau(g)).$$

The derivation of these formulas requires the inversion formula and Flicker's character identity. The first step is to get a character formula between representations $L(\pi,\nu_0)$ and $\pi\nu^{-1}$. This is the bulk of the work and is contained in Sections 1–3. After this, we push over to $SO_{1,2}(\mathbf{F})$ and then stabilize the correspondence.

5.1 Hyperbolic Set

Let $g = diag(x, 1/x)$ and let $\widetilde{g} = (g;\varepsilon)$. By the inversion formula, we have that

$$\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \frac{1}{m}\sum_{\alpha\in\mathbf{F}^\times/\mathbf{F}^{\times 2}} \chi_{\nu_0}(z_\alpha)^{-1}\Theta_{\widetilde{\pi}}(z_\alpha\widetilde{g}).$$

By Proposition 2.72, all but one of the terms in the sum is zero, i.e.,

$$\Theta_{L(\mu,\nu_0)}(\widetilde{g}) = \frac{\varepsilon}{m}(\xi,x)_{\mathbf{F}}\chi_{\nu_0}(\xi I;1)^{-1}\Theta_{\widetilde{\pi}}(diag(\xi x,\xi/x);1) \tag{5.1}$$

for ξ such that $\xi \equiv x$ mod squares. This is a ξ which makes $diag(\xi x, \xi/x)$ a square; without loss of generality we take $\xi = x$.

By definition of χ_{ν_0} and properties of gamma factors (see Proposition 2.42) we have that (recall $\xi = x$)

$$\begin{aligned}(\xi,x)_{\mathbf{F}}\chi_{\nu_0}(\xi I;1)^{-1} &= (-1,x)_{\mathbf{F}}\chi_{\nu_0}(x^{-1}I;(x,x)_{\mathbf{F}})\\ &= \gamma_{\mathbf{F}}(x^{-1},\psi)\nu_0(x^{-1})\\ &= \gamma_{\mathbf{F}}(x,\psi)\nu_0(x)^{-1}.\end{aligned}$$

We put

$$h = diag(x,1) \qquad \text{and} \qquad h' = diag(-x,1).$$

By Flicker's formula on the hyperbolic set (see Definiton 2.63), we have that

$$\begin{aligned}\Theta_{\widetilde{\pi}}\left(diag(\xi x, \xi/x);1\right) &= \Theta_{\widetilde{\pi}}\left(h^2;1\right) \\ &= b\left[\frac{D(h)}{D(h^2)}\Theta_\pi(h) + \frac{D(h')}{D(h^2)}\Theta_\pi(h')\right].\end{aligned}$$

[We remark here that if g is hyperbolic, then $s(g^2)^{-1} = 1$.] By (4.5) and (4.6), this becomes

$$\Theta_{\widetilde{\pi}}\left(diag(\xi x, \xi/x);1\right) = b\left[\Phi_+(g)\Theta_\pi(h) + \Phi_-(g)\Theta_\pi(h')\right].$$

Incorporating these observations in (5.1) we obtain

$$\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \frac{b\varepsilon\nu_0(x)^{-1}\gamma_{\mathbf{F}}(x,\psi)}{m}\left[\Phi_+(g)\Theta_\pi(h) + \Phi_-(g)\Theta_\pi(h')\right]. \qquad (5.2)$$

By Theorem 4.41, we have that $\Gamma_-(g;\varepsilon) = \gamma_{\mathbf{F}}(x,\psi)\varepsilon\Phi_+(g)$ and $\Gamma_+(g;\varepsilon) = \gamma_{\mathbf{F}}(x,\psi)\varepsilon\Phi_-(g)$. Then (5.2) becomes

$$\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \frac{b}{m}\left[\Gamma_-(\widetilde{g})\nu_0(x)^{-1}\Theta_\pi(h) + \Gamma_+(\widetilde{g})\nu_0(-1)\nu_0(-x)^{-1}\Theta_\pi(h')\right].$$

Recall that $2b/m = 1$, so that $b/m = 1/2$. Recall that $\det h = x$ and $\det h' = -x$. We have proved the following proposition.

Proposition 5.3. *Let ν_0 be a character of $\mathbf{F}^\times$ which satisfies $\chi_\pi(xI) = \nu_0(x^2)$. If $g = diag(x, 1/x)$, $h = diag(x,1)$ and $h' = diag(-x,1)$ then*

$$\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \frac{1}{2}\left[\Gamma_-(\widetilde{g})\Theta_{\pi\nu^{-1}}(h) + \nu_0(-1)\Gamma_+(\widetilde{g})\Theta_{\pi\nu^{-1}}(h')\right].$$

5.2 Elliptic Set

Let $\mathbf{E} = \mathbf{F}(\delta)$ be a quadratic extension of $\mathbf{F}$, $\delta^2 = \Delta \in \mathbf{F}$. Fix an embedding ι of $\mathbf{E}^\times$ in $GL_2(\mathbf{F})$ and write $S_\Delta = \iota(\mathbf{E}^1)$ (see Section 2.2). Set $\delta' = \iota(\delta)$.

Suppose g is an elliptic element of $SL_2(\mathbf{F})$. Without loss of generality, $g \in S_\Delta$ for some Δ. By the inversion formula, we have that

$$\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \frac{1}{m} \sum_{\alpha \in \mathbf{F}^\times/\mathbf{F}^{\times 2}} \chi_{\nu_0}(z_\alpha)^{-1} \Theta_{\widetilde{\pi}}\left(z_\alpha \widetilde{g}\right).$$

The analysis is divided into two cases, the case of g a square and g not a square.

5.2.1 Elliptic Set, Case I: $g = h^2$

Suppose that g is the square of an element h in $GL_2(\mathbf{F})$. By Proposition 2.4, ξg is a square if and only if $\xi = 1$ or Δ. In light of Proposition 2.72, the inversion formula becomes

$$\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \frac{1}{m}\left[\Theta_{\widetilde{\pi}}(\widetilde{g}) + \chi_{\nu_0}(\Delta I; 1)^{-1}\Theta_{\widetilde{\pi}}((\Delta I; 1)\widetilde{g})\right] \tag{5.4}$$

By Flicker's formula on the elliptic set, we have that

$$\begin{aligned}\Theta_{\widetilde{\pi}}(\widetilde{g}) &= \varepsilon s(g)\Theta_{\widetilde{\pi}}(g; s(g)^{-1}) \\ &= b\varepsilon s(g)\frac{D(h)}{D(g)}\Theta_\pi(h).\end{aligned}$$

By (4.13) this becomes

$$\Theta_{\widetilde{\pi}}(\widetilde{g}) = b\varepsilon s(g)\Phi_{\det h}(g)\Theta_\pi(h). \tag{5.5}$$

We now consider the term $\chi_{\nu_0}(\Delta I; 1)^{-1}\Theta_{\widetilde{\pi}}((\Delta I; 1)\widetilde{g})$. By definition, we have

$$\chi_{\nu_0}(\Delta I; 1)^{-1} = \nu_0(\Delta)^{-1}\gamma_{\mathbf{F}}(\Delta, \psi)(-1, \Delta)_{\mathbf{F}}. \tag{5.6}$$

Flicker's formula for the character on the elliptic set yields

$$\begin{aligned}\Theta_{\widetilde{\pi}}((\Delta I; 1)\widetilde{g}) &= \varepsilon s(\Delta g)c(\Delta I, g)\Theta_{\widetilde{\pi}}(\Delta g; s(\Delta g)^{-1}) \\ &= b\varepsilon s(\Delta g)c(\Delta I, g)\frac{D(\delta' h)}{D(\Delta g)}\Theta_\pi(\delta' h).\end{aligned}$$

Equation (2.60) implies that the right–hand side equals

$$b\varepsilon s(\Delta g)c(\Delta I,g)\frac{D(\delta' h)}{D(g)}\Theta_\pi(\delta' h).$$

Hence, by (4.15),

$$\Theta_{\widetilde{\pi}}((\Delta I;1)\widetilde{g}) = b\varepsilon s(\Delta g)c(\Delta I,g)\Phi_{-\det h}(g)\Theta_\pi(\delta' h). \tag{5.7}$$

Putting (5.6) and (5.7) together we obtain

$$\chi_{\nu_0}(\Delta I;1)^{-1}\Theta_{\widetilde{\pi}}((\Delta I;1)\widetilde{g}) = \\ b\varepsilon(-1,\Delta)_{\mathbf{F}} s(\Delta g)c(\Delta I,g)\nu_0(\Delta)^{-1}\gamma_{\mathbf{F}}(\Delta,\psi)\Phi_{-\det h}(g)\Theta_\pi(\delta' h).$$

We condense the result using (2.38) and obtain

$$\chi_{\nu_0}(\Delta I;1)^{-1}\Theta_{\widetilde{\pi}}((\Delta I;1)\widetilde{g}) = \\ b\varepsilon s(g)c_\beta(\Delta I,-g)\nu_0(\Delta)^{-1}\gamma_{\mathbf{F}}(\Delta,\psi)\Phi_{-\det h}(g)\Theta_\pi(\delta' h). \tag{5.8}$$

Plugging (5.5) and (5.8) into (5.4), we see

$$\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \frac{b}{m}[\varepsilon s(g)\Phi_{\det h}(g)\Theta_\pi(h) \\ + \varepsilon s(g)\gamma_{\mathbf{F}}(\Delta,\psi)c_\beta(\Delta I,-g)\Phi_{-\det h}(g)\nu_0(\Delta)^{-1}\Theta_\pi(\delta' h)].$$

Recall that $2b/m = 1$, so that $b/m = 1/2$. Then

$$\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \frac{1}{2}[\varepsilon s(g)\Phi_{\det h}(g)\nu_0(\det h)\Theta_{\pi\nu^{-1}}(h) \\ + \varepsilon s(g)\gamma_{\mathbf{F}}(\Delta,\psi)c_\beta(\Delta I,-g)\Phi_{-\det h}(g)\nu_0(-\det h)\Theta_{\pi\nu^{-1}}(\delta' h)].$$

By Theorem 4.41, we have that $\Gamma_{-\det h}(\widetilde{g}) = \varepsilon s(g)\Phi_{\det h}(g)$ and $\Gamma_{\det h}(\widetilde{g}) = \varepsilon s(g)\gamma_{\mathbf{F}}(\Delta,\psi)c_\beta(\Delta I,-g)\Phi_{-\det h}(g)$. This implies the next proposition.

Proposition 5.9. *Suppose that $g = h^2 \in S_\Delta$ is elliptic. Let ν_0 be a character of $\mathbf{F}^\times$ which satisfies $\chi_\pi(xI) = \nu_0(x^2)$. Put $\lambda = \det h$. Then*

$$\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \frac{1}{2}\left[\Gamma_{-\lambda}(\widetilde{g})\nu_0(\lambda)\Theta_{\pi\nu^{-1}}(h) + \Gamma_\lambda(\widetilde{g})\nu_0(-\lambda)\Theta_{\pi\nu^{-1}}(\delta' h)\right].$$

5.2.2 Elliptic Set, Case II: $g \neq h^2$

Assume now that g is not the square of an element in $GL_2(\mathbf{F})$. Put $\iota(z) = g$. By Proposition 2.4, we may write $\xi_0 z = w^2$ with $\xi_0 = Nw$. Let $h = \iota(w)$ and $\widetilde{g} = (g; \varepsilon)$. Note that $\xi_0 \Delta z = (\delta w)^2$ and if ξz is a square, then $\xi = \xi_0$ or $\xi_0 \Delta$. By Proposition 2.72, the inversion formula becomes

$$\begin{aligned}\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \frac{1}{m} \big[&\chi_{\nu_0}(\xi_0 I; 1)^{-1} \Theta_{\widetilde{\pi}}((\xi_0 I; 1)\widetilde{g}) \\ &+ \chi_{\nu_0}(\xi_0 \Delta I; 1)^{-1} \Theta_{\widetilde{\pi}}((\xi_0 \Delta I; 1)\widetilde{g}) \big]\end{aligned} \tag{5.10}$$

First we calculate

$$\chi_{\nu_0}(\xi_0 I; 1)^{-1} \Theta_{\widetilde{\pi}}((\xi_0 I; 1)\widetilde{g}).$$

By definition, we have that

$$\chi_{\nu_0}(\xi_0 I; 1)^{-1} = \nu_0(\xi_0)^{-1} \gamma_{\mathbf{F}}(\xi_0, \psi)(-1, \xi_0)_{\mathbf{F}}. \tag{5.11}$$

By Flicker's formula on the elliptic set, we have that

$$\begin{aligned}\Theta_{\widetilde{\pi}}((\xi_0 I; 1)\widetilde{g}) &= \varepsilon s(\xi_0 g) c(\xi_0 I, g) \Theta_{\widetilde{\pi}}(\xi_0 g; s(\xi_0 g)^{-1}) \\ &= b \varepsilon s(\xi_0 g) c(\xi_0 I, g) \frac{D(h)}{D(\xi_0 g)} \Theta_{\pi}(h).\end{aligned}$$

By (4.26) this becomes

$$\Theta_{\widetilde{\pi}}((\xi_0 I; 1)\widetilde{g}) = b \varepsilon s(\xi_0 g) c(\xi_0 I, g) \Phi_+(g) \Theta_{\pi}(h). \tag{5.12}$$

Patching (5.11) and (5.12) together, we get

$$\begin{aligned}&\chi_{\nu_0}(\xi_0 I; 1)^{-1} \Theta_{\widetilde{\pi}}((\xi_0 I; 1)\widetilde{g}) = \\ &\qquad b \varepsilon \gamma_{\mathbf{F}}(\xi_0, \psi)(-1, \xi_0)_{\mathbf{F}} s(\xi_0 g) c(\xi_0 I, g) \nu_0(\xi_0)^{-1} \Phi_+(g) \Theta_{\pi}(h).\end{aligned}$$

We condense this using (2.38) and obtain

$$\begin{aligned}&\chi_{\nu_0}(\xi_0 I; 1)^{-1} \Theta_{\widetilde{\pi}}((\xi_0 I; 1)\widetilde{g}) = \\ &\qquad b \varepsilon \gamma_{\mathbf{F}}(\xi_0, \psi) s(g) c_{\beta}(\xi_0 I, -g) \nu_0(\xi)^{-1} \Phi_+(g) \Theta_{\pi}(h).\end{aligned} \tag{5.13}$$

Next we calculate

$$\chi_{\nu_0}(\xi_0\Delta I;1)^{-1}\Theta_{\widetilde{\pi}}((\xi_0\Delta I;1)\widetilde{g}).$$

By definition, we have that

$$\chi_{\nu_0}(\xi_0\Delta I;1)^{-1}=\nu_0(\xi_0\Delta)^{-1}\gamma_{\mathbf{F}}(\xi_0\Delta,\psi)(-1,\xi_0\Delta)_{\mathbf{F}}. \tag{5.14}$$

By Flicker's formula on the elliptic set, we have that

$$\begin{aligned}\Theta_{\widetilde{\pi}}((\xi_0\Delta I;1)\widetilde{g}) &= \varepsilon s(\xi_0\Delta g)c(\xi_0\Delta I,g)\Theta_{\widetilde{\pi}}(\xi_0\Delta g;s(\xi_0\Delta g)^{-1})\\ &= b\varepsilon s(\xi_0\Delta g)c(\xi_0\Delta I,g)\frac{D(\delta' h)}{D(\xi_0\Delta g)}\Theta_{\pi}(\delta' h).\end{aligned}$$

By (4.28) this becomes

$$\Theta_{\widetilde{\pi}}((\xi_0\Delta I;1)\widetilde{g})=b\varepsilon s(\xi_0\Delta g)c(\xi_0\Delta I,g)\Phi_-(g)\Theta_{\pi}(\delta' h). \tag{5.15}$$

Patching (5.14) and (5.15) together, we get

$$\begin{aligned}&\chi_{\nu_0}(\xi_0\Delta I;1)^{-1}\Theta_{\widetilde{\pi}}((\xi_0\Delta I;1)\widetilde{g})=\\ &\qquad b\varepsilon\gamma_{\mathbf{F}}(\xi_0\Delta,\psi)(-1,\xi_0\Delta)_{\mathbf{F}}s(\xi_0\Delta g)c(\xi_0\Delta I,g)\nu_0(\xi_0\Delta)^{-1}\Phi_-(g)\Theta_{\pi}(\delta' h).\end{aligned}$$

We condense this using (2.38) and obtain

$$\begin{aligned}&\chi_{\nu_0}(\xi_0\Delta I;1)^{-1}\Theta_{\widetilde{\pi}}((\xi_0\Delta I;1)\widetilde{g})=\\ &\qquad b\varepsilon\gamma_{\mathbf{F}}(\xi_0\Delta,\psi)s(g)c_{\beta}(\xi_0\Delta I,-g)\nu_0(\xi_0\Delta)^{-1}\Phi_-(g)\Theta_{\pi}(\delta' h).\end{aligned} \tag{5.16}$$

By (5.13) and (5.16), (5.10) becomes

$$\begin{aligned}\Theta_{L(\pi,\nu_0)}(\widetilde{g}) =&\frac{b}{m}[\varepsilon s(g)c_{\beta}(\xi_0 I,-g)\gamma_{\mathbf{F}}(\xi_0,\psi)\Phi_+(g)\nu_0(\xi_0)^{-1}\Theta_{\pi}(h)\\ &+\varepsilon s(g)c_{\beta}(\xi_0\Delta I,-g)\gamma_{\mathbf{F}}(\xi_0\Delta,\psi)\Phi_-(g)\nu_0(\xi_0\Delta)^{-1}\Theta_{\pi}(\delta' h)].\end{aligned}$$

Recall that $2b/m=1$, so that $b/m=1/2$. Then

$$\begin{aligned}\Theta_{L(\pi,\nu_0)}(\widetilde{g}) =&\frac{1}{2}[\varepsilon s(g)c_{\beta}(\xi_0 I,-g)\gamma_{\mathbf{F}}(\xi_0,\psi)\Phi_+(g)\Theta_{\pi\nu^{-1}}(h)\\ &+\varepsilon s(g)c_{\beta}(\xi_0\Delta I,-g)\gamma_{\mathbf{F}}(\xi_0\Delta,\psi)\Phi_-(g)\nu_0(-1)\Theta_{\pi\nu^{-1}}(\delta' h)].\end{aligned}$$

By Theorem 4.41,

$$\Gamma_-(\widetilde{g}) = \varepsilon s(g) c_\beta(\xi_0 I, -g) \gamma_{\mathbf{F}}(\xi_0, \psi) \Phi_+(g)$$

and

$$\Gamma_+(\widetilde{g}) = \varepsilon s(g) c_\beta(\xi_0 \Delta I, -g) \gamma_{\mathbf{F}}(\xi_0 \Delta, \psi) \Phi_-(g).$$

This implies the following proposition.

Proposition 5.17. *Suppose that $g \in S_\Delta$ is regular elliptic and is not a square. Write $\xi_0 g = h^2$ with $\xi_0 = \det h$. Let ν_0 be a character of $\mathbf{F}^\times$ which satisfies $\chi_\pi(xI) = \nu_0(x^2)$. Then*

$$\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \frac{1}{2}[\Gamma_-(\widetilde{g})\Theta_{\pi\nu^{-1}}(h) + \nu_0(-1)\Gamma_+(\widetilde{g})\Theta_{\pi\nu^{-1}}(\delta' h)].$$

5.3 Recapitulation

The following theorem summarizes our work thus far.

Theorem 5.18. *Suppose that π is an irreducible representation of $GL_2(\mathbf{F})$ such that $\chi_\pi(-I) = 1$. Let ν_0 be a character of $\mathbf{F}^\times$ which satisfies $\chi_\pi(xI) = \nu_0(x^2)$. Assume that $g \in SL_2(\mathbf{F})$ and let $\widetilde{g} = (g; \varepsilon)$.*

1. *Assume that $g = diag(x, 1/x)$ is hyperbolic. Let $h = diag(x, 1)$ and $h' = diag(-x, 1)$. Then*

$$\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \frac{1}{2}\left[\Gamma_-(\widetilde{g})\Theta_{\pi\nu^{-1}}(h) + \nu_0(-1)\Gamma_+(\widetilde{g})\Theta_{\pi\nu^{-1}}(h')\right].$$

2. *Assume that $g = h^2 \in S_\Delta$ is elliptic and that $\det h = 1$. Then*

$$\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \frac{1}{2}\left[\Gamma_-(\widetilde{g})\Theta_{\pi\nu^{-1}}(h) + \nu_0(-1)\Gamma_+(\widetilde{g})\Theta_{\pi\nu^{-1}}(\delta' h)\right].$$

3. Assume that $g = h^2 \in S_\Delta$ *is elliptic and that* $\det h = -1$. *Then*

$$\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \frac{1}{2}\left[\nu_0(-1)\Gamma_+(\widetilde{g})\Theta_{\pi\nu^{-1}}(h) + \Gamma_-(\widetilde{g})\Theta_{\pi\nu^{-1}}(\delta' h)\right].$$

4. Assume that $g \in S_\Delta$ *is elliptic and is not a square. Write* $\xi_0 g = h^2$ *with* $\xi_0 = \det h$. *Then*

$$\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \frac{1}{2}[\Gamma_-(\widetilde{g})\Theta_{\pi\nu^{-1}}(h) + \nu_0(-1)\Gamma_+(\widetilde{g})\Theta_{\pi\nu^{-1}}(\delta' h)].$$

Notice that the character of $L(\pi, \nu_0)$ always contains two terms. On the hyperbolic set, this is because the inversion formula contributes a single term while Flicker's character identity contributes two. On the elliptic set, the inversion formula contributes two terms. But on the elliptic elements Flicker's identity consists of only one term, so the total number of terms for the elliptic set is also two.

In Chapter 4, we derived character identities for the oscillator representation restricted to $\widetilde{GL}_2(\mathbf{F})_+$. These are related to the above identities through the following corollary.

Corollary 5.19. *Let* $\omega(\mu)$ *denote the oscillator representation of* $\widetilde{GL}_2(\mathbf{F})$. *Then*

$$\Theta_{L(\mu,\mu_0)}(\widetilde{g}) = \mu_0(-1)\Theta_{\omega(\mu)(\mu_0)}(\widetilde{g}).$$

Proof. This follows from (4.10), (4.23), (4.38) and Theorem 5.18. □

5.4 Pushing Down to $SO_{1,2}(\mathbf{F})$: The Character Formula

We would like to write our formulas in Theorem 5.18 in terms of characters of representations on $SO_{1,2}(\mathbf{F})$. We will do this by "pushing" the representation

$\pi\nu^{-1}$ to a representation of $SO_{1,2}(\mathbf{F})$ via the map p from Section 2.3.

Definition 5.20. *Suppose π is a representation of $GL_2(\mathbf{F})$ with trivial central character. Define a representation π' of $SO_{1,2}(\mathbf{F})$ by pushing π down to $SO_{1,2}(\mathbf{F})$ via p, i.e.,*

$$\Theta_\pi(g) = \Theta_{\pi'}(p(g)).$$

From (2.8), we get

$$\Theta_\pi(diag(a, b)) = \Theta_{\pi'}(diag(a/b, b/a, 1)). \tag{5.21}$$

Suppose $h \in S_\Delta$. Then $h = \iota(w)$ for some $w \in \mathbf{F}(\sqrt{\Delta})$. By Proposition 2.13 it follows that

$$\Theta_\pi(h) = \Theta_\pi(\iota w) = \Theta_{\pi'}(p\iota w) = \Theta_{\pi'}(\iota'\varphi w). \tag{5.22}$$

Proposition 5.23. *Suppose that π is a representation of $GL_2(\mathbf{F})$ with trivial central character.*

1. *Suppose that $g = diag(x, 1/x)$, $h = diag(x, 1)$, and $h' = diag(-x, 1)$. Then*

$$\Theta_\pi(h) = \Theta_{\pi'}(\tau(g)) \tag{5.24}$$

and

$$\Theta_\pi(h') = \Theta_{\pi'}(\tau(-g)). \tag{5.25}$$

2. *Suppose that $g = h^2$ and $\det h = \pm 1$. Then*

$$\Theta_\pi(h) = \Theta_{\pi'}(\tau(\pm g)) \tag{5.26}$$

and

$$\Theta_\pi(\delta' h) = \Theta_{\pi'}(\tau(\mp g)) \tag{5.27}$$

3. *Suppose that g is not a square of an element in $GL_2(\mathbf{F})$. Write $\xi_0 g = h^2$, $\xi_0 = \det h$. Then*

$$\Theta_\pi(h) = \Theta_{\pi'}(\tau(g)) \tag{5.28}$$

and

$$\Theta_\pi(\delta' h) = \Theta_{\pi'}(\tau(-g)). \tag{5.29}$$

Proof. Formulas (5.24) and (5.25) follow immediately from (5.21).

Suppose that $g = h^2$ with $\det h = \pm 1$. If $h = \iota(w)$ then $\det h = Nw$. Consequently, by (5.22),

$$\begin{aligned} \Theta_\pi(h) &= \Theta_{\pi'}(\iota' \varphi w) \\ &= \Theta_{\pi'}(\iota'(w^2/Nw)) \\ &= \Theta_{\pi'}(\tau(g/\det h)) \\ &= \Theta_{\pi'}(\tau(\pm g)) \end{aligned}$$

and (5.26) follows. Also,

$$\begin{aligned} \Theta_\pi(\delta' h) &= \Theta_{\pi'}(\iota' \varphi \delta w) \\ &= \Theta_{\pi'}(\iota'((\delta w)^2/N(\delta w))) \\ &= \Theta_{\pi'}(\tau(\Delta g/-\Delta \det h)) \\ &= \Theta_{\pi'}(\tau(\mp g)). \end{aligned}$$

So (5.27) follows.

Finally, suppose that $\xi_0 g = h^2$ with $\xi_0 = \det h$. Write $h = \iota(w)$. Note that $\iota(w^2/Nw) = g$. By (5.22), we have

$$\begin{aligned} \Theta_\pi(h) &= \Theta_{\pi'}(\iota' \varphi w) \\ &= \Theta_{\pi'}(\iota'(w^2/Nw)) \\ &= \Theta_{\pi'}(\tau(g)) \end{aligned}$$

and (5.28) follows. The proof of (5.29) is similar to the proof of (5.27) and is omitted. ☐

In the character formulas for $L(\pi, \nu_0)$, we required that $\chi_\pi(xI) = \nu_0(x^2)$. Then $\pi\nu^{-1}$ has trivial central character and the representation $(\pi\nu^{-1})'$ of $SO_{1,2}(\mathbf{F})$ is defined. The main character identity which we have been seeking is contained in the following theorem.

Theorem 5.30. *Suppose that π is an irreducible representation of $GL_2(\mathbf{F})$. Suppose that ν_0 is a character of $\mathbf{F}^\times$ satisfying $\chi_\pi(xI) = \nu_0(x^2)$. Then for every regular semisimple $\widetilde{g} \in \widetilde{SL}_2(\mathbf{F})$,*

$$\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \frac{1}{2}\left[\Gamma_-(\widetilde{g})\Theta_{(\pi\nu^{-1})'}(\tau(g)) + \nu_0(-1)\Gamma_+(\widetilde{g})\Theta_{(\pi\nu^{-1})'}(\tau(-g))\right]. \qquad (5.31)$$

Proof. This follows immediately by substituting the equalities from Proposition 5.23 (with π replaced by $\pi\nu^{-1}$) into the character identities we obtained in Theorem 5.18. ☐

We could have stated the next corollary after Theorem 5.18, but it is more convenient to do so here.

Corollary 5.32. *Let β_0 be a character of $\mathbf{F}^\times$ satisfying $\beta_0(-1) = 1$. Then*

$$\Theta_{L(\pi,\nu_0)} = \Theta_{L(\pi\beta,\nu_0\beta_0)}.$$

In particular, for any character β_0,

$$\Theta_{L(\pi\beta^2,\nu_0\beta_0^2)} = \Theta_{L(\pi,\nu_0)}.$$

Definition 5.33. *Let R be the set of all pairs (π, ν_0) where π is an irreducible representation of $GL_2(\mathbf{F})$ satisfying $\chi_\pi(-I) = 1$ and ν_0 is a character of $\mathbf{F}^\times$ satisfying $\chi_\pi(xI) = \nu_0(x^2)$ for all $x \in \mathbf{F}^\times$. We define an equivalence relation $\sim$*

on R by $(\pi,\nu_0)\sim(\sigma,\lambda_0)$ if there exists a character β_0 with $\beta_0(-1)=1$ such that $\sigma=\pi\beta$ and $\lambda_0=\nu_0\beta_0$. We write

$$X = R/\sim .$$

Lemma 5.34. *$L(\pi,\nu_0)=L(\sigma,\lambda_0)$ if and only if $(\pi,\nu_0)\sim(\sigma,\lambda_0)$.*

Proof. Suppose $(\pi,\nu_0)\sim(\sigma,\lambda_0)$. Then $L(\sigma,\lambda_0)=L(\pi,\nu_0)$ by the character formula.

Conversely, suppose that $L(\pi,\nu_0)=L(\sigma,\lambda_0)$. By the condition on central characters, this implies that $\nu_0(-1)=\lambda_0(-1)$.

Recall that

$$\mathrm{Lift}_{\mathrm{F}}(\pi)|_{\widetilde{GL}_2(\mathbf{F})_+} = \sum_{i=1}^{|\mathbf{F}^\times/\mathbf{F}^{\times 2}|}\pi_i, \qquad \mathrm{Lift}_{\mathrm{F}}(\pi)|_{\widetilde{SL}_2(\mathbf{F})} = \sum_{i=1}^{|\mathbf{F}^\times/\mathbf{F}^{\times 2}|}\tau_i$$

and that

$$\mathrm{Lift}_{\mathrm{F}}(\sigma)|_{\widetilde{GL}_2(\mathbf{F})_+} = \sum_{i=1}^{|\mathbf{F}^\times/\mathbf{F}^{\times 2}|}\sigma_i, \qquad \mathrm{Lift}_{\mathrm{F}}(\sigma)|_{\widetilde{SL}_2(\mathbf{F})} = \sum_{j=1}^{|\mathbf{F}^\times/\mathbf{F}^{\times 2}|}\mu_j.$$

Our assumption shows that there exist i and j such that $\tau_i=\mu_j$ and hence that $\sum\tau_i=\sum\mu_j$. Since $\widetilde{GL}_2(\mathbf{F})_+=Z(\widetilde{GL}_2(\mathbf{F})_+)\widetilde{SL}_2(\mathbf{F})$, there exists a character β such that $\sum\sigma_i=(\sum\pi_i)\otimes\beta$, i.e., that $\mathrm{Lift}_{\mathrm{F}}(\sigma)=\mathrm{Lift}_{\mathrm{F}}(\pi)\beta$. But $\mathrm{Lift}_{\mathrm{F}}(\pi)\beta=\mathrm{Lift}_{\mathrm{F}}(\pi\beta^2)$. Since Flicker's lifting is an injection, we have that $\sigma=\pi\beta^2$. Hence $L(\pi,\nu_0)=L(\pi\beta^2,\nu_0)=L(\pi,\nu_0\beta_0^{-2})$. By the central character condition, we may find a quadratic character α_0 such that $\beta_0^2=\alpha_0\lambda_0/\nu_0$. That is, setting $\gamma=\beta^2$, we can take $\sigma=\pi\gamma$, $\lambda_0=\alpha_0\nu_0\gamma_0$, and $\gamma_0(-1)=1$. This implies that $\alpha_0(-1)=1$, since $\nu_0(-1)=\lambda_0(-1)$.

If α_0 is trivial, then we are done. Otherwise, since

$$\begin{aligned} L(\pi, \nu_0) &= L(\sigma, \lambda_0) \\ &= L(\pi\gamma, \lambda_0) \\ &= L(\pi, \lambda_0\gamma_0^{-1}) \end{aligned}$$

and $\lambda_0\gamma_0^{-1} = \alpha_0\nu_0$ we have reduced to the case $L(\pi, \nu_0) = L(\pi, \nu_0\alpha_0)$. As α is quadratic with $\alpha_0(-1) = 1$, we get $L(\pi, \nu_0) = L(\pi\alpha, \nu_0)$. This in turn implies that $L(\pi, \mu_0) = L(\pi\alpha, \mu_0)$ for all μ_0, so that $\mathrm{Lift}_{\mathrm{F}}(\pi) = \mathrm{Lift}_{\mathrm{F}}(\pi\alpha)$. Since Flicker's lifting is an injection, we get $\pi = \pi\alpha$ and $(\pi, \nu_0) \sim (\pi, \nu_0\alpha_0) \sim (\sigma, \lambda_0)$. This completes the proof. □

The character formula obtained in Theorem 5.30 suggests that we examine the map

$$Irr(\widetilde{SL}_2(\mathbf{F})) \to Irr(SO_{1,2}(\mathbf{F}))$$

$$L(\pi, \nu_0) \mapsto (\pi\nu^{-1})'.$$

This map is clearly surjective, since if $\pi' \in Irr(SO_{1,2}(\mathbf{F}))$, then π has trivial central character, and $L(\pi, 1) \mapsto \pi'$. Fix a character β_0 satisfying $\beta_0(-1) = -1$. Suppose that $L(\pi, \nu_0)$ and $L(\pi\beta, \nu_0\beta_0)$ are both nonzero. In general, $L(\pi, \nu_0)$ and $L(\pi\beta, \nu_0\beta_0)$ are inequivalent because they have different central characters. However, they both map to $(\pi\nu^{-1})'$. Therefore, this map is not injective. We remedy this situation in the next section when we "stabilize" (5.31).

5.5 Stable Representations of $\widetilde{SL}_2(\mathbf{F})$

Theorem 5.35. *Let β_0 be a character of $\mathbf{F}^\times$ satisfying $\beta_0(-1) = -1$. Then*

$$\Theta_{L(\pi,\nu_0)+L(\pi\beta,\nu_0\beta_0)}(\widetilde{g}) = \Gamma_-(\widetilde{g})\Theta_{(\pi\nu^{-1})'}(\tau(g)). \tag{5.36}$$

Proof. We apply Theorem 5.30 to (π, ν_0) and $(\pi\beta, \nu_0\beta_0)$. In the second case, we replace ν with $\beta^{-1}\nu^{-1}$, so that $(\pi\beta)(\beta^{-1}\nu^{-1})$ equals $\pi\nu^{-1}$. It follows that

$$\begin{aligned}\Theta_{L(\pi,\nu_0)}(\widetilde{g}) &= \frac{1}{2}\left[\Gamma_-(\widetilde{g})\Theta_{(\pi\nu^{-1})'}(\tau(g)) + \nu_0(-1)\Gamma_+(\widetilde{g})\Theta_{(\pi\nu^{-1})'}(\tau(-g)).\right] \\ \Theta_{L(\pi\beta,\nu_0\beta_0)}(\widetilde{g}) &= \frac{1}{2}\left[\Gamma_-(\widetilde{g})\Theta_{(\pi\nu^{-1})'}(\tau(g)) + \beta_0\nu_0(-1)\Gamma_+(\widetilde{g})\Theta_{(\pi\nu^{-1})'}(\tau(-g)).\right] \\ &= \frac{1}{2}\left[\Gamma_-(\widetilde{g})\Theta_{(\pi\nu^{-1})'}(\tau(g)) - \nu_0(-1)\Gamma_+(\widetilde{g})\Theta_{(\pi\nu^{-1})'}(\tau(-g)).\right]\end{aligned}$$

Adding the first and third equations gives the result. □

Definition 5.37. *Suppose that π is an irreducible representation of $GL_2(\mathbf{F})$ satisfying $\chi_\pi(-I) = 1$. Let ν_0 be a character of $\mathbf{F}^\times$ such that $\chi_\pi(xI) = \nu_0(x^2)$ for all $x \in \mathbf{F}^\times$. Fix a character β_0 of $\mathbf{F}^\times$ satisfying $\beta_0(-1) = -1$. Define*

$$L_{st}(\pi, \nu_0) = L(\pi, \nu_0) + L(\pi\beta, \nu_0\beta_0).$$

As before, let R be the collection of pairs (π, ν_0) where π is an irreducible representation of $GL_2(\mathbf{F})$ and ν_0 is a character of $\mathbf{F}^\times$ with the property that $\chi_\pi(xI) = \nu_0(x^2)$. Let

$$Gr_{st}(\widetilde{SL}_2(\mathbf{F})) = span\{L_{st}(\pi, \nu_0) : (\pi, \nu_0) \in R\}.$$

Any $\rho \in Gr_{st}(\widetilde{SL}_2(\mathbf{F}))$ will be called a stable virtual representation of $\widetilde{SL}_2(\mathbf{F})$.

We repeat Theorem 5.35 in this notation:

$$\Theta_{L_{st}(\pi,\nu_0)}(\widetilde{g}) = \Gamma_-(\widetilde{g})\Theta_{(\pi\nu^{-1})'}(\tau(g)).$$

Definition 5.38. *We define an equivalence relation $\sim_{st}$ on R by $(\pi, \nu_0) \sim_{st} (\sigma, \lambda_0)$ if there exists a character β_0 such that $\sigma = \pi\beta$ and $\lambda_0 = \nu_0\beta_0$. Set*

$$X_{st} = R/\sim_{st}.$$

Lemma 5.39. *$L_{st}(\pi, \nu_0) = L_{st}(\sigma, \lambda_0)$ if and only if $(\pi, \nu_0) \sim_{st} (\sigma, \lambda_0)$.*

Proof. Assume first that $(\pi, \nu_0) \sim_{st} (\sigma, \lambda_0)$. It follows immediately from Theorem 5.35 that $L_{st}(\pi, \nu_0) = L_{st}(\sigma, \lambda_0)$. Conversely, if $L_{st}(\pi, \nu_0) = L_{st}(\sigma, \lambda_0)$ then (5.36) implies that $(\pi\nu^{-1})' = (\sigma\lambda^{-1})'$. The pullback is unique, so $\pi\nu^{-1} = \sigma\lambda^{-1}$. Therefore $\pi = \sigma\lambda^{-1}\nu$. Take $\beta_0 = \lambda_0^{-1}\nu_0$ in the definition of $\sim_{st}$ to conclude that $(\pi, \nu_0) \sim_{st} (\sigma, \lambda_0)$. $\square$

Corollary 5.40. *We have bijections*

1. $X_{st} \leftrightarrow \{L_{st}(\pi, \nu_0)\}$,

2. $X_{st} \leftrightarrow Irr(SO_{1,2}(\mathbf{F}))$ *given by* $(\pi, \nu_0) \mapsto (\pi\nu^{-1})'$.

Proof. The obvious map $X_{st} \to \{L_{st}(\pi, \nu_0)\}$ gives the first bijection. The second follows from the first and Theorem 5.35. $\square$

Corollary 5.41. *The map*

$$\mathcal{L}_{st} : Gr(SO_{1,2}(\mathbf{F})) \to Gr_{st}(\widetilde{SL}_2(\mathbf{F}))$$

defined by

$$\mathcal{L}_{st}(\pi) = L_{st}(\pi \circ p, 1)$$

is a bijection.

Chapter 6

Properties of the Correspondence

In this chapter, we discuss the nature of the bijection

$$\mathcal{L}_{st} : Gr(SO_{1,2}(\mathbf{F})) \to Gr_{st}(\widetilde{SL}_2(\mathbf{F})).$$

We begin with the full principal series representations of $\widetilde{SL}_2(\mathbf{F})$ and $SO_{1,2}(\mathbf{F})$. Then we get the correspondence for one–dimensional representations.We conclude with the special and supercuspidal representation and after this we summarize the results.

6.1 Principal Series Representations

We explain how the principal series fit into our correspondence. The parameterization of the principal series of $\widetilde{SL}_2(\mathbf{F})$ and $SO_{1,2}(\mathbf{F})$ is given in the following definition. (The parameterization of principal series of $GL_2(\mathbf{F})$ and $\widetilde{GL}_2(\mathbf{F})$ has already been given in Section 2.6.)

Recall that

$$A = \{diag(x, 1/x) : x \in \mathbf{F}^\times\},$$

$$N = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} : y \in \mathbf{F} \right\},$$

$$B = AN,$$

and that $\widetilde{B} = \widetilde{A}N$. We also write

$$A' = \{diag(a, 1) : a \in A\},$$

$$N' = \{diag(n, 1) : n \in N\},$$

$$B' = \{diag(b, 1) : b \in B\}.$$

Definition 6.1. *Let*

$$\widetilde{\lambda}(x; \varepsilon) = \gamma_{\mathbf{F}}(x, \psi)\varepsilon;$$

this is a genuine character of $\widetilde{\mathbf{F}}^\times$ *(see Proposition 3.14). Any genuine character of* $\widetilde{A}$ *is of the form* $\widetilde{\nu}_0(diag(x, 1/x); \varepsilon) = \nu_0(x)\widetilde{\lambda}(x; \varepsilon)$*. Extend this trivially to* N *and define*

$$PS_{\widetilde{SL}}(\widetilde{\nu}_0) = Ind_{\widetilde{B}}^{\widetilde{SL}_2(\mathbf{F})}(\widetilde{\nu}_0)$$

for the full principal series representation of $\widetilde{SL}_2(\mathbf{F})$ *with parameter* $\widetilde{\nu}_0$*.*

Definition 6.2. *Let* ν_0 *be any character of* $\mathbf{F}^\times$*. We identify this with a character of* A' *in the obvious way. We extend* ν_0 *to* N' *and write*

$$PS_{SO}(\nu_0) = Ind_{B'}^{SO_{1,2}(\mathbf{F})}(\nu_0)$$

for the full principal series representation of $SO_{1,2}(\mathbf{F})$ *with parameter* ν_0*.*

Let D_{SO}, D_{SL}, and D_{GL} be the Weyl denominators for $SO_{1,2}(\mathbf{F})$, $SL_2(\mathbf{F})$, and $GL_2(\mathbf{F})$ respectively.

Lemma 6.3. *Let $x \in \mathbf{F}^\times$ and put $g = diag(x, 1/x)$, $\widetilde{g} = (g; \varepsilon)$. Then*

$$\Gamma_\pm(\widetilde{g}) = \frac{D_{SO}(\tau(\mp g))}{D_{SL}(g)} \gamma(x, \psi)\varepsilon.$$

Proof. By Lemma 4.1, $D_{SO}(\tau(\mp g))/D_{SL}(g) = |\det(1 \mp g)|^{-1/2}$. This, in conjunction with Theorem 4.41 completes the proof. □

Lemma 6.4. *Suppose $x \in \mathbf{F}^\times$ and put $g = diag(x, 1/x)$, $h = diag(x, 1)$, $h' = diag(-x, 1)$. Then*

$$\frac{D_{SO}(\tau(g))}{D_{GL}(h)} = \frac{D_{SO}(\tau(-g))}{D_{GL}(h')} = 1.$$

Proof. This follows immediately from the definitions of D_{SO} and D_{GL}. □

We state the induced character formulas for $GL_2(\mathbf{F})$, $\widetilde{SL}_2(\mathbf{F})$ and $SO_{1,2}(\mathbf{F})$ in terms of our parameterization of their principal series. Let

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Proposition 6.5 (Induced Character Formula). *We have the following formulas.*

1. *Let $\pi(\eta_1, \eta_2)$ be a principal series representation of $GL_2(\mathbf{F})$ (see Definition 2.51). Suppose g is conjugate to $diag(x, y)$. Then*

   $$\Theta_{\pi(\eta_1,\eta_2)}(diag(x, y)) = \frac{\eta_1(x)\eta_2(y) + \eta_1(y)\eta_2(x)}{D_{GL}(diag(x, y))}.$$

   If g is not conjugate to $diag(x, y)$ for some x and y then $\Theta_{\pi(\eta_1,\eta_2)}(g) = 0$.

2. *Suppose that h is conjugate to $g = diag(x, 1/x)$ and let $\widetilde{h} = (h; \varepsilon)$. Then*

   $$\begin{aligned}\Theta_{PS_{\widetilde{SL}}(\widetilde{\nu}_0)}(\widetilde{h}) &= \frac{\widetilde{\nu}_0(g) + \widetilde{\nu}_0(wgw^{-1})}{D_{SL}(g)} \\ &= \gamma_{\mathbf{F}}(x, \psi)\varepsilon \frac{\nu_0(x) + \nu_0^{-1}(x)}{D_{SL}(g)},\end{aligned}$$

   If h is not conjugate to $g = diag(x, 1/x)$ for some x then $\Theta_{PS_{\widetilde{SL}}(\widetilde{\nu}_0)}(\widetilde{h}) = 0$.

3. Suppose that $h \in SO_{1,2}(\mathbf{F})$ is conjugate to $\tau(g)$. Then

$$\begin{aligned}\Theta_{PS_{SO}(\nu_0)}(h) &= \frac{\nu_0(\tau(g)) + \nu_0(\tau(g)^{\tau(w)})}{D_{SO}(g)} \\ &= \frac{\nu_0(x) + \nu_0^{-1}(x)}{D_{SO}(g)}.\end{aligned}$$

If $h \in SO_{1,2}(\mathbf{F})$ is not conjugate to $\tau(g)$ for some g, then $\Theta_{PS_{SO}(\nu_0)}(h) = 0$.

Corollary 6.6. *For all g regular semisimple, $\widetilde{g} = (g; \varepsilon)$,*

$$\Theta_{PS_{\widetilde{SL}}(\widetilde{\nu}_0)}(\widetilde{g}) = \Gamma_-(\widetilde{g})\Theta_{PS_{SO}(\nu_0)}(\tau(g)). \tag{6.7}$$

Proof. If g is not conjugate to an element of A, then both sides are 0. Otherwise, by the induced character formulas,

$$\Theta_{PS_{\widetilde{SL}}(\widetilde{\nu}_0)}(\widetilde{g}) = \frac{D_{SO}(\tau(g))}{D_{SL}(g)}\gamma_{\mathbf{F}}(x,\psi)\varepsilon\Theta_{PS_{SO}(\nu_0)}(\tau(g)).$$

Then

$$\Theta_{PS_{\widetilde{SL}}(\widetilde{\nu}_0)}(\widetilde{g}) = \Gamma_-(\widetilde{g})\Theta_{PS_{SO}(\nu_0)}(\tau(g))$$

by Lemma 6.3. □

Suppose that $\pi(\eta_1, \eta_2)$ is a strongly even principal series representation of $GL_2(\mathbf{F})$, so that $\mathrm{Lift}_{\mathrm{F}}(\pi(\eta_1, \eta_2))$ is non–zero (see Proposition 2.68). This implies that there exist characters μ_1 and μ_2 of $\mathbf{F}^{\times 2}$ satisfying $\mu_i(x^2) = \eta_i(x)$ for all $x \in \mathbf{F}^\times$. Extend each μ_i arbitrarily to $\mathbf{F}^\times$. In order that $L(\pi(\eta_1, \eta_2), \nu_0)$ be defined, i.e., $\chi_{\pi(\eta_1,\eta_2)}(xI) = \nu_0(x^2)$, we must have that

$$\nu_0^2 = \eta_1\eta_2.$$

This implies that there is a quadratic character α_0 such that

$$\nu_0 = \alpha_0\mu_1\mu_2. \tag{6.8}$$

Let $g = diag(x, 1/x)$, $\widetilde{g} = (g; \varepsilon)$, $h = diag(x, 1)$, $h' = diag(-x, 1)$. Put $\pi = \pi(\eta_1, \eta_2)$. By Theorem 5.18, we have

$$\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \frac{1}{2}\left[\Gamma_-(\widetilde{g})\Theta_{\pi\otimes\nu^{-1}}(h) + \nu_0(-1)\Gamma_+(\widetilde{g})\Theta_{\pi\otimes\nu^{-1}}(h')\right]. \tag{6.9}$$

By Theorem 5.18 and the induced character formula for $GL_2(\mathbf{F})$, $\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = 0$ for all other $\widetilde{g}$ with g not conjugate to $diag(x, 1/x)$ for some x. When we write our character formulas below, we assume that $\widetilde{g} = (diag(x, 1/x); \varepsilon)$ and that the character formulas are zero for all other $\widetilde{g}$ with g not conjugate to an element of this form.

Proposition 6.10. *Suppose that $\pi = \pi(\eta_1, \eta_2)$ is a principal series such that η_1 and η_2 are even. Then*

$$\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \Gamma_-(\widetilde{g})\Theta_{(\pi\otimes\nu^{-1})'}(\tau(g)).$$

That is, if π is a strongly even principal series representation of $GL_2(\mathbf{F})$, we have

$$L(\pi, \nu_0) = L_{st}(\pi, \nu_0)$$

Proof. By Lemma 6.3, we have that

$$\nu_0(-1)\Gamma_+(\widetilde{g})\Theta_{\pi\otimes\nu^{-1}}(h') = \nu_0(-1)\frac{D_{SO}(\tau(-g))}{D_{SL}(g)}\Theta_{\pi\otimes\nu^{-1}}(h').$$

By the induced character formula, this becomes

$$\begin{aligned}\nu_0(-1)\Gamma_+(\widetilde{g})\Theta_{\pi\otimes\nu^{-1}}(h') &= \nu_0(-1)\frac{\eta_1(-x)\nu_0^{-1}(-x) + \eta_2(-x)\nu_0^{-1}(-x)}{D_{GL}(h')} \\ &= \gamma_{\mathbf{F}}(x,\psi)\varepsilon\frac{D_{SO}(\tau(-g))}{D_{SL}(g)}\frac{\eta_1(-x)\nu_0^{-1}(x) + \eta_2(-x)\nu_0^{-1}(x)}{D_{GL}(h')}.\end{aligned}$$

By assumption, $\eta_1(-1) = \eta_2(-1) = 1$. Furthermore, by Lemma 6.4 we have that $D_{SO}(\tau(-g))/D_{GL}(h') = 1$. Incorporating this information, we see

$$\nu_0(-1)\Gamma_+(\widetilde{g})\Theta_{\pi\otimes\nu^{-1}}(h') = \gamma_{\mathbf{F}}(x,\psi)\varepsilon\frac{\eta_1(x)\nu_0^{-1}(x) + \eta_2(x)\nu_0^{-1}(x)}{D_{SL}(g)}.$$

By Lemma 6.4, we have that $D_{SO}(\tau(g))/D_{SL}(g) = 1$. Then the above equation becomes

$$\nu_0(-1)\Gamma_+(\widetilde{g})\Theta_{\pi\otimes\nu^{-1}}(h') = \gamma_{\mathbf{F}}(x,\psi)\varepsilon\frac{D_{SO}(\tau(g))}{D_{SL}(g)}\frac{\eta_1(x)\nu_0^{-1}(x)+\eta_2(x)\nu_0^{-1}(x)}{D_{GL}(g)}.$$

By Lemma 6.3 and the induced character formula, we obtain

$$\Gamma_-(\widetilde{g})\Theta_{\pi\otimes\nu^{-1}}(h) = \nu_0(-1)\Gamma_+(\widetilde{g})\Theta_{\pi\otimes\nu^{-1}}(h').$$

Then (6.9) implies that

$$\Theta_{L(\pi,\nu_0)}(\widetilde{g}) = \Gamma_-(\widetilde{g})\Theta_{\pi\nu^{-1}}(h) = \Gamma_-(\widetilde{g})\Theta_{(\pi\nu^{-1})'}(\tau(g)).$$

The last expression equals $\Theta_{L_{st}(\pi,\nu_0)}$. This completes the proof. □

Lemma 6.11. *Let μ_0 any character of $\mathbf{F}^\times$ for which $L(\pi(\eta_1,\eta_2),\mu_0)$ is defined. Then*

$$L(\pi(\eta_1,\eta_2),\mu_0) = L(\pi(\eta_1\eta_2^{-1},1),\mu_0\eta_2^{-1}).$$

Proof. Write η for $\eta_2 \circ \det$. The computation

$$(\eta_1,\eta_2)\eta^{-1}(diag(x,y)) = \eta_1(x)\eta_2(y)\eta_2^{-1}(xy) = \eta_1\eta_2^{-1}(x)$$

shows that $\pi(\eta_1,\eta_2)\eta^{-1} = \pi(\eta_1\eta_2^{-1},1)$. By the remarks after Definition 5.33, the lemma follows. □

Lemma 6.12. *Let $\nu_0 = \alpha_0\mu_1\mu_2$ be as in (6.8). Then*

$$(\pi(\eta_1,\eta_2)\otimes\nu^{-1})' = PS_{SO}(\nu_0/\eta_2) = PS_{SO}(\eta_1/\nu_0).$$

Remark. $\Theta_{PS_{SO}(\lambda_1)} = \Theta_{PS_{SO}(\lambda_2)}$ if and only if $\lambda_2 = \lambda_1$ or λ_1^{-1}.

Proof. We compute

$$\begin{aligned}(\eta_1, \eta_2)\nu^{-1}(diag(x, y)) &= \eta_1(x)\eta_2(y)\nu_0^{-1}(xy) \\ &= \eta_1(x)\eta_2(y)\alpha_0^{-1}(xy)\mu_1^{-1}(xy)\mu_2^{-1}(xy).\end{aligned}$$

Recall that $\mu_1(x^2) = \eta_1(x)$ and $\mu_2(x^2) = \eta_2(x)$ and that α_0 is quadratic. We get

$$\begin{aligned}(\eta_1, \eta_2)\nu^{-1}(diag(x, y)) &= \alpha_0(x/y)\mu_1(x/y)\mu_2(y/x) \\ &= \alpha_0\mu_1/\mu_2(x/y) \\ &= \nu_0/\eta_2(x/y) = \eta_1/\nu_0(x/y).\end{aligned}$$

This completes the proof. □

Corollary 6.13.

$$\Theta_{L_{st}(\pi(\eta_1,\eta_2),\nu_0)}(\widetilde{g}) = \Gamma_-(\widetilde{g})\Theta_{PS_{SO}(\nu_0/\eta_2)}(\tau(g)).$$

Hence

$$PS_{\widetilde{SL}}(\widetilde{\nu_0/\eta_2}) = L(\pi(\eta_1, \eta_2), \nu_0).$$

Proof. Since μ_0 is a character of the form given in (6.8), Lemma 6.12 in conjunction with Proposition 6.10 implies that

$$\Theta_{L(\pi(\eta_1,\eta_2),\nu_0)}(\widetilde{g}) = \Theta_{L(\pi(\eta_1\eta_2^{-1},1),\nu_0\eta_2^{-1})}(\widetilde{g}) = \Gamma_-(\widetilde{g})\Theta_{PS_{SO}(\nu_0\eta_2^{-1})}(\tau(g)).$$

□

Corollary 6.14.

$$\mathcal{L}_{st}(PS_{SO}(\nu_0)) = L(\pi(\nu_0^2, 1), \nu_0).$$

Proof. By Lemma 6.12, $\pi(\nu_0^2, 1)\nu^{-1}$ is $PS_{SO}(\nu_0)$. □

From these two corollaries, we easily deduce the following statement.

Corollary 6.15. *Let π be a strongly even principal series of $GL_2(\mathbf{F})$. Then the correspondence*

$$L(\pi, \nu_0) \leftrightarrow (\pi\nu^{-1})'$$

is

$$PS_{\widetilde{SL}}(\widetilde{\nu_0}) \leftrightarrow PS_{SO}(\nu_0).$$

Remark. If $\pi = \pi(\eta_1, \eta_2)$ is a principal series of $GL_2(\mathbf{F})$ with the η_i both even, then

$$L_{st}(\pi(\eta_1, \eta_2), \nu_0) = L(\pi(\eta_1, \eta_2), \nu_0).$$

Suppose that η_1 and η_2 are odd. If β_0 is a character of $\mathbf{F}^\times$ with $\beta_0(-1) = -1$,

$$L_{st}(\pi(\eta_1, \eta_2), \nu_0) = L(\pi(\eta_1\beta_0, \eta_2\beta_0), \nu_0\beta_0).$$

If one of the η_i is even and the other is odd, then $L_{st}(\pi(\eta_1, \eta_2), \nu_0)$ is zero. However, in order to get all principal series of $\widetilde{SL}_2(\mathbf{F})$ it suffices to consider $L(\pi(\eta_1, \eta_2), \nu_0)$ with η_1 and η_2 both even.

6.2 One Dimensional Representations and the Dependence on ψ

Suppose that μ' is a one dimensional representation of $SO_{1,2}(\mathbf{F})$. Then μ is a one dimensional representation of $GL_2(\mathbf{F})$ with trivial central character. That is, there is a character μ_0 of $\mathbf{F}^\times$ such that $\mu = \mu_0 \circ \det$ and $\mu(xI) = 1$ for all $x \in \mathbf{F}^\times$. This in turn implies that $\mu_0(x^2) = 1$ for all $x \in \mathbf{F}^\times$, i.e., that μ_0 is a quadratic character. Then there exists y such that $\mu_0(x) = q_y(x)$ for all $x \in \mathbf{F}^\times$. Here $q_y(x) = (x, y)_\mathbf{F}$ as in Proposition 3.19. Hence the one dimensional representations of $SO_{1,2}(\mathbf{F})$ are parameterized by $y \in \mathbf{F}^\times/\mathbf{F}^{\times 2}$. We put $Q_y = q_y \circ \det$ and write Q'_y for the corresponding one–dimensional representation of $SO_{1,2}(\mathbf{F})$.

In Chapter 3, we fixed an additive character ψ of $\mathbf{F}$. The maps $\mathcal{L}_{st}$, L_{st}, Γ_-, Γ_+ and χ_{ν_0} all depended on ψ. We will now vary the choice of ψ, and we write $\mathcal{L}_{st}^\psi$, L_{st}^ψ, Γ_+^ψ, Γ_-^ψ and $\chi_{\nu_0}^\psi$ to indicate the dependence on ψ.

The following proposition lists some formal properties showing the role of the choice of the additive character ψ. At this point we refer the reader to Sections 2.5 and 3.3 where we have established the relevant notation.

Proposition 6.16. *We have that* $\chi_{\nu_0}^{\psi_y} = \chi_{\nu_0 q_y}^{\psi}$ *and*

$$L_{st}^\psi(\pi, \nu_0)^y = L_{st}^\psi(\pi, \nu_0 q_y) = L_{st}^\psi(\pi Q_y, \nu_0) = L_{st}^{\psi_y}(\pi, \nu_0)$$

The relationships between $\Gamma_\pm^\psi$ and $\Gamma_\pm^{\psi_y}$ are given in the next two lemmas.

Lemma 6.17. *Suppose* $g \in SL_2(\mathbf{F})$.

1. *Assume that* $g = diag(x, 1/x)$ *is hyperbolic. Then for each* $y \in \mathbf{F}^\times$,

$$q_y(x)\Gamma_+^\psi(\widetilde{g}) = \Gamma_+^{\psi_y}(\widetilde{g}).$$

2. *Assume that* $g = h^2 \in S_\Delta$ *is elliptic and that* $\det h = 1$. *Then for each* $y \in \mathbf{F}^\times$,

$$q_y(\Delta)\Gamma_+^\psi(\widetilde{g}) = \Gamma_+^{\psi_y}(\widetilde{g}).$$

3. *Assume that* $g = h^2 \in S_\Delta$ *is elliptic and that* $\det h = -1$. *Then for each* $y \in \mathbf{F}^\times$,

$$\Gamma_+^\psi(\widetilde{g}) = \Gamma_+^{\psi_y}(\widetilde{g}).$$

4. *Assume that* $g \in S_\Delta$ *is elliptic and is not a square. Write* $\xi_0 g = h^2$ *with* $\xi_0 = \det h$. *Then for each* $y \in \mathbf{F}^\times$,

$$q_y(\xi_0\Delta)\Gamma_+^\psi(\widetilde{g}) = \Gamma_+^{\psi_y}(\widetilde{g}).$$

Proof. This follows from the fact that

$$q_y(x)\gamma_{\mathbf{F}}(x,\psi) = \gamma_{\mathbf{F}}(x,\psi_y)$$

(see Section 2.5). Suppose that we are in case 1. By Theorem 4.41, we have that

$$\begin{aligned} q_y(x)\Gamma_+^{\psi}(\widetilde{g}) &= \frac{q_y(x)\gamma_{\mathbf{F}}(x,\psi)\varepsilon}{|\det(1-g)|^{1/2}} \\ &= \frac{\gamma_{\mathbf{F}}(x,\psi_y)\varepsilon}{|\det(1-g)|^{1/2}} \\ &= \Gamma_+^{\psi_y}(\widetilde{g}). \end{aligned}$$

For case 2, we have that

$$\begin{aligned} q_y(\Delta)\Gamma_+^{\psi}(\widetilde{g}) &= \frac{\varepsilon s(g)(y,\Delta)_{\mathbf{F}}\gamma_{\mathbf{F}}(\Delta,\psi)c_\beta(\Delta I,-g)}{|\det(1-g)|^{1/2}} \\ &= \frac{\varepsilon s(g)\gamma_{\mathbf{F}}(\Delta,\psi_y)c_\beta(\Delta I,-g)}{|\det(1-g)|^{1/2}} \\ &= \Gamma_+^{\psi_y}(\widetilde{g}). \end{aligned}$$

By Theorem 4.41, case 3 does not depend on ψ so the assertion is clear. The proof of case 4 is similar to the proofs of cases 1 and 2 and is omitted. □

Lemma 6.18. *Suppose $g \in SL_2(\mathbf{F})$.*

1. *Assume that $g = diag(x, 1/x)$ is hyperbolic. Then for each $y \in \mathbf{F}^\times$,*

$$q_y(x)\Gamma_-^{\psi}(\widetilde{g}) = \Gamma_-^{\psi_y}(\widetilde{g}).$$

2. *Assume that $g = h^2 \in S_\Delta$ is elliptic and that $\det h = 1$. Then for each $y \in \mathbf{F}^\times$,*

$$\Gamma_-^{\psi}(\widetilde{g}) = \Gamma_-^{\psi_y}(\widetilde{g}).$$

3. *Assume that $g = h^2 \in S_\Delta$ is elliptic and that $\det h = -1$. Then for each $y \in \mathbf{F}^\times$,*

$$q_y(\Delta)\Gamma_-^{\psi}(\widetilde{g}) = \Gamma_-^{\psi_y}(\widetilde{g}).$$

4. Assume that $g \in S_\Delta$ is elliptic and is not a square. Write $\xi_0 g = h^2$ with $\xi_0 = \det h$. Then for each $y \in \mathbf{F}^\times$,

$$q_y(\xi_0)\Gamma_-^{\psi}(\widetilde{g}) = \Gamma_-^{\psi_y}(\widetilde{g}).$$

Proof. The proof is the same as the proof of the previous lemma. Suppose that we are in case 1. By Theorem 4.41, we have that

$$\begin{aligned} q_y(x)\Gamma_-^{\psi}(\widetilde{g}) &= \frac{q_y(x)\gamma_{\mathbf{F}}(x,\psi)\varepsilon}{|\det(1+g)|^{1/2}} \\ &= \frac{\gamma_{\mathbf{F}}(x,\psi_y)\varepsilon}{|\det(1+g)|^{1/2}} \\ &= \Gamma_-^{\psi_y}(\widetilde{g}). \end{aligned}$$

By Theorem 4.41, case 2 does not depend on ψ so the assertion is clear. Cases 3 and 4 follow similarly by the argument for case 1 and their proofs are omitted. □

Proposition 6.19. *Let ν_0 be a character of $\mathbf{F}^\times$. Then*

$$L^{\psi}(\nu,\nu_0) = \begin{cases} \omega_e(\psi) & \text{if } \nu_0(-1) = 1, \\ -\omega_o(\psi) & \text{if } \nu_0(-1) = -1. \end{cases}$$

Proof. Assume β_0 is a character of $\mathbf{F}^\times$ satisfying $\beta_0(-1) = -1$. We have

$$\Theta_{L^{\psi}(\nu,\nu_0)}(g;\varepsilon) + \Theta_{L^{\psi}(\nu\beta,\nu_0\beta_0)}(g;\varepsilon) = \Theta_{\omega_e(\psi)}(g;\varepsilon) - \Theta_{\omega_o(\psi)}(g;\varepsilon). \tag{6.20}$$

We also have

$$\Theta_{L^{\psi}(\nu,\nu_0)}(-g;\varepsilon) + \Theta_{L^{\psi}(\nu\beta,\nu_0\beta_0)}(-g;\varepsilon) = \Theta_{\omega_e(\psi)}(-g;\varepsilon) - \Theta_{\omega_o(\psi)}(-g;\varepsilon).$$

Recall that $(-g;\varepsilon) = (-I;1)(g;\varepsilon c(-I,g))$. So

$$\begin{aligned} &\gamma_{\mathbf{F}}(-1,\psi)c(-I,g)\nu_0(-1)(\Theta_{L^{\psi}(\nu,\nu_0)}(g;\varepsilon) - \Theta_{L^{\psi}(\nu\beta,\nu_0\beta_0)}(g;\varepsilon)) = \\ &\qquad \gamma_{\mathbf{F}}(-1,\psi)c(-I,g)(\Theta_{\omega_e(\psi)}(g;\varepsilon) + \Theta_{\omega_o(\psi)}(g;\varepsilon)), \end{aligned}$$

i.e.,

$$\nu_0(-1)(\Theta_{L^\psi(\nu,\nu_0)}(g;\varepsilon) - \Theta_{L^\psi(\nu\beta,\nu_0\beta_0)}(g;\varepsilon)) = \Theta_{\omega_e(\psi)}(g;\varepsilon) + \Theta_{\omega_o(\psi)}(g;\varepsilon). \quad (6.21)$$

If $\nu_0(-1) = 1$, then adding (6.20) and (6.21) yields $L^\psi(\nu,\nu_0) = \omega_e(\psi)$. On the other hand if $\nu_0(-1) = -1$, then subtracting (6.21) from (6.20) yields $L^\psi(\nu,\nu_0) = -\omega_o(\psi)$. □

Remark. This is consistent with (4.9), (4.22), and (4.37). Note that this could have been proved using the computations given in Lemmas 6.17 and 6.18.

The next corollary gives the correspondence for one–dimensional representations.

Corollary 6.22. *For each $y \in \mathbf{F}^\times$, we have that*

$$\mathcal{L}^\psi_{st}(Q'_y) = \omega_e(\psi_y) - \omega_o(\psi_y).$$

Proof. This is obvious from Proposition 6.19 because

$$L^\psi_{st}(1,q_x) = L^\psi(1,q_x) + L^\psi(\nu,\nu_0 q_x)$$

with $\nu_0(-1) = -1$. □

We also have the following decompositions which are already well known (see Gelbart and Piatetski–Shapiro [10]).

Corollary 6.23. *Let μ_0 be a character of $\mathbf{F}^\times$ such that $\mu_0(-1) = 1$. Then we have*

$$\omega(\mu)|_{\widetilde{SL}_2(\mathbf{F})} = \sum_{x\in\mathbf{F}^\times/\mathbf{F}^{\times 2}} \omega_e(\psi_x).$$

If $\mu'_0(-1) = -1$ then

$$\omega(\mu')|_{\widetilde{SL}_2(\mathbf{F})} = \sum_{x\in\mathbf{F}^\times/\mathbf{F}^{\times 2}} \omega_o(\psi_x).$$

Proof. The first assertion follows from Proposition 3.19. The second assertion follows easily from the previous corollary. □

6.3 Special and Supercuspidal Representations

6.3.1 Special Representations

A reducible principal series on $GL_2(\mathbf{F})$ takes the form $\pi(\lambda|\cdot|^{1/2}, \lambda|\cdot|^{-1/2})$ (see Definition 2.51). Thus the reducible principal series with trivial central character are in one to one correspondence with $\pi(q_x|\cdot|^{1/2}, q_x|\cdot|^{-1/2})$.

Let $\sigma(q_x|\cdot|^{1/2}, q_x|\cdot|^{-1/2})$ and $\mu(q_x|\cdot|^{1/2}, q_x|\cdot|^{-1/2})$ be the associated special and one–dimensional representations respectively, and $\sigma(q_x|\cdot|^{1/2}, q_x|\cdot|^{-1/2})'$ be the push–down of $\sigma(q_x|\cdot|^{1/2}, q_x|\cdot|^{-1/2})$ to $SO_{1,2}(\mathbf{F})$.

Proposition 6.24.

$$\begin{aligned}\mathcal{L}_{st}(\sigma(q_x|\cdot|^{1/2}, q_x|\cdot|^{-1/2})') &= PS_{\widetilde{SL}}(\widetilde{q_x|\cdot|^{1/2}}) - \omega_e(\psi_x) + \omega_o(\psi_x) \\ &= L_{st}(\pi(q_x|\cdot|^{1/2}, q_x|\cdot|^{-1/2}), q_x) - L_{st}(1, q_x)\end{aligned}$$

Proof. By Jacquet–Langlands [12], p. 275, we have that

$$\Theta_{\pi(q_x|\cdot|^{1/2},q_x|\cdot|^{-1/2})} = \Theta_{\mu(q_x|\cdot|^{1/2},q_x|\cdot|^{-1/2})} + \Theta_{\sigma(q_x|\cdot|^{1/2},q_x|\cdot|^{-1/2})}.$$

We have shown that $\pi(q_x|\cdot|^{1/2}, q_x|\cdot|^{-1/2})$ pushes down to $PS_{SO}(q_x|\cdot|^{1/2})$ and that this corresponds with $PS_{\widetilde{SL}}(\widetilde{q_x|\cdot|^{1/2}})$. On the other hand, Jacquet–Langlands have shown (see [12], p. 102) that $\mu(q_x|\cdot|^{1/2}, q_x|\cdot|^{-1/2})$ is equivalent to the one dimensional representation $Q_x = q_x \circ \det$. Thus $\mu(q_x|\cdot|^{1/2}, q_x|\cdot|^{-1/2})$ pushes down to Q_x'. We have shown that $\mathcal{L}_{st}(Q_x') = \omega_e(\psi_x) - \omega_o(\psi_x)$. Hence

$$\mathcal{L}_{st}(\sigma(q_x|\cdot|^{1/2}, q_x|\cdot|^{-1/2})') = PS_{\widetilde{SL}}(\widetilde{q_x|\cdot|^{1/2}}) - \omega_e(\psi_x) + \omega_o(\psi_x),$$

i.e.,

$$\mathcal{L}_{st}(\sigma(q_x|\cdot|^{1/2}, q_x|\cdot|^{-1/2})') = L_{st}(\pi(q_x|\cdot|^{1/2}, q_x|\cdot|^{-1/2}), q_x) - L_{st}(1, q_x).$$

□

The reducible principal series $PS_{\widetilde{SL}}(\widetilde{q_x|\cdot|^{1/2}})$ of $\widetilde{SL}_2(\mathbf{F})$ is an even oscillator representation plus a special representation. If $\widetilde{\sigma}(q_x|\cdot|^{1/2})$ denotes this special representation, then $PS_{\widetilde{SL}}(\widetilde{q_x|\cdot|^{1/2}}) - \omega_e(\psi_x) + \omega_o(\psi_x) = \omega_o(\psi_x) + \widetilde{\sigma}(q_x|\cdot|^{1/2})$.

Corollary 6.25.

$$\mathcal{L}_{st}(\sigma(q_x|\cdot|^{1/2}, q_x|\cdot|^{-1/2})') = \omega_0(\psi_x) + \widetilde{\sigma}(q_x|\cdot|^{1/2}).$$

In particular, a special representation of $SO_{1,2}(\mathbf{F})$ corresponds with a a sum of a supercuspidal and a special representation of $\widetilde{SL}_2(\mathbf{F})$.

6.3.2 Supercuspidal Representations

Suppose π' is an irreducible supercuspidal representation of $SO_{1,2}(\mathbf{F})$. Then π is an irreducible supercuspidal representation of $GL_2(\mathbf{F})$ and Flicker's lifting $\mathrm{Lift}_{\mathrm{F}}(\pi)$ is supercuspidal by Theorem 2.67. Its restriction to $\widetilde{SL}_2(\mathbf{F})$ breaks up into a sum of supercuspidal representations.

Proposition 6.26. *Suppose π' is an irreducible supercuspidal representation of $SO_{1,2}(\mathbf{F})$. Then $\mathcal{L}_{st}(\pi')$ is the sum of two supercuspidal representations.*

6.4 Summary

Theorem 6.27. *Consider the correspondence*

$$\mathcal{L}_{st}^{\psi} : Gr(SO_{1,2}(\mathbf{F})) \leftrightarrow Gr_{st}(\widetilde{SL}_2(\mathbf{F}))$$
$$(\pi \otimes \nu^{-1})' \leftrightarrow L_{st}^{\psi}(\pi, \nu_0).$$

This satisfies:

- *Principal series on* $\widetilde{SL}_2(\mathbf{F})$ *correspond to principal series on* $SO_{1,2}(\mathbf{F})$*. That is,* $\mathcal{L}_{st}^{\psi}(PS_{SO}(\nu_0)) = PS_{\widetilde{SL}}(\widetilde{\nu}_0)$*.*

- *The one–dimensional representation* Q'_x *of* $SO_{1,2}(\mathbf{F})$ *corresponds with the difference of the two halves of the oscillator representation of* $\widetilde{SL}_2(\mathbf{F})$ *attached to* ψ_x*. That is,* $\mathcal{L}_{st}^{\psi}(Q'_x) = \omega_e(\psi_x) - \omega_o(\psi_x)$*.*

- *A special representation on* $SO_{1,2}(\mathbf{F})$ *corresponds with the sum of an odd oscillator representation and an odd special representation of* $\widetilde{SL}_2(\mathbf{F})$*. More precisely,* $\mathcal{L}_{st}^{\psi}(\sigma(q_x|\cdot|^{1/2}, q_x|\cdot|^{-1/2})') = \omega_o(\psi_x) + \widetilde{\sigma}(q_x|\cdot|^{1/2})$*.*

- *If* π' *is an irreducible supercuspidal representation of* $SO_{1,2}(\mathbf{F})$*, then* $\mathcal{L}_{st}^{\psi}(\pi')$ *is the sum of two supercuspidal representations of* $\widetilde{SL}_2(\mathbf{F})$*.*

BIBLIOGRAPHY

[1] J. Adams, *Characters of non–linear groups*, Preprint.

[2] ———, *The character of the oscillator representation*, Israel. J. Math. **98** (1997), 229–252.

[3] ———, *Lifting of characters on orthogonal and metaplectic groups*, Duke Math. J. **92** (1998), no. 1, 129–178.

[4] C. Blondel, *Les représentations supercuspidales de groupes métaplectiques sur $GL(2)$ et leurs caractères*, Mémoire de la Société Mathématique de France **113** (1985), 1–119.

[5] T. Bröcker and T. tom Dieck, *Representation theory of compact Lie groups*, Graduate Texts in Mathematics, vol. 98, Springer–Verlag, 1985.

[6] Y. Flicker, *Automorphic forms on covering groups of $GL(2)$*, Inventiones math. **79** (1980), 119–182.

[7] Y. Flicker and D.A. Kazhdan, *Metaplectic correspondences*, Publ. Math. I.H.E.S. **64** (1986), 53–110.

[8] S. Gelbart, *Automorphic forms on adele groups*, Annals of Mathematics Studies, vol. 83, Princeton University Press, 1975.

[9] ______, *Weil's representation and the spectrum of the metaplectic group*, Lecture Notes in Mathematics, vol. 530, Springer–Verlag, 1976.

[10] S. Gelbart and I. Piatetski-Shapiro, *Distinguished representations and modular forms of half-integral weight*, Inventiones math. **59** (1980), 145–188.

[11] ______, *Some remarks on metaplectic cusp forms and the correspondences of Shimura and Waldspurger*, Israel. Journ. of Math. **44** (1983), no. 2, 97–126.

[12] H. Jacquet and R. Langlands, *Automorphic forms on* $GL(2)$, Lecture Notes in Mathematics, vol. 114, Springer–Verlag, 1970.

[13] Harish–Chandra, *Harmonic analysis on reductive* p*–adic groups*, Lecture Notes in Mathematics, vol. 162, Springer–Verlag, 1970.

[14] ______, *Admissible invariant distributions on reductive* p*–adic groups*, Queen's Papers in Pure and Applied Math. **48** (1978), 281–347.

[15] R. Howe, *The character of the oscillator representation*, Preprint.

[16] ______, θ*–series and invariant theory*, Proceedings of Symposia in Pure Mathematics, vol. 33, AMS, 1979, pp. 275–285.

[17] N. Jacobson, *Basic Algebra I*, W.H. Freeman and Co., 1985.

[18] J.P. Labesse and R.P. Langlands, L*–indistinguishability for* $SL(2)$, Can. J. Math. **31** (1979), 726–785.

[19] D.A. Kazhdan and S.J. Patterson, *Metaplectic forms*, Publ. Math. I.H.E.S. **59** (1984), 35–142.

[20] ———, *Towards a generalized Shimura correspondence*, Adv. in Math. **60** (1986), 161–234.

[21] A. Knapp and S. Gelbart, *L-indistinguishability and R groups for the special linear group*, Adv. in Math. **43** (1982), no. 2, 101–121.

[22] T. Kubota, *Topological covering of $SL(2)$ over a local field*, J. Math. Soc. Japan **19** (1967), no. 1, 114–121.

[23] ———, *On automorphic functions and the reciprocity law in a number field*, Kinokuniya Book–Store Co., Ltd., 1969.

[24] D. Manderscheid, *Supercuspidal duality for the two–fold cover of $SL(2)$ and the split $O(3)$*, Ph.D. Thesis, Yale University, 1981.

[25] C. Moore, *Extensions and low dimensional cohomology theory of locally compact groups*, Trans. Amer. Math. Soc. **113** (1964), 40–86.

[26] ———, *Group extensions of p–adic and adelic linear groups*, Publ. Math. I.H.E.S. **35** (1968), 157–222.

[27] S. Niwa, *Modular forms of half integral weight and the integral of certain theta–functions*, Nagoya Math. J. **56** (1974), 147–161.

[28] I. Piatetski-Shapiro, *Work of Waldspurger*, Lie Group representations II, vol. 1041, Springer Lecture Notes, 1984, pp. 280–302.

[29] D. Prasad, *On the theta correspondence*, Preprint.

[30] S. Rallis and G. Schiffman, *Représentations supercuspidales du group métaplectique*, J. Math. Kyoto Univ. **17** (1977), no. 3, 567–603.

[31] R. Ranga Rao, *On some explicit formulas in the theory of Weil representation*, Pac. J. Math. **157** (1993), no. 2, 335–371.

[32] J.P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics, vol. 7, Springer–Verlag, 1973.

[33] ______, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer–Verlag, 1979.

[34] G. Shimura, *On modular forms of half integral weight*, Ann. of Math. **97** (1973), 440–481.

[35] T. Shintani, *On construction of holomorphic cusp forms of half integral weight*, Nagoya Math. J. **58** (1975), 83–126.

[36] A. Silberger, *Introduction to harmonic analysis on reductive p–adic groups*, Mathematical Notes, vol. 23, Princeton University Press, 1979.

[37] M. Tádic, *Notes on representations of non–archimedean* $SL(n)$, Pacific J. Math. **152** (1992), no. 2, 375–396.

[38] P. Torasso, *Sur le caractère de la représentation de Shale–Weil de* $Mp(n\mathbf{R})$ *et* $Sp(n, \mathbf{C})$, Math. Ann. **252** (1980), 53–86.

[39] J. L. Waldspurger, *Correspondance de Shimura*, J. Math. pures et appl. **59** (1980), 1–133.

[40] ______, *Sur les coefficients de Fourier des formes modulaires de poids demi–entier*, J. Math. pures et appl. **60** (1981), 375–484.

[41] ______, *Correspondances de Shimura et quaternions*, Forum Math. **3** (1991), 219–307.

[42] A. Weil, *Sur certaines groupes d'opérateurs unitaires*, Acta Math. **11** (1964), 143–211.