Characters of covering groups of SL(n)

Jeffrey Adams University of Maryland

May 5, 2000

Abstract

We study characters of an n-fold cover $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ of $\mathrm{SL}(\mathrm{n},\mathbb{F})$ over a non-archimedean local field. We compute the character of an irreducible representation of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ in terms of the character of an irreducible representation of a cover $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ of $\mathrm{GL}(\mathrm{n},\mathbb{F})$. We define an analogue of L-packets for $\widetilde{\mathrm{SL}}(n,\mathbb{F})$, such that the character of a linear combination of the representations in such a packet is computed in terms of the character of an irreducible representation of $\mathrm{PGL}(\mathrm{n},\mathbb{F})$. This is analogous to stable endoscopic lifting for linear groups. We also prove an inversion formula expressing the character of a genuine irreducible representation of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ as a linear combination of virtual characters, each of which is lifted from $\mathrm{PGL}(\mathrm{n},\mathbb{F})$.

1 Introduction

Let $\mathbb{G}$ be a reductive linear group defined over a local field $\mathbb{F}$ of characteristic 0, and let $G=\mathbb{G}(\mathbb{F})$. One of the ingredients of the local portion of the Langlands program for G is the study of the characters of admissible representations of G. These are used on one side of the trace formula, and provide information about automorphic representations of $\mathbb{G}$ over a global field.

Important examples of automorphic representations involve reductive groups which are not linear, such as the oscillator representation of the metaplectic group, the two–fold cover of $Sp(2n,\mathbb{F})$. We refer to a finite central extension $\tilde{G}$ of G which is not itself a linear group as a *non-linear* group. It would be interesting to understand the representation theory of such groups, and to extend the Langlands program to the study of their automorphic representations.

A representation π of $\tilde{G}$ is said to be *genuine* if it does not factor to any proper quotient of $\tilde{G}$. One approach to the representation theory of $\tilde{G}$ is to relate genuine representations of $\tilde{G}$ to representations of a linear group via character theory. There are a number of examples of this approach. See the references and [2] for a survey.

Now assume the cardinality of the n^{th} roots of unity $\mu_n=\mu_n(\mathbb{F})$ is n. We consider a central extension $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ of $\mathrm{SL}(\mathrm{n},\mathbb{F})$ by μ_n (cf. Section 2).

Arbitrary covers $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ of $\mathrm{GL(n,\mathbb{F})}$ have been studied extensively [5], [8], [7], [4]. Flicker, Kazhdan and Patterson relate character theory of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ to that of $\mathrm{GL(n,\mathbb{F})}$. The group $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ is a subgroup of a corresponding group $\widetilde{\mathrm{GL}}(n,\mathbb{F})$, and a natural approach is to study representations of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ by restricting representations of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$. The corresponding problem for $SL(n,\mathbb{F})$ and $GL(n,\mathbb{F})$ is quite difficult [17], [12]. For example the case of $n=2$ is the first example of endoscopy and is highly non–trivial [6].

Surprisingly the corresponding restriction problem for genuine representations of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ is very easy, and character theory of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ reduces to that of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$. Our first step is to write a formula (Theorem 3.1) for the character of an irreducible genuine representation π of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ in terms of the character of an irreducible representation of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ which contains π in its restriction.

Flicker, Kazhdan and Patterson have defined a lifting theory, conjecturally taking an irreducible unitary representation π of $\mathrm{GL(n,\mathbb{F})}$ to an irreducible genuine unitary representation $t_*(\pi)$ of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ or 0 [7], [4], [8]. The character of $t_*(\pi)$ is computed in terms of the character of π. Together with Theorem 3.1, restriction from $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ to $\widetilde{\mathrm{SL}}(n,\mathbb{F})$, this expresses the character of an irreducible constituent of $t_*(\pi)$ restricted to $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ in terms of characters of $\mathrm{GL(n,\mathbb{F})}$.

A constituent of $t_*(\pi)$ restricted to $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ is determined by a character ν of $\mathbb{F}^*$ for which ν^n equals the central character of π. We write $L(\pi,\nu)$ for this constituent. For any character α, $L(\pi\alpha^n,\nu\alpha^n)\approx L(\pi,\nu)$; we sum over $\widehat{\mathbb{F}^*}/\widehat{\mathbb{F}^{*n}}\approx\widehat{\mu_n}$ and define (cf. Definition 5.3):

$$L_{st}(\pi,\nu)=\sum_{\alpha\in\widehat{\mu_n}}L(\pi\alpha,\nu\alpha).$$

Now $\pi\nu^{-1}$ factors to $\mathrm{PGL(n,\mathbb{F})}$, and it turns out that the character of $L_{st}(\pi,\nu)$ may be computed in terms of the character of $\pi\nu^{-1}$. The main result is (Theorem 8.1):

$$\Theta_{L_{st}(\pi,\nu)}(g)=\sum_{\substack{h\in\mathrm{PGL(n,\mathbb{F})}\\ \phi(h)=p(g)}}\Delta_\mu(h,g)\Theta_{\pi\nu^{-1}}(h).$$

Here g is a regular semisimple element of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$, and we identify the character of a representation with a function on the regular semisimple elements. Also ϕ is the *orbit correspondence*: $\phi(g)=\det(g^{-1})g^n\in\mathrm{SL(n,\mathbb{F})}$ (Section 6), p is projection from $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ to $\mathrm{SL(n,\mathbb{F})}$, and $\Delta_\mu(h,g)$ is a *transfer factor* (Section 7). The group $\mathrm{PGL(n,\mathbb{F})}$ is the one predicted by the Hecke algebra isomorphism of [13].

The set $\Pi(\pi,\nu)=\{L(\pi\alpha,\nu\alpha)\,|\,\alpha\in\widehat{\mu_n}\}$ appearing here is analogous to an L-packet for a linear group G. The character formula is is analogous to endoscopic lifting from the quasisplit form G_{qs} to G, and $L_{st}(\pi,\nu)$ is analogous to the stable lift of $\pi\nu^{-1}$, although since $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ is non-linear the notion of stable distribution is not defined.

An L-packet for $\mathrm{SL}(n,\mathbb{F})$ is conjecturally the set of constituents of the restriction of an irreducible representation of $\mathrm{GL}(n,\mathbb{F})$ to $\mathrm{SL}(n,\mathbb{F})$. The character of the sum of these representations is stable, i.e. invariant by conjugation by $\mathrm{GL}(n,\mathbb{F})$, and these sets satisfy other required properties of L–packets. It is interesting to note that $\Pi(\pi,\nu)$ is *not* the set of constituents of the restriction of a representation of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$. In particular (see the Remark following Theorem 8.1) $\Theta_{L_{st}(\pi,\nu)}$ is typically not $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ conjugation invariant. It would be interesting to find an intrinsic characterization of the virtual characters $L_{st}(\pi,\nu)$.

In the theory of endoscopy for a linear group an irreducible representation π contained in an L-packet Π may be expressed as a linear combination of virtual characters, in the span of the elements of Π, each of which is stable, or lifted from a stable character on a smaller endoscopic group. This is known as *inversion*. By analogy with this we seek to write $L(\pi,\nu)$ as a linear combination of virtual representations, in the span of the elements of $\Pi(\pi,\nu)$, each of which is computed in terms of characters of a linear group. For $\zeta\in\mu_n$ let

$$L_\zeta(\pi,\nu)=\sum_{\alpha\in\widehat{\mu_n}}\alpha(\zeta)L(\pi\alpha,\nu\alpha).$$

We obtain an inversion formula (Theorem 9.1):

$$\begin{aligned}\Theta_{L(\pi,\nu)}(g)&=\frac{1}{n}\sum_{\zeta\in\mu_n}L_\zeta(\pi,\nu)\\&=\frac{1}{n}\sum_{\zeta\in\mu_n}\chi^{-1}(z_\zeta)\Theta_{L_{st}(\pi,\nu)}(z_\zeta g).\end{aligned}$$

Here χ is the central character of $L(\pi,\nu)$ and z_ζ is an element of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ with image $\zeta I\in\mathrm{SL}(n,\mathbb{F})$.

These results all hold as stated for $\mathbb{F}=\mathbb{R}$ and $n=2$, in which case they are equivalent to a special case of [3].

Similar results hold for certain other N-fold covers of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$. One would not expect the general N-fold cover to be amenable to these methods, as the case $N=1$ makes abundantly clear.

The case of $n=2$, worked out in detail, is the subject of the University of Maryland thesis of Jason Schultz [14]. This gives an intrinsic characterization of the local lift of Waldspurger [18]. In this case the set Π containing a genuine discrete series representation π consists of two elements π,π' where π' is the "Waldspurger involution" [18] applied to π. This goes back to the Shimura correspondence for modular forms of half–integral weight which is the origin of the theory of non-linear groups.

1.1 Desiderata

We consider covering groups:

$$1 \to \mu_n \to \tilde{G} \xrightarrow{p} G \to 1$$

with μ_n central in $\tilde{G}$ (cf. Section 2). We write χ_π for the central character of a representation π. We say a representation π of $\tilde{G}$ is genuine if π has a central character χ_π whose restriction to μ_n is injective. If π is not genuine then π factors to a representation of a cover of G with kernel a subgroup of μ_n. If $\iota : \mu_n \hookrightarrow \mathbb{C}^n$ is an embedding we say π is of type ι if $\chi_\pi|_{\mu_n} = \iota$.

An important role is played by the exact sequences

$$1 \to \mu_n \to \mathbb{F}^* \xrightarrow{n} \mathbb{F}^{*n} \to 1 \tag{1}$$

$$1 \to \mathbb{F}^{*n} \xrightarrow{\iota} \mathbb{F}^* \to \mathbb{F}^*/\mathbb{F}^{*n} \to 1 \tag{2}$$

and their Pontriagin duals:

$$1 \to \widehat{\mathbb{F}^{*n}} \to \widehat{\mathbb{F}^*} \xrightarrow{\mathrm{res}} \widehat{\mu_n} \to 1 \tag{3}$$

$$1 \to \widehat{\mathbb{F}^*/\mathbb{F}^{*n}} \to \widehat{\mathbb{F}^*} \xrightarrow{\mathrm{res}} \widehat{\mathbb{F}^{*n}} \to 1 \tag{4}$$

Suppose μ_n is in the kernel of $\lambda \in \widehat{\mathbb{F}^*}$. Then by (3) $\lambda(x) = \mu(x^n)$ for some character μ of $\widehat{\mathbb{F}^{*n}}$, which by (4) extends to $\tau \in \widehat{\mathbb{F}^*}$. This gives the following well-known lemma which we use repeatedly:

Lemma 1.1 *Let $\lambda \in \widehat{\mathbb{F}^*}$. Then $\lambda = \mu^n$ for some $\mu \in \widehat{\mathbb{F}^*}$ if and only if $\lambda(\zeta) = 1$ for all $\zeta \in \mu_n$.*

We identify the center Z of $\mathrm{GL(n,\mathbb{F})}$ with $\mathbb{F}^*$ and the central character χ_π of a representation of $\mathrm{GL(n,\mathbb{F})}$ with an element of $\widehat{\mathbb{F}^*}$.

For $\alpha \in \widehat{\mathbb{F}^*}$ we write α for the character $\alpha \circ \det$ of of $\mathrm{GL(n,\mathbb{F})}$, and also for the character $\alpha \circ p$ of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$. Note that for π a representation of $\mathrm{GL(n,\mathbb{F})}$ (with a central character)

$$\chi_{\pi\alpha} = \chi_\pi \alpha^n. \tag{5}$$

We write Θ_π for the global character of a representation π, considered as a function on the set of regular semisimple elements.

2 Group Structure

We continue with the notation of Section 1. We first define the group $\widetilde{\mathrm{SL}}(n,\mathbb{F})$(cf. [10], [9], [16]): this is a topological group which fits in an exact sequence:

$$1 \to \mu_n \xrightarrow{\iota} \widetilde{\mathrm{SL}}(n,\mathbb{F}) \xrightarrow{p} \mathrm{SL(n,\mathbb{F})} \to 1 \tag{6}$$

with ι, p continuous, ι closed and p open. The classes of such extensions are parametrized by the group of (bilinear) Steinberg cocycles with values in μ_n. Let $(\ ,\ )_n : \mathbb{F}^* \times \mathbb{F}^* \to \mu_n$ denote the n^{th} norm residue symbol for $\mathbb{F}$. For properties of $(\ ,\ )_n$ see [15], ([7], §0.1). In particular $(\ ,\ )_n$ is a perfect pairing on $\mathbb{F}^*/\mathbb{F}^{*n}$ and gives an isomorphism of $\mathbb{F}^*/\mathbb{F}^{*n}$ with $\widehat{\mathbb{F}^*/\mathbb{F}^{*n}}$. Each Steinberg cocycle is given by $c(x,y) = (x,y)_n^k$ for some k. Write $G[k]$ for the group defined by the cocycle $(x,y)_n^k$. Then $G[k]$ and $G[k']$ are equivalent extensions if and only if $k \equiv k' \mod (n)$.

The commutator subgroup $G[k]_c$ of $G[k]$ is a covering group of $\mathrm{SL(n,\mathbb{F})}$ with kernel a subgroup of μ_n. If $G[k]$ is not perfect then $G[k] = G[k]_c\mu_n$ and the representations of $G[k]$ of type ι are in bijection with the representations of $G[k]_c$ of type $\iota|_{\mu_n \cap G[k]_c}$. For this reason we assume $G[k]$ is perfect, which holds if and only if $gcd(k,n) = 1$.

The map $G[k] \ni (g,\zeta) \to (g,\zeta^j) \in G[kj]$ is a homomorphism, and is an isomorphism if $gcd(j,n) = 1$. In particular if $gcd(k,n) = 1$ then $G[k]$ is isomorphic to $G[1]$ (although not equivalent as an extension unless $k \equiv 1 \mod (n)$). We let $\widetilde{\mathrm{SL}}(n,\mathbb{F}) = G[1]$. Once and for all we fix an embedding

$$\iota : \mu_n(\mathbb{F}) \hookrightarrow \mathbb{C}^*$$

and we identify μ_n with its image. Henceforth we assume all genuine representations are of type ι.

The Steinberg cocycle defines a cover $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ of $\mathrm{GL(n,\mathbb{F})}$ by [7], and $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ is a subgroup of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ (we are taking $c = 0$ in the notation of [7]).

We write $c(\ ,\ )$ for the cocycle defining $\widetilde{\mathrm{GL}}(n,\mathbb{F})$. Then

$$\widetilde{\mathrm{GL}}(n,\mathbb{F}) = \{(g,\zeta)\,|\,g \in \mathrm{GL(n,\mathbb{F})}, \zeta \in \mu_\mathrm{n}\}$$

with multiplication $(g,\zeta)(g',\zeta') = (gg',\zeta\zeta' c(g,g'))$.

An essential role is played by the commutator. Suppose g and h are commuting elements of $\mathrm{GL(n,\mathbb{F})}$. Choose any inverse images $\tilde{g}, \tilde{h}$ of g, h in $\widetilde{\mathrm{GL}}(n,\mathbb{F})$. Then $\eta = \tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1} \in \mu_n$ is independent of the choices of $\tilde{g}$ and $\tilde{h}$. We write $\{g,h\} = \eta$.

An important property of the commutator is ([7], proof of Proposition 0.1.1):

$$\{xI, g\} = (x, \det(g))_n^{-1} \tag{7}$$

2.1 Centers

Let

$$\begin{aligned}\mathrm{GL(n,\mathbb{F})}_+ &= \{g \in \mathrm{GL(n,\mathbb{F})} \mid \mathrm{det(g)} \in \mathbb{F}^{*\mathrm{n}}\} \\ &= \mathrm{SL(n,\mathbb{F})Z}.\end{aligned}$$

Write $\tilde{H}$ for the inverse image in $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ of a subgroup H of $\mathrm{GL(n,\mathbb{F})}$. The following Lemma follows immediately from (7) and properties of the norm residue symbol.

Lemma 2.1 *Let* $Z_+ = \{xI \mid x \in \mathbb{F}^{*n}\}$.

(1) The center of $\mathrm{GL(n,\mathbb{F})}$ *is* $\widetilde{Z_+}$,

(2) The center of $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$ *is* $\tilde{Z}$,

(3) $Cent_{\widetilde{\mathrm{GL}}(n,\mathbb{F})}(\widetilde{\mathrm{GL}}(n,\mathbb{F})_+) = \tilde{Z}$ *and* $Cent_{\widetilde{\mathrm{GL}}(n,\mathbb{F})}(\tilde{Z}) = \widetilde{\mathrm{GL}}(n,\mathbb{F})_+$.

Thus $\tilde{Z}$ and $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$ form a dual pair in the sense of Howe.

Therefore $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+ = \widetilde{\mathrm{SL}}(n,\mathbb{F})\tilde{Z}$, and $\tilde{Z}$ is the center of $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$. Consequently an irreducible representation of $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$ restricts to an irreducible representation of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$, and every irreducible representation of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ is obtained this way. For many purposes we may replace $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ by $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$. This is analogous to the corresponding situation for the linear groups. Note $\widetilde{\mathrm{GL}}(n,\mathbb{F})/\widetilde{\mathrm{GL}}(n,\mathbb{F})_+ \approx \mathrm{GL(n,\mathbb{F})}/\mathrm{GL(n,\mathbb{F})}_+ \approx \mathbb{F}^*/\mathbb{F}^{*\mathrm{n}}$.

The cocycle restricted to Z_+ is trivial so $\widetilde{Z_+} \approx \mathbb{F}^{*n} \times \mu_n$. The cocycle restricted to Z is given by $c(xI, yI) = \prod_{i<j}(x,y)_n = (x,y)_n^{n(n-1)/2}$. This is equal to 1 if n is odd, or ± 1 if n is even.

For later use we note there exists a (genuine) character μ of $\tilde{Z}$ satisfying

$$\mu|_{\widetilde{Z_+}} = \iota. \tag{8}$$

In fact we may take

$$\mu(xI,\zeta) = \begin{cases} \zeta & n \text{ odd} \\ \gamma(x,\psi)\zeta & n \text{ even} \end{cases} \tag{9}$$

Here ψ is a non-trivial additive character of $\mathbb{F}$ and $\gamma(x,\psi) \in \{\pm 1, \pm i\}$ is the Weil index ([11], appendix). In particular $\mu^n = 1$ (n odd), and $\mu^{2n} = 1$ (n even). We only use this explicit formula for (25).

Given μ, the genuine characters of $\tilde{Z}$ are in bijection with $\widehat{\mathbb{F}^*}$; given $\nu \in \widehat{\mathbb{F}^*}$ let

$$\chi_\nu(z) = \mu(z)\nu(x) \quad (z \in \tilde{Z}, p(z) = xI) \tag{10}$$

i.e.

$$\chi_\nu(xI,\zeta) = \mu(xI,\zeta)\nu(x) = \mu(xI,1)\zeta\nu(x). \tag{11}$$

2.2 Cartan Subgroups

We define a Cartan subgroup of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ or $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ to be the inverse image of a Cartan subgroup of the corresponding linear group. These groups are in general non-abelian, and an important role is played by the center. We say an element of a covering group is semisimple (respectively regular) if its image in the linear group is semisimple (resp. regular).

Lemma 2.2 *Let T be a Cartan subgroup of $GL(n)$ with inverse image $\tilde{T}$ in* $\widetilde{\mathrm{GL}}(n,\mathbb{F})$.

(1) The center of $\tilde{T}$ *is* $p^{-1}(T^n)$,

(2) The center of $\tilde{T} \cap \widetilde{\mathrm{SL}}(n,\mathbb{F})$ *is* $p^{-1}(ZT^n \cap \mathrm{SL}(\mathrm{n},\mathbb{F}))$.

Proof. (1) is proved in ([4],§3), and (2) follows from this as well. We will sketch another proof of (2) in Section 3. ■

We say a regular semisimple element $g \in \tilde{T}$ is is *relevant* if it is contained in the center of $\tilde{T}$ [2]. It is a basic fact that if π is a genuine representation of $\tilde{G}$ then $\Theta_\pi(g) = 0$ if g is not relevant [5], ([2], Proposition 2.7).

3 Restriction from $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ to $\widetilde{\mathrm{SL}}(n,\mathbb{F})$

We compute the character of an irreducible representation of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ or $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$ in terms of a character of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ (Theorem 3.1). The main point is that Clifford theory for restriction of a genuine representation Π of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ to $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$ is very easy: each such representation restricts to a direct sum of $|\mathbb{F}^*/\mathbb{F}^{*n}|$ distinct irreducible representations which are permuted by the action of $\widetilde{\mathrm{GL}}(n,\mathbb{F})/\widetilde{\mathrm{GL}}(n,\mathbb{F})_+ \approx \mathbb{F}^*/\mathbb{F}^{*n}$. Furthermore the character of each summand may be computed in terms of the character of Π using Fourier inversion on $\tilde{Z}/\widetilde{Z_+} \approx \mathbb{F}^*/\mathbb{F}^{*n}$.

Let π be a genuine representation of $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$. Write $g : \pi \to \pi^g$ for the action (by conjugation on $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$) of $g \in \widetilde{\mathrm{GL}}(n,\mathbb{F})$ on representations of $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$. Assume π has a central character χ_π. We compute χ_{π^g}. Let $z \in \tilde{Z}$ with $p(z) = xI$. Then

$$\begin{aligned}
\chi_{\pi^g}(z) &= \chi_\pi(gzg^{-1}) \\
&= \chi_\pi(\{p(g), xI\}z) \\
&= \chi_\pi((x,\det(g))_n z) \quad \text{(by (7))} \\
&= \chi_\pi(z)(x,\det(g))_n \quad \text{(since } \pi \text{ is genuine).}
\end{aligned} \tag{12}$$

By non-degeneracy of the symbol, if $\det(g) \notin \mathbb{F}^{*n}$ there exists x such that $(x,\det(g))_n \neq 1$. Therefore if $g \notin \widetilde{\mathrm{GL}}(n,\mathbb{F})_+$, $\chi_{\pi^g} \neq \chi_\pi$, and *a fortiori* $\pi^g \not\approx \pi$. Note the assumption π is genuine is essential; the corresponding result is demonstrably false for representations of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ which factor to $\mathrm{GL}(\mathrm{n},\mathbb{F})$.

Let $\Pi = Ind_{\widetilde{\mathrm{GL}}(n,\mathbb{F})_+}^{\widetilde{\mathrm{GL}}(n,\mathbb{F})}(\pi)$. By (12) and Clifford theory $\widetilde{\mathrm{GL}}(n,\mathbb{F})/\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$ acts simply transitively on the set of constituents of Π restricted to $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$. For each $x \in \mathbb{F}^*/\mathbb{F}^{*n}$ choose $g_x \in \widetilde{\mathrm{GL}}(n,\mathbb{F})$ with $\det(g) \equiv x \mod (\mathbb{F}^{*n})$. Let $\pi^x = \pi^{g_x}$; the isomorphism class of π^{g_x} is independent of the choice of g_x. Thus:

$$\Pi|_{\widetilde{\mathrm{GL}}(n,\mathbb{F})_+} = \sum_{x \in \mathbb{F}^*/\mathbb{F}^{*n}} \pi^x. \tag{13}$$

If π' is a constituent of the restriction of Π to $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$ then $\chi_{\pi'}$ (a character of $\tilde{Z}$) restricted to $\widetilde{Z_+}$ is equal to χ_Π. The set of extensions of χ_Π to $\tilde{Z}$ is in bijection with $\widehat{\mathbb{F}^*/\mathbb{F}^{*n}}$. By (13) the constituents of this restriction are in bijection with $\widehat{\mathbb{F}^*/\mathbb{F}^{*n}}$. This proves:

Proposition 3.1 *Let Π be an irreducible genuine representation of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$. Let S be the set of extensions of χ_Π to $\widetilde{Z_+}$, this set is in bijection with $\widehat{\mathbb{F}^*/\mathbb{F}^{*n}}$. For $\lambda\in S$ let Π_λ be the λ eigenspace of Π.*

For all λ, Π_λ is an irreducible representation of $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$ and

$$\Pi|_{\widetilde{\mathrm{GL}}(n,\mathbb{F})_+}=\sum_{\lambda\in S}\Pi_\lambda.$$

Fix an irreducible constituent π of this restriction. Then

$$\Pi|_{\widetilde{\mathrm{GL}}(n,\mathbb{F})_+}=\sum_{x\in\mathbb{F}^*/\mathbb{F}^{*n}}\pi^x$$

and the central character of π^x is $\chi_\pi(\cdot,x)_n$.

Remark: A similar result holds for $\widetilde{\mathrm{SL}}(n,\mathbb{F})$: $\Pi|_{\widetilde{\mathrm{SL}}(n,\mathbb{F})}=\sum_x\pi^x$ as above. However the π^x are not necessarily distinct: in some cases $\pi\approx\pi^x$ (this implies $(x,\zeta)_n=1$ for all $\zeta\in\mu_n$).

We strengthen this result using Fourier inversion on $\mathbb{F}^*/\mathbb{F}^{*n}$ to write Θ_π in terms of Θ_Π.

For $z\in\tilde{Z}, z'\in\widetilde{Z_+}$, $\chi_\pi(zz')^{-1}\Theta_\Pi(zz'g)=\chi_\pi(z)\Theta_\Pi(zg)$. Thus $\chi_\pi(z)^{-1}\Theta_\Pi(zg)$ is well defined for $z\in\tilde{Z}/\widetilde{Z_+}$. We compute

$$\begin{aligned}\sum_{z\in\tilde{Z}/\widetilde{Z_+}}\chi_\pi(z)^{-1}\Theta_\Pi(zg)&=\sum_{z\in\tilde{Z}/\widetilde{Z_+}}\chi_\pi(z)^{-1}\sum_{x\in\mathbb{F}^*/\mathbb{F}^{*n}}\Theta_{\pi^x}(zg)\quad\text{(by (13))}\\&=\sum_{z\in\tilde{Z}/\widetilde{Z_+}}\sum_{x\in\mathbb{F}^*/\mathbb{F}^{*n}}\chi_\pi(z)^{-1}\chi_{\pi^x}(z)\Theta_{\pi^x}(g).\end{aligned}$$

Now $\chi_\pi(z)^{-1}\chi_{\pi^x}(z)$ factors to $\tilde{Z}/\widetilde{Z_+}\approx\mathbb{F}^*/\mathbb{F}^{*n}$, and by orthogonality of characters the right hand side equals $|\mathbb{F}^*/\mathbb{F}^{*n}|\Theta_\pi(g)$. Explicitly, by (12)

$$\chi_\pi(z)^{-1}\chi_{\pi^x}(z)=\chi_\pi(z)^{-1}\chi_\pi(z)(y,x)_n=(y,x)_n$$

where $p(z)=yI$. As z runs over $\tilde{Z}/\widetilde{Z_+}$, y runs over representatives of $\mathbb{F}^*/\mathbb{F}^{*n}$, and this gives

$$\sum_{y\in\mathbb{F}^*/\mathbb{F}^{*n}}\sum_{x\in\mathbb{F}^*/\mathbb{F}^{*n}}(y,x)_n\Theta_{\pi^x}=|\mathbb{F}^*/\mathbb{F}^{*n}|\Theta_\pi(g)$$

since $(\ ,\ )_n$ is a perfect pairing. This proves:

Theorem 3.1 *Let π be a genuine representation of $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$ with a central character. Let $\Pi = Ind_{\widetilde{\mathrm{GL}}(n,\mathbb{F})_+}^{\widetilde{\mathrm{GL}}(n,\mathbb{F})}(\pi)$. This is a genuine representation of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ and is irreducible if π is irreducible. Assume Θ_π exists. Then for $g \in \widetilde{\mathrm{GL}}(n,\mathbb{F})_+$ (a regular semisimple element)*

$$\begin{aligned}\Theta_\pi(g) &= \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|}\sum_{z\in\tilde{Z}/\widetilde{Z_+}} \chi_\pi(z)^{-1}\Theta_\Pi(zg) \\ &= \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|}\sum_{x\in\mathbb{F}^*/\mathbb{F}^{*n}} \chi_\pi(z_x)^{-1}\Theta_\Pi(z_x g).\end{aligned} \tag{14}$$

In the first sum z runs over any set of coset representatives of $\tilde{Z}/\widetilde{Z_+}$. In the second the sum runs over any coset representatives of $\mathbb{F}^*/\mathbb{F}^{*n}$, and for each x $z_x \in \tilde{Z}$ is any element satisfying $p(z_x) = xI$. Each term is independent of the choices.

Essentially the same result holds for genuine representations of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$. Let π be a genuine representation of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$, and extend π to a genuine representation π' of $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$ (cf. Section 2). Then (3.1) holds for $g \in \widetilde{\mathrm{SL}}(n,\mathbb{F})$ and π replaced by π'. Each summand is independent of the choice of π'.

We can now sketch a proof of Lemma 2.2 (2).

Sketch of Proof. By the Theorem if π is a genuine representation of $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$ then $\Theta_\pi(g) = 0$ unless $p(zg) \in T^n$ for some z by Lemma 2.2 (1), i.e. $p(g) \in ZT^n$. Conversely if g satisfies this condition then there exists a genuine representation for which $\Theta_\pi(g) \neq 0$, by the property that characters separate points. This proves the result for regular semisimple elements. For general elements apply a continuity argument. Alternatively apply the argument of Theorem 3.1 directly to an irreducible finite–dimensional genuine representation π of $\tilde{T}$, in which case Θ_π is defined for all $g \in \tilde{T}$. ■

4 Lifting from $\mathrm{GL}(\mathrm{n},\mathbb{F})$ to $\widetilde{\mathrm{GL}}(n,\mathbb{F})$

In this section we summarize results on lifting of characters from $\mathrm{GL}(\mathrm{n},\mathbb{F})$ to $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ [7],[4], [8].

We first define transfer factors in this setting. Recall the Weyl denominator for $\mathrm{GL}(\mathrm{n},\mathbb{F})$ is given by $\Delta(g) = \prod_{i<j}|x_i - x_j|_{\overline{\mathbb{F}}}/|x_i x_j|_{\overline{\mathbb{F}}}^{\frac{1}{2}}$ if g is a regular semisimple element with (distinct) eigenvalues x_i (in an algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$).

Definition 4.1 *Suppose $h \in \mathrm{GL}(\mathrm{n},\mathbb{F})$, $g \in \widetilde{\mathrm{GL}}(n,\mathbb{F})$ are regular semisimple elements satisfying $h^n = p(g)$.*

Let

$$\tau(h,g) = gs(h)^{-n}u(h). \tag{15}$$

Here $u(h)=\pm 1\in\mu_n$ is defined by ([8], §2) (we take $u(h)=1$ if n is odd), and $s:\mathrm{GL}(\mathrm{n},\mathbb{F})\to\widetilde{\mathrm{GL}}(\mathrm{n},\mathbb{F})$ is any section. Note that $p\tau(h,g)=1$ and we consider $\tau(h,g)$ to be an element of μ_n.

Let

$$\Delta(h,g)=|n^n|_{\mathbb{F}}^{-\frac{1}{2}}\tau(h,g)\frac{\Delta(h)}{\Delta(g)}. \tag{16}$$

Let π be a representation of $GL(n)$ with central character χ_π satisfying $\chi_\pi(\zeta I)=1$ for all $\zeta\in\mu_n$. Suppose g is a regular semisimple element of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$, so $p(g)$ is contained in a Cartan subgroup T of $\mathrm{GL}(\mathrm{n},\mathbb{F})$. Let

$$t_*(\Theta_\pi)(g)=\sum_{\substack{h\in T\\ h^n=p(g)}}\Delta(h,g)\Theta_\pi(h). \tag{17}$$

This is a conjugation invariant function on the regular semisimple elements of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$.

This is a special case of ([4], 26.1), and we have written it in a different form. We use the notation of [4]. To see that (17) agrees with [4] first note that in our case the center $\widetilde{Z_+}$ of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ is equal to $s(Z^n)\mu_n$, and it follows that the supplementary choice of $\tilde{\omega}$ of [4] is unnecessary. The summand in [4] is over

$$\{h\in T\,|\,h^{*-1}g\in\widetilde{Z_+}\}/Z.$$

Given $\overline{h}$ in this set, choose a representative $h\in T$, and write $h^*z=g$ for $z\in\widetilde{Z_+}$.

Equivalently the sum is over

$$A=\{h\in T\,|\,(hz)^n=p(g)\text{ for some }z\in Z\}/Z.$$

On the other hand we have written the sum over

$$B=\{h\in T\,|\,h^n=p(g)\}.$$

There is an $n-1$ surjective map from B to A given by $h\to\overline{h}$. Finally if $h^n=p(g)$ then $h^{*-1}g=s(h)^{-n}u(h)g=\tau(g,h)$, and since this is an element of μ_n, $\tilde{\omega}(\tau(g,h))=\tau(g,h)$. We have incorporated this term, together with the constant b of ([4], §24) (divided by n because of the difference between A and B) into the transfer factor $\Delta(h,g)$.

Flicker, Kazhdan and Patterson conjecture that for π an irreducible unitary representation $t_*(\pi)$ is either 0 or $\pm$ the character of a genuine irreducible unitary representation of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$. We refine this conjecture into two hypotheses for later use.

Hypothesis I Let π be an irreducible representation of $\mathrm{GL}(\mathrm{n},\mathbb{F})$ such that $\chi_\pi(\zeta I)=1$ for all $\zeta\in\mu_n$. We say *Hypothesis I holds for* π if $t_*(\pi)$ is 0 or $\pm$ the character of an irreducible representation of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$. If this holds we define the virtual representation $t_*(\pi)$ by $t_*(\Theta_\pi)=\Theta_{t_*(\pi)}$. Furthermore if $t_*(\pi)\neq 0$ define $\epsilon(\pi)=\pm 1$ so that $\epsilon(\pi)t_*(\pi)$ is a representation.

Hypothesis II Every genuine irreducible unitary representation of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ is isomorphic to $\epsilon(\pi)t_*(\pi)$ for some irreducible unitary representation π satisfying Hypothesis I.

Hypotheses I and II are true for $n = 2$ [5]. Hypothesis I is true if π is a discrete series representation, and t_* is a bijection between a subset of the discrete series of $\mathrm{GL}(\mathrm{n},\mathbb{F})$ and the genuine discrete series of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ ([4], §26). Hence Hypothesis II holds in the context of discrete series representations. For π a discrete series representation $\epsilon(\pi) = 1$. If $t_*(\pi)$ is supercuspidal then π is supercuspidal, but not conversely.

Hypothesis I holds if π is tempered [4], with the caveat that this statement depends on ([4], Proposition 26.2), and in some cases there is a technical obstruction to this result holding as stated. In any event if π is tempered and satisfies Hypothesis I then $t_*(\pi)$ is tempered and $\epsilon(\pi) = 1$. Subject to the preceding caveat Hypothesis II holds for tempered representations and t_* is a bijection between the genuine irreducible tempered representations of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ and a subset of the irreducible tempered representations of $\mathrm{GL}(\mathrm{n},\mathbb{F})$ ([4], Theorem 27.3).

Assuming Hypothesis II holds for tempered representations, then the Grothendieck group of genuine representations of $\mathrm{GL}(\mathrm{n},\mathbb{F})$ is spanned by the $t_*(\pi)$ for π satisfying Hypothesis I. Furthermore the non-zero $t_*(\pi)$ as π runs over all standard modules for $\mathrm{GL}(\mathrm{n},\mathbb{F})$ is a basis of the Grothendieck group of genuine representations of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$.

We are particularly interested in non-tempered π satisfying Hypothesis I. For example Hypothesis I holds for any character α satisfying $\alpha(\zeta) = 1$ for all $\zeta \in \mu_n$. In this case $t_*(\alpha)$ is a singular unitary quotient of a minimal principal series with a one-dimensional space of Whittaker functionals [5],([7], Corollary I.3.6). Hypothesis I should hold for all characters α. For example for $n = 2$, $-t_*(\alpha)$ is the supercuspidal constituent of the oscillator representation if $\alpha(-1) = -1$ [5].

The central characters of π and $t_*(\pi)$ are related by

$$\chi_{t_*(\pi)}(x^n I, 1) = \chi_\pi(x). \tag{18}$$

We also have for any $\alpha \in \widehat{\mathbb{F}^*}$

$$t_*(\pi\alpha^n) = t_*(\pi)\alpha. \tag{19}$$

These follow immediately from (17).

5 Parameters for $\widetilde{\mathrm{SL}}(n,\mathbb{F})$

We put lifting from $\mathrm{GL}(\mathrm{n},\mathbb{F})$ to $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ together with restriction from $\widetilde{\mathrm{GL}}(n,\mathbb{F})$ to $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ to obtain a character formula relating $\mathrm{GL}(\mathrm{n},\mathbb{F})$ and $\widetilde{\mathrm{SL}}(n,\mathbb{F})$.

We first consider $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$. Suppose for the moment that Hypothesis II is true. We parametrize the genuine irreducible unitary representations of $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$ as follows.

Fix a genuine irreducible unitary representation Π of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$. By Proposition 3.1 a constituent of the restriction of Π to $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$ is determined by a character λ of $\tilde{Z}$ satisfying $\lambda|_{\widetilde{Z_+}} = \chi_\Pi$, i.e.

$$\lambda(x^n,1) = \chi_\Pi(x^n,1) \quad (x \in \mathbb{F}^*).$$

By Hypothesis II there exists an irreducible representation π of $\mathrm{GL(n,\mathbb{F})}$, with $\chi_\pi(\mu_n) = 1$, such that $t_*(\pi) = \pm\Pi$. By (18)

$$\chi_\Pi(x^n,1) = \chi_\pi(x)$$

so we have

$$\lambda(x^n,1) = \chi_\pi(x). \tag{20}$$

Fix a genuine character μ of $\tilde{Z}$ satisfying (8). Then the set of characters λ of $\tilde{Z}$ satisfying (20) is (cf. 10):

$$\{\chi_\nu \,|\, \nu^n = \chi_\pi\}.$$

Note that by Lemma 1.1 and (4) the set of such ν is parametrized by $\widehat{\mathbb{F}^*/\mathbb{F}^{*n}}$, and by Proposition 3.1 this parametrizes the constituents of $\Pi|_{\widetilde{\mathrm{GL}}(n,\mathbb{F})_+}$.

This motivates the following definition.

Definition 5.1 *Let X be the set of pairs (π,ν) where:*

(1) π is an irreducible representation of $\mathrm{GL(n,\mathbb{F})}$, with central character χ_π satisfying $\chi_\pi(\zeta I) = 1$ for all $\zeta \in \mu_n$,

(2) ν is a character of $\mathbb{F}^*$ satisfying $\nu^n = \chi_\pi$.

Let $(\pi,\nu) \in X$, and assume Hypothesis I holds for π.

(3) Let $L_0(\pi,\nu)$ be the constituent of $t_*(\pi)$ restricted to $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$ with central character χ_ν (cf. (10)).

(4) Let $L(\pi,\nu)$ be the restriction of $L_0(\pi,\nu)$ to $\widetilde{\mathrm{SL}}(n,\mathbb{F})$.

Remark: L and L_0 depend on the choice of μ satisfying (8).

By definition $\epsilon(\pi)L(\pi,\nu)$ is the character of a representation. Assuming Hypothesis II every genuine irreducible unitary representation of $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$ is isomorphic to $\epsilon(\pi)L(\pi,\nu)$ for some $(\pi,\nu) \in X$.

If $(\pi,\nu) \in X$ then by (5)

$$\chi_{\pi\nu^{-1}} = \chi_\pi\nu^{-n} = 1, \tag{21}$$

i.e. $\pi\nu^{-1}$ factors to a representation of $\mathrm{PGL(n,\mathbb{F})}$. If π is a representation of $\mathrm{GL(n,\mathbb{F})}$ with trivial central character let $\overline{\pi}$ be the corresponding representation of $\mathrm{PGL(n,\mathbb{F})}$.

Definition 5.2 *For $(\pi,\nu) \in X$, let $M(\pi,\nu)$ be the irreducible representation $\overline{\pi\nu^{-1}}$ of* $\mathrm{PGL(n,\mathbb{F})}$.

Thus X is the graph of a correspondence between irreducible genuine representations of $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$ or $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ and $\mathrm{PGL(n,\mathbb{F})}$. That is for π an irreducible representation of $\widetilde{\mathrm{GL}}(n,\mathbb{F})_+$ or $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ and π' an irreducible representation of $\mathrm{PGL(n,\mathbb{F})}$ we say π corresponds to π' if there exists $x=(\pi,\nu)\in X$, with π satisfying Hypothesis I, such that $L_0(x)=\pi$ or $L(x)=\pi$, and $M(x)=\pi'$. Assuming Hypothesis II every genuine irreducible unitary representation of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ is in the image of the correspondence.

Lemma 5.1

(1) If $(\pi,\nu)\in X$ then $(\pi\alpha,\nu\alpha)\in X$ for all $\alpha\in\widehat{\mathbb{F}^*}$. Thus $x=(\pi,\nu)\to\alpha x=(\pi\alpha,\nu\alpha)$ defines an action of $\widehat{\mathbb{F}^*}$ on X.

For all $\alpha\in\widehat{\mathbb{F}^*}$ and $x\in X$,

(2) $M(\alpha x)=x$,

(3) $L_0(\alpha^n x)=L_0(x)\alpha$,

(4) $L(\alpha^n x)=L(x)$.

Proof. (1) and (2) are immediate. By (19) $t_*(\alpha^n\pi)=t_*(\pi)\alpha$, and by (5) $L_0(\alpha^n x)$ and $L_0(x)\alpha$ have the same central character; (3) follows and (4) is an immediate consequence of (3). ■

Remark If $\beta\in\widehat{\mathbb{F}^*}$ is non–trivial on μ_n, then $\beta\notin\widehat{\mathbb{F}^*}^n$, and there is no elementary relationship between $L_0(\beta x)$ and $L_0(x)$.

Remark The action of $\widehat{\mathbb{F}^*}/\widehat{\mathbb{F}^{*n}}\approx\widehat{\mu_n}$ on genuine representations of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ given by $\alpha: L(x)\to L(\alpha x)$ generalizes the "Waldspurger involution" for $\widetilde{SL}(2,\mathbb{F})$ [18].

We compute the set of representations of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ corresponding to a given irreducible representation of $\mathrm{PGL(n,\mathbb{F})}$.

Fix $x=(\pi,\nu)\in X$. If $M(x')=M(x)$ then $x'=\alpha x$ for some α. By Lemma 5.1 (4) if $\alpha\in(\widehat{\mathbb{F}^*})^n\approx\widehat{\mathbb{F}^{*n}}$ then $L(\alpha x)=L(x)$. Therefore the irreducible representations of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ corresponding to $M(x)$ are the $L(\alpha x)$ for $\alpha\in\widehat{\mathbb{F}^*}/\widehat{\mathbb{F}^{*n}}$ (not to be confused with $\widehat{\mathbb{F}^*/\mathbb{F}^{*n}}$), which by (3) is isomorphic to $\widehat{\mu_n}$.

Definition 5.3 *Let π be an irreducible representation of* $\mathrm{GL(n,\mathbb{F})}$ *with central character trivial on μ_n. Suppose Hypothesis I holds for $\pi\alpha$ for all $\alpha\in\widehat{\mathbb{F}^*}$.*

(1) For $(\pi,\nu)\in X$

$$L_{st}(\pi,\nu)=\sum_\alpha L(\pi\alpha,\nu\alpha)$$

where the sum runs over a set of representatives of $\widehat{\mathbb{F}^*}/\widehat{\mathbb{F}^{*n}}\approx\widehat{\mu_n}$.

(2) Let π be an irreducible representation of $\mathrm{PGL(n,\mathbb{F})}$*, and let π' denote π pulled back to* $\mathrm{GL(n,\mathbb{F})}$*. Assume $\pi'\alpha$ satisfies Hypothesis I for all $\alpha\in\widehat{\mathbb{F}^*}$. Define $L_{st}(\pi)=L_{st}(\pi',1)$.*

Remark $L_{st}(\pi,\nu) = L_{st}(\pi\alpha,\nu\alpha)$ for all α, and in particular

$$L_{st}(\pi,\nu) = L_{st}(\pi\nu^{-1},1) = L_{st}(\pi\nu^{-1}).$$

As discussed in the Introduction $L_{st}(\pi)$, respectively $\Pi(\pi,\nu) = \{L(\pi\alpha,\nu\alpha)\,|\,\alpha \in \widehat{\mu_n}\}$, is our candidate for a "stable" virtual character, respectively L-packet, of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$. Note that the non-zero $L(\pi\alpha,\nu\alpha)$ in $\Pi(\pi,\nu)$ are distinct, and in fact have distinct central characters on $\widetilde{\mathrm{SL}}(n,\mathbb{F})$. One could define a stable virtual character of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ to be in the span of the $L_{st}(\pi)$. It is not clear how to characterize the stable virtual characters intrinsically.

Not all $L(\pi\alpha,\nu\alpha)$ are necessarily non-zero. For example suppose π is the principal series defined by the character $\lambda(diag(h_1,\dots,h_n))) = \prod \lambda_i(h_i)$. This has central character trivial on μ_n if $\prod_i \lambda_i(\zeta) = 1$ for all $\zeta \in \mu_n$. On the other hand $t_*(\pi) = 0$ unless $\lambda_i(\zeta) = 1$ for all $i,\zeta \in \mu_n$. Assume this holds. Then $L(\pi,\alpha)$ is a principal series of $\widetilde{\mathrm{GL}}(n,\mathbb{F})$, and $L(\pi\alpha,\nu\alpha) = 0$ for all $\alpha \neq 1$, so $L(\pi,\nu) = L_{st}(\pi,\nu)$.

If π is a discrete series representation with $\chi_\pi(\mu_n) = 1$ π satisfies Hypothesis I, $t_*(\pi\alpha) \neq 0$ for all α, and $L(\pi\alpha,\nu\alpha) \neq 0$ for all α. Therefore in this case $|\Pi(\pi,\alpha)| = n$.

In the case $n = 2$ $L(1,1)$ is non-tempered, and is isomorphic to ω_e, the even half of the oscillator representation $\omega = \omega_e \oplus \omega_o$ of $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ (ω depends on an additive character ψ, which is determined by μ). If $\alpha(-1) = -1$ then $L(\alpha,\alpha) = -\omega_o$ is supercuspidal, and $L_{st}(1) = \omega_e - \omega_o$. See [5], [14] and [1].

6 Orbit Correspondence

For $g \in \mathrm{GL(n,\mathbb{F})}$ write $\overline{g}$ for the image of g in $\mathrm{PGL(n,\mathbb{F})}$.

Definition 6.1 *For $h \in \mathrm{GL(n,\mathbb{F})}$ let*

$$\phi(h) = det(h^{-1})h^n \in \mathrm{SL(n,\mathbb{F})}.$$

Then $\phi(zg) = \phi(g)$ for all $z \in Z$, so ϕ factors to a map from $\mathrm{PGL(n,\mathbb{F})}$ to $\mathrm{SL(n,\mathbb{F})}$.

Thus $\mathrm{GL(n,\mathbb{F})}$ is the graph of a correspondence between $\mathrm{PGL(n,\mathbb{F})}$ and $\mathrm{SL(n,\mathbb{F})}$ via the maps the maps $g \to \overline{g} \in \mathrm{PGL(n,\mathbb{F})}$ and $g \to \phi(g) \in \mathrm{SL(n,\mathbb{F})}$. The following Lemma is immediate.

Lemma 6.1

(1) For all $h \in \mathrm{PGL(n,\mathbb{F}), g \in GL(n,\mathbb{F})}$, $\phi(\overline{g}h\overline{g}^{-1}) = g\phi(h)g^{-1}$.

(2) If h is a regular semisimple element then $\phi(h)$ is relevant (cf. Lemma 2.2).

We also need the *weak* orbit correspondence. Suppose $h \in \mathrm{GL(n,\mathbb{F}), g \in SL(n,\mathbb{F})}$ satisfy

$$h^n = zg \quad (z \in Z).$$

Multiplying both sides by $\det(h^{-1})$ shows this is equivalent to

$$\phi(h) = \det(h^{-1})zg \quad (z \in Z).$$

Since $\phi(h)$ and g have determinant one this gives

$$\det(h^{-1})z = \phi(h)g^{-1} = \zeta I \quad (\zeta \in \mu_n). \tag{22}$$

Definition 6.2 *We say* $h \in \mathrm{PGL}(\mathrm{n}, \mathbb{F}), \mathrm{g} \in \mathrm{SL}(\mathrm{n}, \mathbb{F})$ *weakly correspond, written* $h \overset{weak}{\leftrightarrow} g$*, if for any (equivalently all)* $h' \in \mathrm{GL}(\mathrm{n}, \mathbb{F})$ *with* $\overline{h'} = h$*,*

$$h'^n = zg \quad (z \in Z).$$

Equivalently

$$g = \zeta\phi(h) \quad (\zeta \in \mu_n).$$

If $h \overset{weak}{\leftrightarrow} g$ *define* $\zeta(h, g) \in \mu_n$ *by*

$$g = \zeta(h, g)\phi(h). \tag{23}$$

We give an alternative description of the orbit correspondences in terms of roots and weights. This is not needed for what follows. Given a Cartan subgroup of $\mathrm{GL}(\mathrm{n}, \mathbb{F})$, we identify the root and weight lattices of the corresponding Cartan subgroups of $\mathrm{PGL}(\mathrm{n}, \mathbb{F})$ and $\mathrm{SL}(\mathrm{n}, \mathbb{F})$.

Lemma 6.2 *Let* h, g *be semisimple elements.*

(1) $h \overset{weak}{\leftrightarrow} g$ *if and only if* $\alpha(h^n) = \alpha(g)$ *for all roots* α*,*

(2) $\phi(h) = g$ *if and only if* $(n\lambda)(h) = \lambda(g)$ *for all weights* λ*.*

Proof. Part (1) is immediate. For (2) we need to show for $h \in \mathrm{GL}(\mathrm{n}, \mathbb{F}), \zeta \in \mu_{\mathrm{n}}$,

$$(n\lambda)(h) = \lambda(\zeta det(h^{-1})h^n) \quad \text{for all weights } \lambda \Leftrightarrow \zeta = 1.$$

The subtlety is that $\lambda(h)$ is not defined for arbitrary elements of $\mathrm{GL}(\mathrm{n}, \mathbb{F})$. If $h \in \mathrm{SL}(\mathrm{n}, \mathbb{F})\mathrm{Z} = \mathrm{GL}(\mathrm{n}, \mathbb{F})_+$ then $\lambda(h)$ is defined and this is immediate. It is enough to work over the algebraic closure $\overline{\mathbb{F}}$, in which case $\mathrm{GL}(\mathrm{n}, \overline{\mathbb{F}})_+ = \mathrm{GL}(\mathrm{n}, \overline{\mathbb{F}})$, proving the result. ∎

Remark If g is in the split torus then $|\{h \mid \phi(h) = g\}| = n^{n-2}$. In general the cardinality of the inverse image of a (regular semisimple) $g \in \mathrm{SL}(\mathrm{n}, \mathbb{F})$ depends on the Cartan subgroup containing g.

7 Transfer Factors

We continue with the notation of §6. Fix a character μ of $\widetilde{Z}$ satisfying (8). We define transfer factors $\Delta_\mu(h, g)$ (Definition 7.2). These satisfy one of the standard requirements of transfer fators: $|\Delta_\mu(h, g)| = |\Delta(h)/\Delta(g)|$ (25), up to a constant which is 1 for almost every residual characteristic.

Recall the definition of Δ (Definition 4.1).

Definition 7.1 *Suppose* $h \in \mathrm{GL(n,\mathbb{F})}, \mathrm{g} \in \widetilde{\mathrm{SL}}(\mathrm{n},\mathbb{F})$ *satisfy*

$$h^n = p(zg) \quad (z \in \tilde{Z}). \tag{24}$$

Let

$$\Delta_\mu(h,g) = \frac{n^2}{|\mathbb{F}^*/\mathbb{F}^{*n}|}\mu(z)^{-1}\Delta(h,zg).$$

This is independent of the choice of z satisfying (24).

Lemma 7.1 *For all* $\lambda \in \mathbb{F}^*$

$$\Delta_\mu(\lambda h, g) = \Delta_\mu(h,g).$$

Definition 7.2 *Suppose* $h \in \mathrm{PGL(n,\mathbb{F})}, \mathrm{g} \in \widetilde{\mathrm{SL}}(\mathrm{n},\mathbb{F})$ *satisfy*

$$h \overset{weak}{\leftrightarrow} p(g).$$

Choose $h' \in \mathrm{GL(n,\mathbb{F})}$ *satisfying* $\overline{h'} = h$*. Let*

$$\Delta_\mu(h,g) = \Delta_\mu(h',g).$$

By the Lemma this is independent of the choice of h'*.*

Given h, g as in Definition 7.2, choose $h' \in \mathrm{GL(n,\mathbb{F})}$ satisfying $\overline{h'} = h$, and choose $z \in \tilde{Z}$ with $h'^n = p(zg)$. Recall τ is given by Definition 4.1, and $|\mathbb{F}^*/\mathbb{F}^{*n}| = n^2/|n|_{\mathbb{F}}$ ([7], Lemma 0.3.2). This gives

$$\begin{aligned}
\Delta_\mu(h,g) &= \frac{n^2}{|\mathbb{F}^*/\mathbb{F}^{*n}|}\mu(z)^{-1}\Delta(h',zg) \\
&= |n|^{1-n/2}\mu(z)^{-1}\tau(h',zg)\frac{\Delta(h)}{\Delta(g)} \\
&= |n|^{1-n/2}\mu(z)^{-1}zgs(h')^{-n}u(h')\frac{\Delta(h)}{\Delta(g)}.
\end{aligned}$$

This is independent of the choices.

Proof of the Lemma. Choose $z \in \tilde{Z}$ satisfying $h^n = p(zg)$, and $w \in \tilde{Z}$ satisfying $p(w) = \lambda^n I$. Then $(\lambda h)^n = p(wzg)$. We need to show $\Delta_\mu(h,g) = \Delta_\mu(\lambda h, g)$, i.e.

$$\mu(z)^{-1}zgs(h)^{-n}u(h) = \mu(wz)^{-1}wzgs(\lambda h)^{-n}u(\lambda h).$$

After cancellations this is equivalent to

$$s(\lambda h)^n u(\lambda h) = \mu(w)^{-1}ws(h)^n u(h).$$

By ([4], §4) $s(\lambda h)^n u(\lambda h) = s(h)^n u(h) s_0(\lambda^n)$, where s_0 is the distinguished section, i.e. $s_0(g) = (g, 1)$. Inserting this we are reduced to showing $s_0(\lambda^n) = \mu(w)^{-1} w$, which is precisely the fact that $\mu|_{\widetilde{Z_+}} = \iota$. ■

Remark If $n = 2$ or the residual characteristic of $\mathbb{F}$ does not divide n then by (9)

$$\left(\frac{\Delta_\mu(h, g)}{\Delta(h)/\Delta(g)}\right)^N = 1 \tag{25}$$

with $N = n$ (n odd) or $N = 2n$ (n even).

Remark If μ' is another character satisfying (8), then $\mu'(z) = \mu(z)(y, x)_n$ for some y, where $p(z) = xI$, and

$$\frac{\Delta_{\mu'}}{\Delta_\mu}(h, g) = \frac{\mu}{\mu'}(h) = (\det(h), y)_n$$

($\det(h)$ is a well defined element of $\mathbb{F}^*/\mathbb{F}^{*n}$).

Although we will not need it we state the invariance property of Δ_μ. Suppose $h \overset{\text{weak}}{\leftrightarrow} g$. For $y \in \widetilde{\mathrm{GL}}(n, \mathbb{F})$ let $y_0 = \overline{p(y)} \in \mathrm{PGL(n, \mathbb{F})}$.

Lemma 7.2

$$\Delta_\mu(y_0 h y_0^{-1}, y g y^{-1}) = \Delta_\mu(h, g)(det(h)\zeta(h, g), det(y))_n$$

Proof. A straightforward computation which is left to the reader. ■

8 Stable Character Formula

We state the character formula relating the character of an irreducible representation π of $\mathrm{PGL(n, \mathbb{F})}$ to the character of the virtual genuine representation $L_{st}(\pi)$ of $\widetilde{\mathrm{SL}}(n, \mathbb{F})$.

Fix μ as in (8), define L_{st} as in Definition 5.3, ϕ as in Section 6 and Δ_μ as in Section 7.

Theorem 8.1 (Main Theorem) *Let π be an irreducible representation of* $\mathrm{PGL(n, \mathbb{F})}$*, for which $L_{st}(\pi)$ is defined (Definition 5.3). Then for g a regular semisimple element of* $\widetilde{\mathrm{SL}}(n, \mathbb{F})$*,*

$$\Theta_{L_{st}(\pi)}(g) = \sum_{\substack{h \in \mathrm{PGL(n, \mathbb{F})} \\ \phi(h) = p(g)}} \Delta_\mu(h, g)\Theta_\pi(h). \tag{26}$$

Recall the hypothesis on π is that $t_*(\pi\alpha)$ is defined for all $\alpha \in \widehat{\mathbb{F}^*}$ (we have pulled π back to GL(n, $\mathbb{F}$)).

Remark By Lemma 7.2 the right hand side of (26) is *a priori* $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ conjugation invariant. We do not need this, and it is a consequence of the Theorem. Note that $\Theta_{L_{st}(\pi)}$ is not necessarily invariant by conjugation by $\widetilde{\mathrm{GL}}(n,\mathbb{F})$, since Δ_μ is only $\widetilde{\mathrm{SL}}(n,\mathbb{F})$ conjugation invariant (Lemma 7.2).

Proof.

We first give a formula for $\Theta_{L(\pi,\nu)}(g)$ for arbitrary $(\pi,\nu) \in X$ (with π satisfying Hypothesis I).

By Theorem 3.1 $\Theta_{L(\pi,\nu)}(g) = \sum_{z\in\tilde{Z}/\widetilde{Z_+}} \chi_\nu(z)^{-1}\Theta_{t_*(\pi)}(zg)$ (sum over any set of coset representatives). Inserting (17) gives

$$\Theta_{L(\pi,\nu)}(g) = \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \sum_{z\in\tilde{Z}/\widetilde{Z_+}} \sum_{h^n=p(zg)} \chi_\nu(z)^{-1}\Delta(h,zg)\Theta_\pi(h). \qquad (27)$$

Write the summand as follows:

$$\begin{aligned}\chi_\nu(z)^{-1}\Delta(h,zg)\Theta_\pi(h) &= \mu(z)^{-1}\nu(z)^{-1}\Delta(h,zg)\Theta_\pi(h) \quad \text{(by (11))}\\ &= \frac{|\mathbb{F}^*/\mathbb{F}^{*n}|}{n^2}\nu(z)^{-1}\Delta_\mu(h,g)\Theta_\pi(h) \quad \text{(Definition 7.1)}\\ &= \frac{|\mathbb{F}^*/\mathbb{F}^{*n}|}{n^2}\nu(h)\nu(z)^{-1}\Delta_\mu(h,g)\Theta_{\pi\nu^{-1}}(h).\end{aligned}$$

By (21) and Lemma 7.1 $\Delta(h,g)$ and $\Theta_{\pi\nu^{-1}}(h)$ only depend on the image $\overline{h} \in$ PGL(n, $\mathbb{F}$) of h. By (22) and (23) $\nu(h)\nu(z)^{-1} = \nu(\phi(h)^{-1}g) = \nu(\zeta(\overline{h},g))$. This gives

$$\frac{|\mathbb{F}^*/\mathbb{F}^{*n}|}{n^2}\nu(\zeta(\overline{h},g))\Delta_\mu(\overline{h},g)\theta_{\pi\nu^{-1}}(\overline{h}).$$

Inserting this in (27) and changing the order of summation gives the following intermediate result.

Proposition 8.1

$$\begin{aligned}\Theta_{L(\pi,\nu)}(g) &= \frac{1}{n}\sum_{\substack{h\in\mathrm{PGL(n,\mathbb{F})}\\ h\overset{weak}{\leftrightarrow} g}} \nu(\zeta(h,g))\Delta_\mu(h,g)\theta_{\pi\nu^{-1}}(h)\\ &= \frac{1}{n}\sum_{\zeta\in\mu_n}\nu(\zeta)\sum_{\substack{h\in\mathrm{PGL(n,\mathbb{F})}\\ \phi(h)=\zeta g}} \Delta_\mu(h,g)\theta_{\pi\nu^{-1}}(h).\end{aligned}$$

Replace (π,ν) with $(\pi\alpha,\nu\alpha)$. On the right hand side only the term $\nu(\zeta(h,g))$ is affected. Summing over α gives:

$$\Theta_{L_{st}(\pi,\nu)}(g) = \frac{1}{n} \sum_{\zeta \in \mu_n} \sum_{\alpha \in \widehat{\mu_n}} \nu(\zeta)\alpha(\zeta) \sum_{\substack{h \in \mathrm{PGL(n,\mathbb{F})} \\ \phi(h)=\zeta g}} \Delta_\mu(h,g)\theta_{\pi\nu^{-1}}(h).$$

By orthogonality of characters for μ_n this equals

$$\sum_{\substack{h \in \mathrm{PGL(n,\mathbb{F})} \\ \phi(h)=g}} \Delta_\mu(h,g)\theta_{\pi\nu^{-1}}(h).$$

This completes the proof. ∎

9 Inversion

We continue in the setting of the preceding section. Suppose $\pi\alpha$ satisfies Hypothesis I for all α.

Definition 9.1 *For $\zeta \in \mu_n$ let*

$$L_\zeta(\pi,\nu) = \sum_{\alpha \in \widehat{\mu_n}} \alpha(\zeta) L(\pi\alpha, \nu\alpha).$$

This is a virtual character in which we allow rational coefficients, and $L_1(\pi,\nu) = L_{st}(\pi,\nu)$.

By Fourier inversion on μ_n:

$$L(\pi,\nu) = \frac{1}{n} \sum_{\zeta \in \mu_n} L_\zeta(\pi,\nu). \tag{28}$$

Recall the central character of $L(\pi\alpha,\nu\alpha)$ is $\chi_{\alpha\nu}$, i.e.

$$\chi_{L(\pi\alpha,\nu\alpha)}(z_\zeta) = \chi_\nu(z_\zeta)\alpha(\zeta)$$

where $p(z_\zeta) = \zeta I$. That is

$$\alpha(\zeta)\Theta_{L(\pi\alpha,\nu\alpha)}(g) = \chi_\nu^{-1}(z_\zeta)\Theta_{L(\pi\alpha,\nu\alpha)}(z_\zeta g).$$

Inserting this into the definition gives

Lemma 9.1 *For all $\zeta \in \mu_n$,*

$$\Theta_{L_\zeta(\pi,\nu)}(g) = \chi_\nu^{-1}(z_\zeta)\Theta_{L_{st}(\pi,\nu)}(z_\zeta g)$$

for any choice of z_ζ satisfying $p(z_\zeta) = \zeta I$.

Inserting this in (28) gives:

Theorem 9.1 (Inversion) *Suppose* $(\pi, \nu) \in X$, *and* $\pi\alpha$ *satisfies Hypothesis I for all* $\alpha \in \widehat{\mathbb{F}^*}$. *Then*

$$\begin{aligned}\Theta_{L(\pi,\nu)}(g) &= \frac{1}{n}\sum_{\zeta\in\mu_n} L_\zeta(\pi,\nu)(g) \\ &= \frac{1}{n}\sum_{\zeta\in\mu_n} \chi_\nu^{-1}(z_\zeta)\Theta_{L_{st}(\pi,\nu)}(z_\zeta g).\end{aligned} \tag{29}$$

By Theorem 8.1 each term on the right hand side of (29) may be expressed in terms the character $\Theta_{\pi\nu^{-1}}$ of PGL(n, $\mathbb{F}$). The resulting formula is Proposition 8.1.

We record the analogue of Theorem 8.1 for $L_\zeta(\pi, \nu)$.

$$\Theta_{L_\zeta(\pi,\nu)}(g) = \nu(\zeta)^{-1} \sum_{\substack{h \overset{\text{weak}}{\leftrightarrow} g \\ \zeta(h,g)=\zeta^{-1}}} \Delta_\mu(h,g)\Theta_{\pi\nu^{-1}}(h).$$

References

[1] J. Adams. Character of the oscillator representation. *Israel J. Math.*, 98:229–252, 1997.

[2] J. Adams. Characters of non-linear groups. *Proceedings of Conf. on Representation Theory and Harmonic Analysis, Okayama, Japan*, 26:1–18, 2000.

[3] Jeffrey Adams. Lifting of characters on orthogonal and metaplectic groups. *Duke Math. J.*, 92(1):129–178, 1998.

[4] Y. Flicker D. A. Kazhdan. Metaplectic correspondence. *Inst. Hautes Études Sci. Publ. Math.*, 64:53–110, 1986.

[5] Y. Flicker. Automorphic forms on covering groups of GL(2). *Invent. Math.*, 57:119–182, 1980.

[6] R.P. Langlands J.-P. Labesse. L-indistinguishability for SL(2). *Canad. J. Math.*, 31:726–785, 1979.

[7] D. A. Kazhdan and S. J. Patterson. Metaplectic forms. *Inst. Hautes Études Sci. Publ. Math.*, (59):35–142, 1984.

[8] D. A. Kazhdan and S. J. Patterson. Towards a generalized Shimura correspondence. *Adv. in Math.*, 60(2):161–234, 1986.

[9] H. Matsumoto. Sur les sous groupes arithmétiques des groupes semi–simples. *Ann. Sci. École Norm. Sup. (4)*, pages 1–62, 1969.

[10] C. Moore. Group extensions of p-adic and adelic linear groups. *Inst. Hautes Études Sci. Publ. Math.*, (35):157–222, 1968.

[11] R. Ranga Rao. On some explicit formulas in the theory of the weil representation. *Pacific J. Math.*, 157:335–371, 1993.

[12] A. Knapp S. Gelbart. L-indistinguishability and R groups for the special linear group. *Adv. in Math.*, 43:101–121, 1982.

[13] G. Savin. Local Shimura correspondence. *Math. Ann.*, 280:185–190, 1988.

[14] J. Schultz. *Lifting of Characters on* $\widetilde{SL}(2)$ *and* $SO(2,1)$ *over a non-archimedean local field.* PhD thesis, University of Maryland, 1998. Thesis.

[15] J.-P. Serre. *Local Fields.* Number 67 in Graduate Texts in Mathematics. Springer-Verlag, Berlin-Heidelberg-New York, 1979.

[16] R. Steinberg. Générateurs, relations et revêtements de groupes algébriques. *Colloq. Théorie des Groupes Algébriques, Bruxelles (1962)*, pages 113–127, 1962.

[17] M. Tadić. Notes on representations of non-archimedean $\mathrm{SL}(n)$. *Pacific J. Math.*, 152(2):375–396, 1992.

[18] J.-L. Waldspurger. Correspondances de Shimura et quaternions. *Forum Math.*, 3:219–307, 1991.