

# Examples of the Atlas of Lie Groups and Representations

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## 1 Introduction

The Atlas of Lie Groups and Representations is a project in representation theory of real reductive groups. The main goal of the `atlas` computer software, currently under development, is to compute the unitary dual of any real reductive Lie group  $G$ . As a step in this direction it currently computes the admissible representations of  $G$ .

The underlying mathematics of the software is described in *Algorithms for Representation Theory of Real Reductive Groups* [1]. This paper is a complement, and contains examples illustrating the algorithm and the software.

### 1.1 How to read this paper

A brief summary of the main constructions and results of *Algorithms for Representation Theory of Real Reductive Groups* is given in Section 2. We recommend the reader have the paper [1] available. Unexplained notation is as in [1]. We also recommend the reader download the software from the Atlas web site, [www.liegroups.org](http://www.liegroups.org); also see the [www.liegroups.org/papers](http://www.liegroups.org/papers). The online help commands are another a useful source of information.

The software is currently in an early stage of development (version 0.2.6). There will be substantial changes to the interface for version 1.0, and some additional features, which we hope to release in fall 2008. Most of the changes in the interface will be in the input methods, and we do not expect the format of output to change significantly.

We plan to publish a manual for the software at the time version 1.0 is released.

## 2 Summary of the Algorithm

Here is a summary of *Algorithms for Representation Theory of Real Reductive Groups* [1].

The starting point of the algorithm is a complex reductive algebraic group  $G$  together with an *inner class of real forms* of  $G$ . The latter is determined by an involution  $\gamma \in \text{Out}(G)$ , the group of outer automorphisms of  $G$ . We refer to  $(G, \gamma)$  as *basic data* [1, Section 4].

We define a semidirect product  $G^\Gamma = G \rtimes \Gamma$ , where  $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \delta\}$ . We let  $\delta$  act on  $G$  by a “distinguished” involution  $\tau \in \text{Aut}(G)$  mapping to  $\gamma \in \text{Out}(G)$ , via the map  $\text{Aut}(G) \rightarrow \text{Out}(G)$ . (A distinguished involution is one which fixes a splitting datum  $(B, H, \{X_\alpha\})$ ; it is the “most compact” involution in this inner class.) See [1, Section 5].

A *real form* of  $G$  in this inner class is a conjugacy class of involutions  $\theta \in \text{Aut}(G)$  mapping to  $\gamma \in \text{Out}(G)$ . If  $\theta$  is an involution of  $G$  let  $\sigma$  be an antiholomorphic involution of  $G$  commuting with  $\theta$ . Then  $G(\mathbb{R}) = G^\sigma$  is a real group in the usual sense, and  $G(\mathbb{R})^\theta$  is a maximal compact subgroup of  $G(\mathbb{R})$ . See [1, Section 5].

By a *strong involution* for  $(G, \gamma)$  we mean an element  $\xi \in G^\Gamma \setminus G$  satisfying  $\xi^2 \in Z$ , where  $Z$  is the center of  $G$ . A *strong real form* is an equivalence (conjugacy) class of strong involutions. If  $\xi$  is a strong involution then  $\theta_\xi = \text{int}(\xi)$  is an involution of  $G$  in the inner class of  $\gamma$ , and this gives a surjective map from strong real forms to real forms in this inner class. If  $G$  is adjoint this map is bijective, but not otherwise.

A representation of a strong involution is a pair  $(\xi, \pi)$  where  $\xi$  is a strong involution and  $\pi$  is a  $(\mathfrak{g}, G^\xi)$ -module. A representation of a strong real form is an equivalence class of such pairs. The natural map from representation of strong real forms to representations of forms is a bijection if  $G$  is adjoint. See [1, Section 6].

An alternative definition of strong involutions is obtained by choosing a set  $Z^r \subset Z$  of representatives of  $Z^\Gamma / (1+\theta)Z$ . Here  $(1+\theta)Z = \{z\theta(z) \mid z \in Z\}$ , where  $\theta = \text{int}(\xi)$  for any strong involution  $\xi$ . (See the *reduced parameter space* below.) This is a finite set. We define a strong involution to be an element  $\xi$  satisfying  $\xi^2 \in Z^r$ , with the same notion of equivalence as before. The map

from strong real forms to real forms is still surjective. The fibers of this map are smaller (in particular finite). See [1, Section 13].

We fix once and for all a Cartan subgroup  $H$  of  $G$ . The main combinatorial object is the *(one sided) parameter space*:

$$(2.1) \quad \mathcal{X} = \{g \in \text{Norm}_{G^\Gamma \backslash G}(H) \mid g^2 \in Z\}/H$$

(the quotient is by the conjugation action of  $H$ ). See [1, Section 9]. The *reduced parameter space* [1, Section 13]  $\mathcal{X}^r$ , which is finite, is defined the same way, with  $Z$  replaced by  $Z^r$ . We let  $\tilde{\mathcal{X}}$  be the numerator of this expression (before taking the quotient by  $H$ ) and  $\tilde{\mathcal{X}}^r$  similarly.

Fix a set of representatives of strong real forms in this inner class, in the sense of the reduced parameter space. This is a set of representatives  $\{\xi_i \mid i \in I\}$  of

$$(2.2) \quad \{\xi \in G^\Gamma \backslash G \mid \xi^2 \in Z^r\}/G.$$

This is a finite set. For each  $i \in I$  let  $x_i$  be its image in  $\mathcal{X}^r$ ,  $\theta_i = \text{int}(\xi_i)$  and  $K_i = G^{\theta_i}$ . See [1, 5.16].

Let  $W^\Gamma = \text{Norm}_{G^\Gamma}(H)/H$  and

$$\mathcal{I}_W = \{\tau \in W^\Gamma \backslash W \mid \tau^2 = 1\}.$$

This is (identified with) the space of *twisted involutions* in the Weyl group (see [1, (9.14)]). There is a natural map  $\tilde{p} : \tilde{\mathcal{X}}^r \rightarrow \mathcal{I}_W$ , which factors to a map  $p : \mathcal{X}^r \rightarrow \mathcal{I}_W$ . These maps are surjective [1, Lemma 9.12]. The fiber of  $p$  over an element  $\tau \in \mathcal{I}_W$  is denoted  $\mathcal{X}_\tau^r$ , respectively.

The group  $\text{Norm}_{G^\Gamma}(H)$  acts by conjugation on  $\tilde{\mathcal{X}}^r$ , and this factors to an action of  $W$  on  $\mathcal{X}^r$ . Because of its relationship to the cross action of [2, Definition 8.3.1] we call this the *cross action* of  $W$ , and denote it  $w \times x$ . This action is equivariant for  $p$  and the conjugation action of  $W$  on  $\mathcal{I}_W$ :  $p(w \times x) = wp(x)w^{-1}$ .

*Cayley transforms* play an important role in representation theory. These appear in the space  $\mathcal{X}^r$  as follows (see [1, Section 14] for details). Fix  $x \in \mathcal{X}^r$  and let  $\tau = p(x) \in \mathcal{I}_W$ . Then  $\tau$  acts on the roots, and a root  $\alpha$  is classified as *imaginary*, *real* or *complex* with respect to  $\tau$  if  $\tau(\alpha) = \alpha, -\alpha$  or neither, respectively. Furthermore let  $\xi$  be a preimage of  $x$  in  $\tilde{\mathcal{X}}^r$ . If  $\alpha$  is imaginary

we say it is *compact* or *noncompact* with respect to  $x$  if this holds with respect to  $\theta_\xi$  (independent of the choice of  $\xi$ ).

Now suppose  $\alpha$  is a non-compact imaginary root with respect to  $x$ . Then associated to  $\alpha$  and  $x$  is a new element of  $\mathcal{X}^r$  denoted  $c^\alpha(x)$ . It satisfies  $p(c^\alpha(x)) = s_\alpha p(x)$ , and  $\alpha$  is real with respect to  $s_\alpha \tau$ . Its inverse is the single or double-valued *real* Cayley transform  $c_\alpha(x)$ : if  $\alpha$  is real with respect to  $x$  then  $c_\alpha(x)$  is a set with one or two elements.

Given  $x \in \mathcal{X}$  let

$$\mathcal{X}[x] = \{x' \in \mathcal{X} \mid x' \text{ is } G\text{-conjugate to } x\}$$

(see [1, (9.7)]). Choose a preimage  $\xi$  of  $x$  in  $\tilde{\mathcal{X}}$  and let  $\theta = \text{int}(\xi)$ ,  $K = G^\theta$ . Then  $\mathcal{X}[x]$  is isomorphic to  $K \backslash G/B$ . In fact, one of the important properties of  $\mathcal{X}^r$  is that it captures information about the  $K$  orbits on  $G/B$  for all  $K$  in this inner class:

$$\mathcal{X}^r = \coprod_{i \in I} \mathcal{X}[x_i] \simeq \coprod_{i \in I} K_i \backslash G/B.$$

The “real” Weyl group  $W(K, H) = \text{Norm}_K(H)/H \cap K$  plays an important role. It is isomorphic to  $W(G(\mathbb{R}), H(\mathbb{R})) = \text{Norm}_{G(\mathbb{R})}(H(\mathbb{R}))/H(\mathbb{R})$  where  $G(\mathbb{R})$  is a real form of  $G$  corresponding to  $K$ , and  $H(\mathbb{R})$  is the corresponding real form of  $H$ . See [1, Section 12].

The final ingredient in the algorithm is the *two-sided parameter space*. Let  $G^\vee$  be the dual group of  $G$ . The involution  $\gamma \in \text{Out}(G)$  defines an involution  $\gamma^\vee \in \text{Out}(G^\vee)$  (cf. [1, Definition 2.9]), and  $(G^\vee, \gamma^\vee)$  is also basic data. Let  $\mathcal{X}^\vee$  be the one-sided parameter space defined by  $(G^\vee, \gamma^\vee)$ .

Suppose  $x \in \mathcal{X}$ , and suppose  $\xi \in \tilde{\mathcal{X}}$  is a pre-image of  $x$  in  $\tilde{\mathcal{X}}$ . Then  $\theta_\xi$  restricted to  $H$  is independent of the choice of  $\xi$ , and we denote it  $\theta_{x,H}$ . We also write  $\theta_{x,H} \in \text{End}(\mathfrak{h})$  for its differential. There is a natural pairing between  $\mathfrak{h}$  and  $\mathfrak{h}^\vee$  (a Cartan subalgebra on the dual side); the adjoint  $\theta_{x,H}^t$  is an element of  $\text{End}(\mathfrak{h}^\vee)$ .

We can now define the *two-sided parameter space*:

$$(2.3) \quad \mathcal{Z} = \{(x, y) \in \mathcal{X} \times \mathcal{X}^\vee \mid \theta_{x,H}^t = -\theta_y\}.$$

We define the *reduced* two-sided parameter space  $\mathcal{Z}^r$  by replacing  $\mathcal{X}$  and  $\mathcal{X}^\vee$  with the corresponding reduced spaces. This is a finite set.

With the obvious notation  $W \times W^\vee$  acts on  $\mathcal{Z}$  and  $\mathcal{Z}^r$ ; this is the cross action. By the condition relating  $x$  and  $y$  in (2.3), if  $\alpha$  is a real root with respect to  $x$  then  $\alpha^\vee$  is an imaginary root with respect to  $y$ , and vice-versa. Thus Cayley transforms are defined on  $\mathcal{Z}^r$ , via a real root on one side, and an imaginary noncompact root on the other.

The main result in [1] is that the space  $\mathcal{Z}$  parametrizes the irreducible representations of strong real forms of  $G$ , with regular integral infinitesimal character, up to “translation”. Because of our emphasis on the reduced parameter space, we give a slightly different version here. See [1, Theorem 10.4 and Section 13].

Apply the construction in the paragraph preceding (2.1) to choose a subset  $Z^{\vee r}$  of the center  $Z^\vee$  of  $G^\vee$ . The pairing of  $\mathfrak{h}$  and  $\mathfrak{h}^\vee$  gives an isomorphism  $\mathfrak{h}^* \simeq \mathfrak{h}^\vee$ , and let

$$(2.4) \quad L = \{\lambda \in \mathfrak{h}^* \mid \exp(2\pi i\lambda) \in Z^{\vee r}\}.$$

Choose a set  $\Lambda$  of representatives of  $L/X^*(H^\vee)$ , satisfying  $\langle \lambda, \alpha^\vee \rangle \neq 0$  for all  $\lambda \in \Lambda$  and all roots  $\alpha$ . Compare [1, Theorem 7.17].

For each  $i \in I$  (2.2) let  $G_i(\mathbb{R})$  be the real form of  $G$  defined by  $\xi_i$ , and let  $\Pi(G_i(\mathbb{R}), \Lambda)$  be the set of irreducible admissible representations of  $G_i(\mathbb{R})$  with infinitesimal character contained in  $\Lambda$ .

**Theorem 2.5** *There is a natural bijection*

$$(2.6) \quad \mathcal{Z} \xleftrightarrow{1-1} \prod_{i \in I} \Pi(G_i(\mathbb{R}), \Lambda).$$

Fix  $(x, y) \in \mathcal{Z}^r$ , and consider the representations associated to the pairs  $(x', y')$  in the subset  $\mathcal{X}[x] \times \mathcal{X}^\vee[y]$  of  $\mathcal{Z}^r$ . Choose  $i \in I$  so that  $x$  is  $G$ -conjugate to  $x_i$ . Then these are representations of  $G_i(\mathbb{R})$ , all with the same infinitesimal character. In fact this set of irreducible representations is a *block* in the sense of [2, Chapter 9], and every block is obtained this way.

The entire construction is obviously symmetric in  $G$  and  $G^\vee$ , so an element  $(x, y) \in \mathcal{Z}$ , which defines a representation  $\pi$  of a real form of  $G$ , also defines a representation  $\pi^\vee$  of a real form of  $G^\vee$ . The map  $\pi \rightarrow \pi^\vee$  is a version of *Vogan Duality*; see [3] and [1, (1.35) and Corollary 10.9]. In particular this is a duality of *blocks*: the set  $\mathcal{X}[x] \times \mathcal{X}^\vee[y]$  defines a block  $\mathcal{B}$  of a real form of  $G$ , a block  $\mathcal{B}^\vee$  of a real form of  $G^\vee$ , and gives a bijection  $\mathcal{B} \leftrightarrow \mathcal{B}^\vee$ .

### 3 Defining basic data

The first step in using the `atlas` software is to define basic data  $(G, \gamma)$ , i.e. a connected complex reductive group  $G$  and an involution  $\gamma \in \text{Out}(G)$ . See Section 2.

This proceeds in three steps:

- (1) Define a complex reductive Lie algebra  $\mathfrak{g}$ , and let  $G^*$  be the product of a complex torus and a simply connected, semisimple complex group with Lie algebra  $\mathfrak{g}$ ;
- (2) Choose a finite subgroup  $A$  of  $Z(G^*)$ , and set  $G = G^*/A$ ;
- (3) Choose an inner class of real forms of  $G$ .

All three steps are accomplished by the `type` command. We break this up into the three steps above.

#### 3.1 Defining $\mathfrak{g}$ and $G^*$

A complex reductive Lie algebra is given by a list of types  $A_n, B_n, \dots, E_8, T_n$ , where  $T_n$  is the abelian Lie algebra  $\mathbb{C}^n$ . In response to the `type` command, the software asks for the `Lie type:`. The user then enters such a list, with terms separated by a period. The order is irrelevant here, although it plays a role in steps (2) and (3). The entry  $T_2$  is the same as  $T_1.T_1$ .

This defines the complex reductive Lie algebra  $\mathfrak{g}$ , and group  $G^*$  (step (1) above).

**Example 3.1** Here are some simple examples. We start with  $SL(2, \mathbb{C})$ :

```
main: type
Lie type: A1
```

Here is  $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ :

```
main: type
Lie type: A1.A1
```

Here is  $SL(2, \mathbb{C}) \times \mathbb{C}^\times$ :

```
main: type
Lie type: A1.T1
```

and finally  $SL(2, \mathbb{C})^2 \times Spin(5, \mathbb{C}) \times Sp(4, \mathbb{C}) \times Sp(6, \mathbb{C}) \times \mathbb{C}^{\times 4}$ .

```
main: type
Lie type: A1.T1.B2.C2.T1.C3.T2.A1
```

### 3.2 Defining a complex group $G$

The second step is to pick a finite subgroup of the center of  $Z^*$  of  $G^*$ .

The center of each simple factor of  $G$  is a finite cyclic group, except in type  $D_{2n}$  in which case it is  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . The finite cyclic group of order  $n$  is denoted  $\mathbb{Z}/n$ , and is viewed as the group

$$\frac{1}{n}\mathbb{Z}/\mathbb{Z} = \left\{ \frac{0}{n}, \frac{1}{n}, \dots, \frac{n-1}{n} \right\}$$

The elements of finite order in  $\mathbb{C}^\times$  are isomorphic to  $\mathbb{Q}/\mathbb{Z}$ , and an element of this group is given by an element of  $\mathbb{Q}$ .

Once the user has given the Lie type, the software prompts the user for a finite subgroup of  $Z^*$ , generated by a set of elements of  $Z^*$ . Each such element is given by a list of fractions, one for each term in the Lie type (two for  $D_{2n}$ ), separated by commas. Each element is given on a single line; the empty line terminates this aspect of the input. For example simply typing `return` in response to this prompt takes  $A = 1$  and  $G = G^*$ . Typing `sc` has the same effect. Typing `ad` gives the adjoint group (actually a torus times the adjoint group of the derived group of  $G^*$ ).

**Example 3.2** For example here is the group  $SL(2, \mathbb{C})$ :

```
main: type
Lie type: A1
elements of finite order in the center of the simply connected group:
Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
sc
```

Entering a carriage return alone also gives  $SL(2, \mathbb{C})$ .

**Example 3.3** To define  $PSL(2, \mathbb{C})$  take  $A$  to be the center of  $SL(2, \mathbb{C})$ , which is generated by the element of order 2:

```

Lie type: A1
elements of finite order in the center of the simply connected group:
Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2

```

Entering ad instead of 1/2 has the same effect.

**Example 3.4** Here is  $SO(10, \mathbb{C})$ , which is the quotient of  $Spin(10, \mathbb{C})$  by the element of  $Z^*$  of order 2:

```

main: type
Lie type: D5
elements of finite order in the center of the simply connected group:
Z/4
enter kernel generators, one per line
(ad for adjoint, ? to abort):
2/4

```

Often a reductive group is a quotient of  $G^*$  by a “diagonal” subgroup.

**Example 3.5** Here is  $GL(2, \mathbb{C}) \simeq SL(2, \mathbb{C}) \times \mathbb{C}^*/\{(-I, -1)\}$ :

```

main: type
Lie type: A1.T1
elements of finite order in the center of the simply connected group:
Z/2.Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,1/2

```

The center of  $Spin(4n, \mathbb{C})$  is not cyclic, and therefore requires two terms.

**Example 3.6** Here is  $SO(8, \mathbb{C}) \simeq Spin(8, \mathbb{C})/A$  where  $A$  is the “diagonal” subgroup of  $Z^* = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ :

```

main: type
Lie type: D4
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,1/2

```

**Example 3.7** We can also take the quotient of  $Spin(8, \mathbb{C})$  by a non-diagonal subgroup  $A \simeq \mathbb{Z}/2\mathbb{Z}$ :

```

main: type
Lie type: D8
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,0/2

```

This is not isomorphic to  $SO(8, \mathbb{C})$ .

This defines the group  $G = G^*/A$  and completes step 2.

### 3.3 Defining an inner class of real forms

We next define an inner class of real forms. Recall (Section 2) this is determined by an involution in  $\text{Out}(G)$ .

The trivial element of  $\text{Out}(G)$  corresponds to the inner class of real forms containing a compact Cartan subgroup. This is the *compact* inner class, and is denoted  $\mathfrak{c}$ , and also  $\mathfrak{e}$  for *equal rank*. In particular if  $G$  has a torus factor its real points are isomorphic to  $S^1 \times \cdots \times S^1$ .

Another natural inner class is that of the split real form. This class is denoted  $\mathfrak{s}$ . In particular if  $G$  has a torus factor its real points are isomorphic to  $\mathbb{R}^\times \cdots \times \mathbb{R}^\times$ . In many cases the classes  $\mathfrak{c}$  and  $\mathfrak{s}$  are the same, for example if  $\text{Out}(G) = 1$ .

Now suppose  $G = G_1 \times G_1$ . The outer automorphism switching the two factors corresponds to the inner class of the real form  $G_1(\mathbb{C})$ , viewed as a real group. This class is denoted  $\mathfrak{C}$ .

Suppose  $G$  is simple. Then  $\text{Out}(G)$  is isomorphic to a subgroup of the automorphism group of the Dynkin diagram of  $G$  (with equality if  $G$  is simply connected or adjoint). In type  $A_n, D_n$  ( $n \geq 2$ ) and  $E_6$  the inner class of real forms which are not of equal rank is denoted  $\mathbf{u}$ . In types  $B_n, C_n, E_7, E_8, F_4$  and  $G_2$   $\text{Out}(G) = 1$ , and there is only one inner class of real forms.

Still assuming  $G$  is simple, in most cases each inner class is either compact or split, and the inner class  $\mathbf{u}$  is not needed. The only exception is type  $D_{2n}$ , in which case compact and split inner classes coincide, i.e.  $\mathbf{c} = \mathbf{s} \neq \mathbf{u}$ . See Example 3.14.

Now suppose  $G = G^*$ . An inner class of real forms of  $G$  is specified by choosing  $\mathbf{c}, \mathbf{e}, \mathbf{s}, \mathbf{u}$  for each simple or  $T^1$  factor, or  $C$  for each pair of identical (simple or  $T^1$ ) factors. In general an inner class of real forms of  $G$  is given by an *allowed* inner class of real forms of  $G^*$ : the involutions in this inner class must factor to  $G$ .

To summarize, to specify an inner class of real forms of  $G$ , give a list of choices for each simple or torus factor, or pair of identical entries in the case of  $C$ :

- $\mathbf{c}$ : compact
- $\mathbf{e}$ : equal rank (same as  $\mathbf{c}$ )
- $\mathbf{s}$ : split
- $\mathbf{u}$ : unequal rank
- $C$ : complex (for an identical pair of entries).

The order of the choices corresponds to the order in which the simple and torus factors of  $G^*$  were specified. A factor  $C$  allowed only if there are two subsequent identical factors, or a torus factor  $T^2$ . If  $G \neq G^*$  some choices may not be allowed.

Here are some examples. Getting slightly ahead of ourself, the `showrealforms` command lists the real forms in the given inner class.

**Example 3.8** The group  $SL(2, \mathbb{C})$  has two real forms  $SL(2, \mathbb{R})$  and  $SU(2)$ , both of which are in the same inner class. Thus  $\mathbf{c}=\mathbf{e}=\mathbf{s}=\mathbf{u}$  in this case.

```

empty: type
Lie type: A1
elements of finite order in the center of the simply connected group:
Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
sc
enter inner class(es): s
main: showrealforms
(weak) real forms are:
0: su(2)
1: sl(2,R)

```

**Example 3.9** The smallest simple group with two inner classes is type  $A_2$ .

```

main: type
Lie type: A2
elements of finite order in the center of the simply connected group:
Z/3
enter kernel generators, one per line
(ad for adjoint, ? to abort):
enter inner class(es): s
main: showrealforms
there is a unique real form: sl(3,R)

```

This is the split inner class, containing only  $SL(3, \mathbb{R})$ .

```

main: type
Lie type: A2
elements of finite order in the center of the simply connected group:
Z/3
enter kernel generators, one per line
(ad for adjoint, ? to abort):

enter inner class(es): c
main: showrealforms
(weak) real forms are:
0: su(3)
1: su(2,1)

```

This is the equal rank inner class, consisting of  $SU(3)$  and  $SU(2, 1)$ .

**Example 3.10** Here is  $SL(2, \mathbb{C})$  viewed as a real group:

```
empty: type
Lie type: A2.A2
elements of finite order in the center of the simply connected group:
Z/3.Z/3
enter kernel generators, one per line
(ad for adjoint, ? to abort):

enter inner class(es): C
main: showrealforms
(weak) real forms are:
0: sl(3,C)
```

**Example 3.11** If the group is a product specify the inner class as a list. For example here is the inner class of  $SL(3, \mathbb{R}) \times SU(3)$ :

```
Lie type: A2.A2
elements of finite order in the center of the simply connected group:
Z/3.Z/3
enter kernel generators, one per line
(ad for adjoint, ? to abort):

enter inner class(es): sc
main: showrealforms
(weak) real forms are:
0: sl(3,R).su(3)
1: sl(3,R).su(2,1)
```

**Example 3.12** Here is the inner class of  $GL(2, \mathbb{R})$ :

```
main: type
Lie type: A1.T1
elements of finite order in the center of the simply connected group:
Z/2.Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
```

1/2,1/2

```
enter inner class(es): ss
main: showrealforms
(weak) real forms are:
0: su(2).gl(1,R)
1: sl(2,R).gl(1,R)
enter your choice:
```

**Example 3.13** On the other hand here is the inner class of  $U(1, 1)$ :

```
main: type
Lie type: A1.T1
elements of finite order in the center of the simply connected group:
Z/2.Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,1/2
```

```
enter inner class(es): cc
main: showrealforms
(weak) real forms are:
0: su(2).u(1)
1: sl(2,R).u(1)
enter your choice:
```

**Example 3.14** The group  $SO(12, \mathbb{C})$  has the following real forms, in two inner classes. One inner class consists of the groups  $SO(12, 0)$ ,  $S(10, 2)$ ,  $SO(8, 4)$ ,  $SO(6, 6)$  and  $SO^*(12)$ . This is both the compact inner class (it contains  $SO(12, 0)$ ) and split (it contains  $SO(6, 6)$ ), and is therefore the class  $c=s$ . There is another inner class consisting of  $SO(11, 1)$ ,  $SO(9, 3)$  and  $SO(7, 5)$ . This is the inner class  $u$  of unequal rank. This is an example where  $u$  is needed.

Here is the compact and split inner class:

```
main: type
Lie type: D6 sc s
main: showrealform
(weak) real forms are:
```

```

0: so(12)
1: so(10,2)
2: so*(12)[1,0]
3: so*(12)[0,1]
4: so(8,4)
5: so(6,6)

```

Here is the unequal rank inner class:

```

main: type
Lie type: D6 sc u
main: showrealforms
(weak) real forms are:
0: so(11,1)
1: so(9,3)
2: so(7,5)

```

For an explanation of the two version of  $\mathfrak{so}^*(12)$  see Example 4.4.

**Example 3.15** Here is an example in which the inner class for  $G^*$  is not defined for  $G$ :

```

main: type
Lie type: A1.A1
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,0/2

enter inner class(es): C
sorry, that inner class is not compatible with the weight lattice

```

What this means is: the automorphism which switches the two factors in  $G^* = SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$  does not preserve  $A$ , and so does not factor to  $G = PSL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ .

**Example 3.16** A slightly more interesting example is the “non-diagonal” quotient of  $Spin(8, \mathbb{C})$  (see Example 3.7):

```

main: type
Lie type: D4
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,0/2

enter inner class(es): u
sorry, that inner class is not compatible with the weight lattice

```

**Remark 3.17** Here is a very useful shortcut. You can do steps (1-3) all in one step in many cases. Give the command `type`, and at the prompt enter the Lie type, the isogeny (`sc` or `ad`) and the inner class.

For example here is the inner class of  $SL(2, \mathbb{R})$ :

```

main: type
Lie type: A1 sc s
main:

```

or  $PSL(2, \mathbb{R})$ :

```

main: type
Lie type: A1 ad s
main:

```

## 4 Defining a real group

We now suppose the user has completed the `type` command, which defines  $G$  and an inner class of real forms. We now specify a particular real form of  $G$  in the inner class. In the terminology of `atlas` these are *weak* real forms; we drop this terminology for now, and turn to the notion of strong real forms in the next section.

Once an inner class has been chosen, a real form of  $G$  in this inner class is determined by a real form of  $\mathfrak{g}$  (equivalently  $G^*$ ) in the given inner class. There is a finite list of such real forms. The command `showrealforms` gives a list of the real forms of  $\mathfrak{g}$  in the given inner class; the command `realform` gives the same list, and the user can choose the real form from the list.

The real forms of the classical Lie algebras are given in the usual notation. Some examples are  $\mathfrak{sl}(4, \mathbb{R})$ ,  $\mathfrak{su}(3, 1)$ ,  $\mathfrak{su}(4)$ ,  $\mathfrak{sl}(4, \mathbb{H})$ ,  $\mathfrak{sp}(3, 2)$ ,  $\mathfrak{so}(3, 2)$ , and  $\mathfrak{so}^*(10)$ . For a torus the real form is specified by  $\mathfrak{gl}(1, \mathbb{R})$ ,  $\mathfrak{u}(1)$  or  $\mathfrak{gl}(1, \mathbb{C})$ , corresponding to  $\mathbb{R}^\times$ ,  $S^1$  and  $\mathbb{C}^\times$ , respectively.

For each exceptional group the real form is specified by specifying the type of the maximal compact subgroup, except that the split real form is denoted R. For example the real forms of  $E_7$  are  $\mathfrak{e7}$  (compact),  $\mathfrak{e7}(\mathfrak{e6.u}(1))$ ,  $\mathfrak{e7}(\mathfrak{so}(12).\mathfrak{su}(2))$  and  $\mathfrak{e7}(\mathbb{R})$  (split).

Here are some examples.

**Example 4.1** Here is the group  $SL(2, \mathbb{R})$ :

```
main: type
Lie type: A1 sc s
main: realform
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
```

**Example 4.2** Here is  $PSL(2, \mathbb{R}) \simeq PGL(2, \mathbb{R}) \simeq SO(2, 1)$ .

```
main: type
Lie type: A1 ad s
main: realform
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
```

Note that the weak real forms listed don't depend on the fact that the complex group is  $PSL(2, \mathbb{C})$  here instead of  $SL(2, \mathbb{C})$ ; the real form is determined by a real form of  $\mathfrak{g}$ .

**Example 4.3** Here are the equal rank real forms of  $SO(10, \mathbb{C})$ :

```
main: type
Lie type: D5
elements of finite order in the center of the simply connected group:
```

```
Z/4
enter kernel generators, one per line
(ad for adjoint, ? to abort):
2/4
```

```
enter inner class(es): c
main: realform
(weak) real forms are:
0: so(10)
1: so(8,2)
2: so*(10)
3: so(6,4)
```

**Example 4.4** Type  $D_{2n}$  is a bit different; here are the equal rank real forms of  $SO(12, \mathbb{C})$  (see Example 3.14)

```
real:main: type
Lie type: D6
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,1/2
```

```
enter inner class(es): c
main: realform
(weak) real forms are:
0: so(12)
1: so(10,2)
2: so*(12)[1,0]
3: so*(12)[0,1]
4: so(8,4)
5: so(6,6)
```

The two copies of  $\mathfrak{so}^*(12)$  illustrate a technical point. We say two real forms are equivalent if they are conjugate by  $G$ . See [1, Section 5]. The usual definition defines equivalence to be conjugacy by  $\text{Aut}(G)$ . If  $G$  is simple the two notions agree in most cases.

Suppose  $G = SO(2n, \mathbb{C})$ . The outer automorphism group is  $\mathbb{Z}/2\mathbb{Z}$ . In our example there are two real forms (in our sense) of  $SO(12, \mathbb{C})$  corresponding to the real group  $SO^*(12)$ . In other words there are two subgroups  $G_1, G_2$  of  $SO(12, \mathbb{C})$ , each isomorphic to  $SO^*(12)$ , which are not conjugate to each other. They *are* related by an outer automorphism of  $SO(12, \mathbb{C})$ . (In the Kac classification of real forms each is labelled by a 2 on one of the branches of the fork in the Dynkin diagram). These two groups are denoted  $SO^*(12)[0, 1]$  and  $SO^*(12)[0, 1]$ . Lest the reader think this distinction is mere sophistry, we consider another example.

If  $G = Spin(2n, \mathbb{C}), SO(2n, \mathbb{C})$  or  $PSO(2n, \mathbb{C})$  the two real corresponding real forms (locally isomorphic to  $SO^*(2n)$ ) are isomorphic, and interchanged by an outer automorphism of  $G$ . However in the “non-diagonal” quotient of  $Spin(2n, \mathbb{C})$  this fails, as we see by the next example.

### Example 4.5

```

main: type
Lie type: D6
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,0/2

enter inner class(es): c
main: realform
(weak) real forms are:
0: so(12)
1: so(10,2)
2: so*(12)[1,0]
3: so*(12)[0,1]
4: so(8,4)
5: so(6,6)
enter your choice: 2
real: components
component group is (Z/2)^1
real: realform

```

```

(weak) real forms are:
0: so(12)
1: so(10,2)
2: so*(12)[1,0]
3: so*(12)[0,1]
4: so(8,4)
5: so(6,6)
enter your choice: 3
real: components
group is connected

```

These two groups are *not* isomorphic.

This illustrates the command `components` which we had not previously discussed. It gives the component group of the real group, which is an elementary abelian two-group.

In  $D_4$  the situation is even more interesting. Let  $G = Spin(8, \mathbb{C})$  and consider the three real forms, corresponding to  $Spin(6, 2)$  and the two versions of  $Spin^*(8)$ . In fact these are isomorphic;  $Spin(6, 2) \simeq Spin^*(8)$ , and these three groups are interchanged by the outer automorphism group of  $Spin(8, \mathbb{C})$ , which is  $S_3$ . Similar statements hold in  $PSO(8, \mathbb{C})$ . We leave it to the reader to analyze real forms of  $SO(8, \mathbb{C})$  and the “non-diagonal” quotient of  $Spin(8, \mathbb{C})$ , in which this symmetry is broken in different ways. See Examples 3.6 and 3.7.

## 5 Cartan subgroups

Fix basic data  $(G, \gamma)$ . Let  $\theta_{qs}$  be a quasisplit involution in this inner class and let  $K_{qs} = G^{\theta_{qs}}$ . Thus  $K_{qs}$  is the complexified maximal compact subgroup of the quasisplit form  $G_{qs}(\mathbb{R})$  of  $G$ . The conjugacy classes of Cartan subgroups of  $G_{qs}(\mathbb{R})$ , equivalently the  $K_{qs}$ -conjugacy classes of  $\theta_{qs}$ -stable Cartan subgroups of  $G$ , are in natural bijection with  $\mathcal{I}_W/W$  (twisted involutions in  $W$  modulo the action of  $W$ ). The conjugacy classes of Cartan subgroups for any real form of  $G$  are a subset of those for the quasisplit real form. See [1, Propositions 12.9 and 12.12].

The `cartan` command gives a list of Cartan subgroups for a given inner form. It displays the following information about each Cartan subgroup: its structure as a real torus, the size of the corresponding  $W$ -orbit on the space

of twisted involutions, information about the Weyl group, and information about the corresponding fibers  $\mathcal{X}_\tau$ .

Any Cartan subgroup is isomorphic to  $(\mathbb{R}^\times)^a \times (S^1)^b \times (\mathbb{C}^\times)^c$  for integers  $(a, b, c)$ ; these are the `split`, `compact` and `complex` entries in the output of `cartan`.

Each Cartan subgroup corresponds to a  $W$ -orbit of twisted involutions. The number of such involutions is the `twisted involution orbit size`.

Associated to any Cartan subgroup are the sets of real and imaginary roots; each of these is a root system. These are given by `imaginary root system` and `real root system`. The `complex factor` line in the output has to do with the Weyl group; see Section 8. The lines beginning `real form...` give information about  $\mathcal{X}_\tau$ . See Section 7; for now we observe only that for a given Cartan only the real forms which contain this Cartan subgroup are displayed.

**Example 5.1** Let  $G = SL(2, \mathbb{C})$ , which has a unique inner class of real forms. Then  $\mathcal{I}_W = W$ , with the trivial action, so  $\mathcal{I}_W/W = \{1, s\}$ . There are two conjugacy classes of Cartan subgroups of  $SL(2, \mathbb{R})$ , compact and split.

```
main: type
Lie type: A1 sc s
main: cartan
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
Name an output file (return for stdout, ? to abandon):

Cartan #0:
split: 0; compact: 1; complex: 0
twisted involution orbit size: 1
imaginary root system: A1
real root system is empty
complex factor is empty
real form #1: [0] (1)
real form #0: [1] (1)
```

```

Cartan #1:
split: 1; compact: 0; complex: 0
twisted involution orbit size: 1
imaginary root system is empty
real root system: A1
complex factor is empty
real form #1: [0] (1)

```

Cartan #0 is always the fundamental (most compact) Cartan subgroup, in this case  $S^1$ . The corresponding twisted involution is the identity, and its orbit is itself. Both real forms contain this Cartan subgroup.

Cartan #1 is the split Cartan subgroup  $\mathbb{R}^\times$ , and this occurs only in real form #1 ( $SL(2, \mathbb{R})$ ).

**Example 5.2** Here is the complex group  $SL(2, \mathbb{C})$ :

```

real: type
Lie type: A1.A1 sc C
main: cartan
there is a unique real form: sl(2,C)
Name an output file (return for stdout, ? to abandon):

```

```

Cartan #0:
split: 0; compact: 0; complex: 1
twisted involution orbit size: 2
imaginary root system is empty
real root system is empty
complex factor: A1
real form #0: [0] (1)

```

There is one real form, and one conjugacy class of Cartan subgroups, isomorphic to  $\mathbb{C}^\times$ .

For a complex group the twisted involutions are in bijection with the Weyl group, in this case there are 2.

**Example 5.3** Here are the Cartan subgroups of  $Sp(4, \mathbb{R})$ :

```

main: type
Lie type: C2 sc s

```

```
main: cartan
(weak) real forms are:
0: sp(2)
1: sp(1,1)
2: sp(4,R)
enter your choice: 2
Name an output file (return for stdout, ? to abandon):
```

```
Cartan #0:
split: 0; compact: 2; complex: 0
twisted involution orbit size: 1
imaginary root system: B2
real root system is empty
complex factor is empty
real form #2: [0,1] (2)
real form #1: [2] (1)
real form #0: [3] (1)
```

```
Cartan #1:
split: 0; compact: 0; complex: 1
twisted involution orbit size: 2
imaginary root system: A1
real root system: A1
complex factor is empty
real form #2: [0] (1)
real form #1: [1] (1)
```

```
Cartan #2:
split: 1; compact: 1; complex: 0
twisted involution orbit size: 2
imaginary root system: A1
real root system: A1
complex factor is empty
real form #2: [0] (1)
```

```

Cartan #3:
split: 2; compact: 0; complex: 0
twisted involution orbit size: 1
imaginary root system is empty
real root system: B2
complex factor is empty
real form #2: [0] (1)

```

There are four conjugacy classes of Cartan subgroups, isomorphic to  $(S^1)^2, \mathbb{C}^\times, S^1 \times \mathbb{R}^\times$  and  $\mathbb{R}^\times \times \mathbb{R}^\times$ . All four are contained in the split group  $Sp(4, \mathbb{R})$ ; two of them are contained in  $Sp(1, 1)$ , and only the compact Cartan subgroup occurs in  $Sp(2, 0)$ .

**Example 5.4** Here is the real form of  $E_6$  with  $K$  of type  $F_4$ :

```

main: type
Lie type: E6 sc s
main: realform
(weak) real forms are:
0: e6(f4)
1: e6(R)
enter your choice: 0
real: cartan
Name an output file (return for stdout, ? to abandon):

```

```

Cartan #0:
canonical twisted involution:
split: 0; compact: 2; complex: 2
twisted involution orbit size: 45
imaginary root system: D4
real root system is empty
complex factor: A2
real form #1: [0,1,2] (3)
real form #0: [3] (1)

```

This group has a unique conjugacy class of Cartan subgroups. See Example 10.9.

We discuss Cartan subgroups of the Classical groups.

**Example 5.5** Type  $A_n$ .

First let  $\gamma = 1$ , so  $W^\Gamma = S_n \times \mathbb{Z}/2\mathbb{Z}$ . It is well known that the conjugacy classes of involutions in  $S_n$  are parametrized by ordered pairs  $(a, b) \in \mathbb{N}^2$  satisfying  $a + 2b = n$ : in cycle notation take  $w = (1, 2)(3, 4) \dots (2b - 1, 2b)$ .

The quasisplit group in this inner class is the quasisplit unitary group  $SU(m, m)$  ( $n = 2m - 1$ ) or  $SU(m + 1, m)$  ( $n = 2m$ ). The Cartan corresponding to  $(a, b)$  is isomorphic to  $(S^1)^a \times (\mathbb{C}^\times)^b$ ; the identity element corresponds to the compact Cartan subgroup.

Now suppose  $\gamma$  is given by the unique non-trivial automorphism of the Dynkin diagram of type  $A_n$  ( $n \geq 2$ ). This corresponds to the split group  $SL(n + 1, \mathbb{R})$ . It turns out that the twisted involutions in  $W$  are also parametrized by pairs  $(a, b)$  with  $a + 2b = n$ ; in this case the corresponding Cartan subgroup is isomorphic to  $(\mathbb{R}^\times)^a \times (\mathbb{C}^\times)^b$ . (This is an aspect of Vogan duality, which relates  $(SL(n + 1, \mathbb{C}), \gamma)$  and  $(PSL(n + 1, \mathbb{C}), 1)$ ).

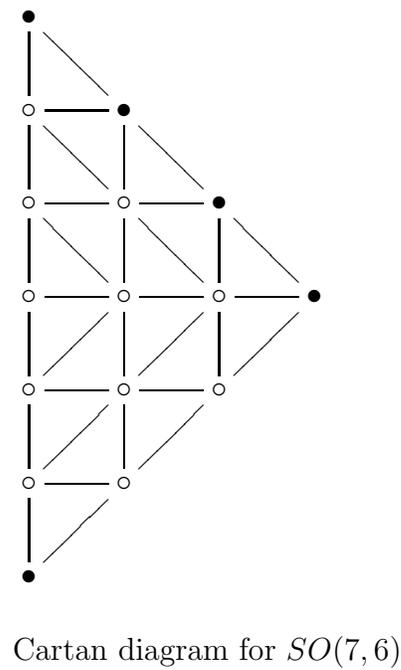
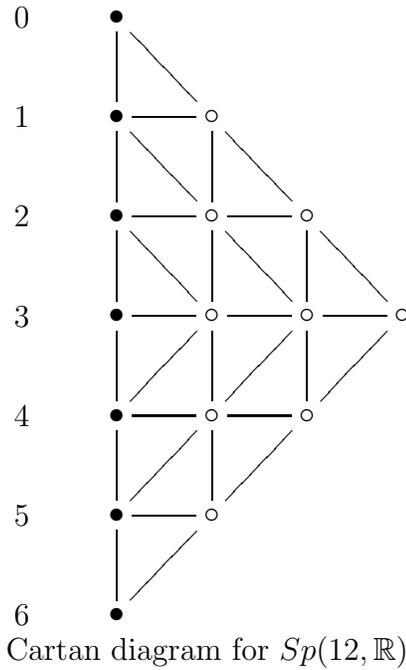
**Example 5.6** Types  $B_n$  and  $C_n$ .

In this case  $\gamma$  is necessarily trivial, and the Cartan subgroups of  $SO(n + 1, n)$  or  $Sp(2n, \mathbb{R})$  are parametrized by conjugacy classes of involutions in  $W \simeq S_n \times (\mathbb{Z}/2\mathbb{Z})^n$ . These are parametrized by  $(a, b, c)$  with  $a + b + 2c = n$ , and the corresponding Cartan subgroup is isomorphic to  $(S^1)^a \times (\mathbb{R}^\times)^b \times (\mathbb{C}^\times)^c$ .

It is interesting to consider the Hasse diagram of these Cartan subgroups. This is the graph, with one node for each Cartan subgroup, and an edge for each Cayley transform relating two Cartan subgroups. We make a node black if the corresponding Cartan subgroup is the most split Cartan subgroup of a real form of  $G$ . The split rank is given in the first column.

**Example 5.7** Type  $D_n$ .

In this case (for  $n \geq 2$ ) there are two choices of  $\gamma$ , corresponding to the quasisplit groups  $SO(n, n)$  and  $SO(n + 1, n - 1)$  ( $\gamma = 1$  corresponds to  $SO(n, n)$  if  $n$  is even, and  $SO(n + 1, n - 1)$  if  $n$  is odd). It is most convenient to group these two real forms together. Then the Cartan subgroups are parametrized by  $(a, b, c)$  with  $a + b + 2c = n$ , except that  $(0, 0, c)$  is counted twice. (These two Cartan subgroups are conjugate by the outer automorphism of  $SO(n, n)$  coming from  $O(n, n)$ .) Again the corresponding Cartan subgroup is isomorphic to  $(S^1)^a \times (\mathbb{R}^\times)^b \times (\mathbb{C}^\times)^c$ . If  $a$  is even this Cartan subgroup occurs in  $SO(n, n)$  and in  $SO(n + 1, n - 1)$  otherwise.



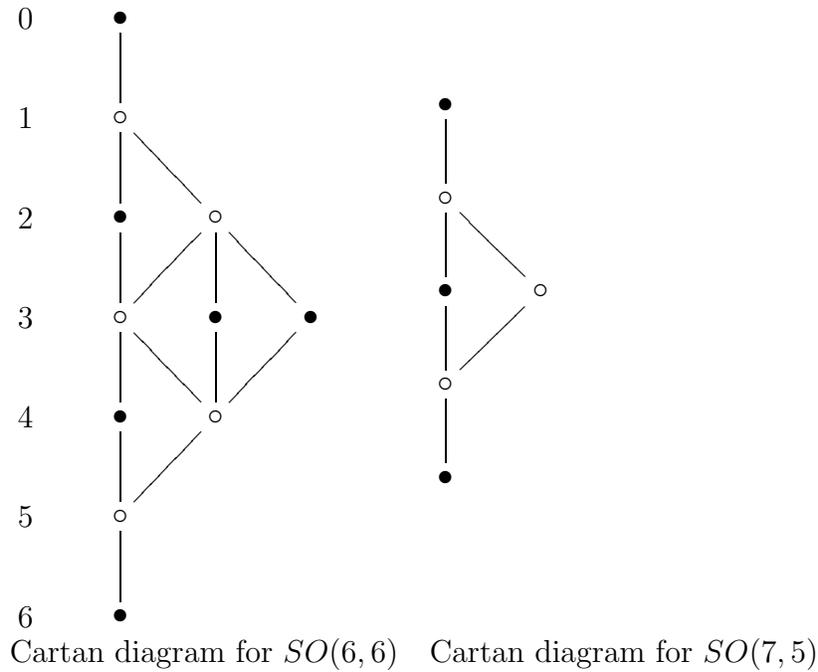
**Example 5.8** Note that the Cartan subgroups in  $G(\mathbb{R})$  only depend on  $W^\Gamma$ , and are therefore independent of isogeny (this is not true in the p-adic case). While the list of Cartan subgroups is independent of isogeny, the description as a real torus is not, and the nature of the torus can change unexpectedly under isogenies.

For example suppose  $G$  is of type  $D_2 \simeq A_1 \times A_1$  and  $\gamma \neq 1$ . There is one Cartan subgroup  $H(\mathbb{R})$  in this case. The possibilities are:

$G$	$G(\mathbb{R})$	$H(\mathbb{R})$
$Spin(4, \mathbb{C}) \simeq SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$	$Spin(3, 1) \simeq SL(2, \mathbb{C})$	$\mathbb{C}^\times$
$SO(4, \mathbb{C})$	$SO(3, 1)$	$\mathbb{R}^\times \times S^1$
$PSO(4, \mathbb{C}) \simeq PSL(2, \mathbb{C}) \times PSL(2, \mathbb{C})$	$PSO(3, 1) \simeq PSL(2, \mathbb{C})$	$\mathbb{C}^\times$

For example here is the Cartan subgroup of  $Spin(3, 1) \simeq SL(2, \mathbb{C})$ :

```
main: type
Lie type: A1.A1 sc C
main: cartan
```



there is a unique real form:  $\mathfrak{sl}(2, \mathbb{C})$

Name an output file (return for stdout, ? to abandon):

Cartan #0:

split: 0; compact: 0; complex: 1

...

For  $SO(3,1)$  we get:

split: 1; compact: 1; complex: 0

and for  $PSO(3,1) \simeq PSL(2, \mathbb{C})$ :

split: 0; compact: 0; complex: 1

Note that  $G(\mathbb{R})$  is a connected complex group in the case of  $Spin(4, \mathbb{C})$  and  $PSO(4, \mathbb{C})$ , but not  $SO(4, \mathbb{C})$ .

**Remark 5.9** If  $G(\mathbb{R})$  is a real form of  $SL(n, \mathbb{C})$ ,  $Sp(2n, \mathbb{C})$  or  $SO(n, \mathbb{C})$  then two Cartan subgroups are isomorphic if and only if they are conjugate by

$G(\mathbb{R})$ , or by an automorphism of  $G(\mathbb{R})$  in the case of  $(\mathbb{C}^\times)^n \subset SO(2n, 2n)$  (cf. Example 5.7). It is perhaps surprising that this does not necessarily hold for isogenous groups, as the following example shows.

Let  $G(\mathbb{R}) = Spin(n, n)$  or  $PSO(n, n)$  with  $n$  even. There are three Cartan subgroups isomorphic to  $\mathbb{R}^\times \times S^1 \times (\mathbb{C}^\times)^{\frac{n}{2}-1}$ . Two of these are interchanged by an outer automorphism of  $G(\mathbb{R})$ . (These are the two black dots in the middle row of the Cartan diagram of  $SO(6, 6)$ . The corresponding Cartan subgroups of  $SO(n, n)$  are isomorphic to  $(\mathbb{C}^\times)^{\frac{n}{2}}$ .) The third one is not related to the others: it has a different real Weyl group.

Here are the three Cartan subgroups in question for  $Spin(6, 6)$ :

Cartan #4:

split: 1; compact: 1; complex: 2  
 twisted involution orbit size: 180  
 imaginary root system: A1.A1.A1  
 real root system: A1.A1.A1  
 complex factor: A1  
 real form #5: [0] (1)  
 real form #4: [1] (1)

Cartan #5:

split: 1; compact: 1; complex: 2  
 twisted involution orbit size: 60  
 imaginary root system: A1.A1.A1  
 real root system: A1.A1.A1  
 complex factor: A2  
 real form #5: [0] (1)  
 real form #3: [1] (1)

Cartan #6:

split: 1; compact: 1; complex: 2  
 twisted involution orbit size: 60  
 imaginary root system: A1.A1.A1  
 real root system: A1.A1.A1  
 complex factor: A2  
 real form #5: [0] (1)

real form #2: [1] (1)

The second and third are interchanged by the outer automorphism of  $Spin(6, 6)$ .

A similar phenomenon holds in the split real forms of  $F_4$  and  $G_2$ .

## 6 Adjoint Fiber

Fix basic data  $(G, \gamma)$  where  $G$  is adjoint, and a twisted involution  $\tau \in \mathcal{I}_W$ . The fiber  $\mathcal{X}_\tau$  of the map from  $\mathcal{X}$  to  $\mathcal{I}_W$  plays an important role in the algorithm. While it doesn't translate directly to something with representation theoretic meaning, it is useful to understand something about it.

Let  $H^{-\tau} = \{h \in H \mid h\tau(h) = 1\}$ . There is a natural simply transitive action of  $H^{-\tau}/(H^{-\tau})^0$  on  $\mathcal{X}_\tau$  [1, Section 11]. Note that  $H^{-\tau}/(H^{-\tau})^0 \simeq (\mathbb{Z}/2\mathbb{Z})^b$  where  $b$  is the number of  $S^1$  factors of the real form of  $H$ . Fix  $x \in \mathcal{X}_\tau$ . Via a choice of basepoint in  $\mathcal{X}_\tau$  the output of `cartan` identifies  $\mathcal{X}_\tau$  with  $\{0, 1, \dots, 2^b - 1\}$ . (The element of  $\mathbb{Z}/2\mathbb{Z}^b$  corresponding to  $0 \leq k \leq 2^b - 1$  is the binary expansion of  $k$ .)

The imaginary Weyl group  $W_i$  acts on  $\mathcal{X}_\tau$ , and the orbits are in one to one correspondence with real forms containing this Cartan subgroup. The `cartan` command gives this decomposition. For example a line

```
real form #2: [0,1,2] (3)
```

means that corresponding to real form #2 (from the output of the `realform` command) are three elements of  $\mathcal{X}_\tau$ , labelled 0,1,2 (the number in parentheses is the number of elements).

The fibers  $\mathcal{X}_\tau$  for  $\tau$  in a given conjugacy class may be canonically identified.

**Example 6.1** Here is the adjoint group  $PSL(2, \mathbb{C})$ :

```
main: type
Lie type: A1 ad s
main: cartan
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
```

Name an output file (return for stdout, ? to abandon):

```
Cartan #0:
split: 0; compact: 1; complex: 0
twisted involution orbit size: 1
imaginary root system: A1
real root system is empty
complex factor is empty
real form #1: [0] (1)
real form #0: [1] (1)
```

```
Cartan #1:
split: 1; compact: 0; complex: 0
twisted involution orbit size: 1
imaginary root system is empty
real root system: A1
complex factor is empty
real form #1: [0] (1)
```

This means that for  $\tau = 1$ , corresponding to the compact Cartan subgroup,  $|\mathcal{X}_\tau|$  has two elements, 0 and 1, and the action of  $W_i = W$  is trivial. The element 0 corresponds to  $SL(2, \mathbb{R})$ , and 1 to  $SU(2)$ . For  $\tau = s$  (the non-trivial element of  $W$ ) the corresponding Cartan subgroup is split, occurs only in  $SL(2, \mathbb{R})$ , and the fiber  $\mathcal{X}_\tau$  is a single point. See [1, Example 12.20].

The preceding example is too small to be really helpful; here is a more interesting one.

**Example 6.2** Here are the equal rank real forms of  $PSL(4, \mathbb{C})$ :

```
main: type
Lie type: A3 ad c
main: cartan
(weak) real forms are:
0: su(4)
1: su(3,1)
2: su(2,2)
enter your choice: 0
```

Name an output file (return for stdout, ? to abandon):

Cartan #0:  
canonical twisted involution:  
split: 0; compact: 3; complex: 0  
twisted involution orbit size: 1  
imaginary root system: A3  
real root system is empty  
complex factor is empty  
real form #2: [0,2,5] (3)  
real form #1: [1,3,4,6] (4)  
real form #0: [7] (1)

This is the fundamental fiber  $\tau = \delta$ . In this case  $W_i = W$ . This fiber has  $2^3 = 8$  elements. There is a fixed point of the action of  $W$  on  $\mathcal{X}_\delta$ , corresponding to the compact real form  $PSU(4)$  (realform #0). The largest orbit, of order 4, corresponds to the group  $SU(3,1)$ .

Here are the remaining Cartan subgroups:

Cartan #1:  
canonical twisted involution: 1,2,3,2,1  
split: 0; compact: 1; complex: 1  
twisted involution orbit size: 6  
imaginary root system: A1  
real root system: A1  
complex factor is empty  
real form #2: [0] (1)  
real form #1: [1] (1)

There are 6 elements  $\tau$  here for which  $\mathcal{X}_\tau$  has order 2; all but the compact real form appear.

Cartan #2:  
canonical twisted involution: 2,1,3,2  
split: 1; compact: 0; complex: 1  
twisted involution orbit size: 3  
imaginary root system is empty  
real root system: A1.A1  
complex factor: A1  
real form #2: [0] (1)

Finally only the quasisplit group  $PSU(3, 3)$  contains the maximally split Cartan, in which case there are 3 twisted involutions  $\tau$ , each with  $|\mathcal{X}_\tau| = 1$ .

Note that  $\mathcal{X}$  therefore has  $1 \times 8 + 6 \times 2 + 3 \times 1 = 23$ . See Example 9.6

## 7 Strong Real Forms and Fibers

Recall (Section 2) a strong involution  $\xi$  is an element of  $G^\Gamma \backslash G$  satisfying  $\xi^2 \in Z^r$ , and a real form is a conjugacy class of strong involutions. The map  $\xi \rightarrow \theta_\xi$  taking strong real forms to real forms is surjective, but (if  $G$  is not adjoint) not necessarily injective.

The `strongreal` command gives information about strong real forms, which may also be interpreted as information about the fiber  $\mathcal{X}_\tau$  of the map  $p : \mathcal{X}^r \rightarrow \mathcal{I}_W$  over  $\tau \in \mathcal{I}_W$ .

Fix a twisted involution  $\tau \in \mathcal{I}_W$ . Let

$$(7.1) \quad \begin{aligned} H_1(\tau) &= \{h \in H \mid h\tau(h) \in Z^r\} \\ H_2(\tau) &= \{h \in H \mid h\tau(h) = 1\}. \end{aligned}$$

Then  $\mathcal{X}_\tau \simeq H_1(\tau)/H_2(\tau)^0$  [1, Proposition 11.2].

The strong real forms “containing”  $(H, \tau)$  are parametrized by  $\mathcal{X}_\tau^r/W_{i,\tau}$ , where  $W_{i,\tau}$  is the Weyl group of the  $\tau$ -imaginary roots. In particular every strong real form contains the fundamental Cartan subgroup  $(H, \delta)$ , so the strong real forms are parametrized by  $\mathcal{X}_\delta^r/W_{i,\delta}$ . Things are particularly simple if  $G$  has equal rank, in which case

$$(7.2) \quad \mathcal{X}_\delta^r/W \simeq \{h \in H \mid h^2 = 1\}/W.$$

Note that  $|\mathcal{X}_\delta^r| = |Z^r|2^n$  in this case.

The map  $H_1(\tau) \ni h \rightarrow h\tau(h) \in Z^r$  is a group homomorphism. It is not necessarily surjective. The software labels the image  $\{z_0, \dots, z_{r-1}\}$ , denoted `class #0`, `class #1`, ..., `class #r-1`. This is a subgroup of the two-group  $Z^\Gamma/(1 + \theta)Z$ , so  $r = 2^s \leq |Z^r|$  for some  $s$ . For each  $z_j$  the set

$$(7.3) \quad \mathcal{X}_\tau^r(z_j) = \{x \in \mathcal{X}_\tau \mid x^2 = z_j\}$$

is isomorphic to  $H_2(\tau)/H_2(\tau)^0$  (independent of  $j$ ) which has cardinality  $2^b$ . Here  $b$  is the number of  $S^1$  factors in the real group  $H(\mathbb{R})$  defined by  $\tau$ . See

[1, Section 11]. The group  $W_{i,\tau}$  acts on  $\mathcal{X}_\tau(z_j)$ , and the orbits correspond to strong real forms. In particular

$$(7.4) \quad |\mathcal{X}_\tau^r| = 2^b r = 2^{b+s}.$$

All of this information is given by the `strongreal` command.

Note that the order of  $\mathcal{X}^r$  may be computed by summing over Cartan subgroups, taking into account the size of the conjugacy classes of twisted involutions.

**Example 7.5** Here are strong real forms of  $SL(2)$ . (See [1, Example 12.20], especially the Figure at the end). The set  $\mathcal{X}_\delta$  consists of four elements  $\pm I, \pm \text{diag}(i, -i)$ . In this case  $Z^r = Z/Z^2 = Z = \{\pm I\}$ .

```
main: type
Lie type: A1 sc s
main: strongreal
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
cartan class (one of 0,1):
sorry, value must be one of 0,1
try again (? to abort): 0
Name an output file (return for stdout, ? to abandon):

there are 2 real form classes:

class #0:
real form #1: [0,1] (2)

class #1:
real form #0: [0] (1)
real form #0: [1] (1)
```

The element `class #0` of  $Z$  is  $-I$ . The two elements  $\pm \text{diag}(i, -i)$  of  $H$  map to  $-I$ . These form a single  $W$ -orbit, denoted `[0,1]`, corresponding to `real form #1`, i.e.  $SL(2, \mathbb{R})$ ; we have  $\mathcal{X}_\tau(z) = \{\pm \text{diag}(i, -i)\}$ .

On the other hand class #1 is the element  $I \in Z$ . In this case  $\mathcal{X}_\tau(I) = \{\pm I\}$ , which consists of two  $W$ -orbits. This gives two strong real forms, both mapping to the compact real form  $SU(2)$ .

Here are the strong real forms containing the split Cartan subgroup:

```
real: strongreal
cartan class (one of 0,1): 1
Name an output file (return for stdout, ? to abandon):
```

```
real form #1: [0] (1)
```

In this case  $\mathcal{X}_\tau = \mathcal{X}_\tau(-I)$  is a single element, and  $\mathcal{X}_\tau(I) = \emptyset$ . Only the split real form occurs.

**Example 7.6** Now consider the equal rank inner class of  $SL(4, \mathbb{C})$ . Compare Example 6.2.

```
main: type
Lie type: A3 sc c
main: strongreal
(weak) real forms are:
0: su(4)
1: su(3,1)
2: su(2,2)
enter your choice: 0
there is a unique conjugacy class of Cartan subgroups
Name an output file (hit return for stdout):
```

```
there are 2 real form classes:
```

```
class #0:
real form #2: [0,1,2,4,5,6] (6)
real form #0: [3] (1)
real form #0: [7] (1)
```

```
class #1:
real form #1: [0,2,3,5] (4)
real form #1: [1,4,6,7] (4)
```

In this case  $Z \simeq \mathbb{Z}/4\mathbb{Z}$  and  $Z/Z^2 \simeq \mathbb{Z}/2\mathbb{Z}$ , so  $|Z^r| = 2$ . In this case  $b$  (the number of  $S^1$  factors) is 3, so  $|\mathcal{X}_\tau(z)| = 8$  for each  $z \in Z^r$  (it is never 0 in this case), and  $|\mathcal{X}_\tau| = 16$ .

For **class #0** we can take  $I$ . There are three orbits of  $W = W_i$  on  $\mathcal{X}_\tau(I)$ . The  $W$ -orbit of  $\text{diag}(1, 1, -1, -1)$  has 6 elements; given in the line **real form #2: [0,1,2,4,5,6] (6)**; the corresponding real form (**#2**) is  $SU(2, 2)$ . The elements  $\pm I$  are each fixed by  $W$ , and correspond to the two lines beginning **real form #0**. (The software does not specify which of these is  $I$ , and which  $-I$ ).

Note that  $-I \simeq I \bmod Z^2$ , so we could have chosen  $z = -I$  instead. Then  $SU(2, 2)$  would be given by  $iI$  times the preceding ones, i.e. the  $W$ -orbit of  $\text{diag}(i, i, -i, -i)$ . Similarly the two strong real forms mapping to  $SU(2)$  would be  $\pm iI$ . The software does not make an actual choice of  $Z^r$ ; the output of the software, and the combinatorics of the algorithm, are independent of any such choice. See [1, Section 13].

For **class #1** we can choose  $z = iI$ . These are the elements  $\zeta(\pm 1, \dots, \pm 1)$  where  $\zeta = e^{\pi i/4}$ , with an odd number of plus signs. The 8 such elements constitute 2  $W$ -orbits, hence the two strong real forms mapping to  $SU(3, 1)$  (**real form #1**).

Here are the strong real forms containing the other two Cartan subgroups in this example.

```
real: strongreal
cartan class (one of 0,1,2): 1
Name an output file (return for stdout, ? to abandon):
```

there are 2 real form classes:

```
class #0:
real form #2: [0,1] (2)
```

```
class #1:
real form #1: [0] (1)
real form #1: [1] (1)
```

The compact real form **real form #0** only contains the compact Cartan subgroup and doesn't occur. In this case  $|\mathcal{X}_\tau(I)| = |\mathcal{X}_\tau(iI)| = 2$ ,  $|\mathcal{X}_\tau| = 4$ , and both  $SU(2, 2)$  and  $SU(3, 1)$  occur.

```

real: strongreal
cartan class (one of 0,1,2): 2
Name an output file (return for stdout, ? to abandon):

```

```

real form #2: [0] (1)

```

For the most split Cartan subgroup  $|\mathcal{X}_\tau(I)| = 0$ ,  $|\mathcal{X}_\tau(iI)| = 1$ , and only  $SU(2, 2)$  occurs.

The number of twisted involutions in each conjugacy class are 1, 6 and 3, as given by the `cartan` command. Therefore the cardinality of  $\mathcal{X}^r$  is

$$(7.7) \quad 16 \times 1 + 4 \times 6 + 1 \times 3 = 43.$$

See Example 9.4.

**Example 7.8** Type  $A_{2n}$  is a little different than  $A_{2n+1}$ . Here are the fundamental fiber and the equal rank strong real forms of  $SL(5, \mathbb{C})$ :

```

Lie type: A4 sc c
main: strongreal
(weak) real forms are:
0: su(5)
1: su(4,1)
2: su(3,2)
enter your choice: 0
there is a unique conjugacy class of Cartan subgroups
Name an output file (hit return for stdout):

```

```

real form #2: [0,1,2,4,5,6,8,10,11,13] (10)
real form #1: [3,9,12,14,15] (5)
real form #0: [7] (1)

```

In this case  $Z \simeq \mathbb{Z}/5\mathbb{Z}$  and  $Z/Z^2 = 1$ . We can take  $z = I$  in this case, and the map from strong real forms to real forms is bijective.

We next consider the inner class of  $SL(n, \mathbb{R})$ . If  $n$  is odd this is the unique real form in this inner class; if  $n$  is even there is one other real form  $SL(n/2, \mathbb{H})$ . We can take  $\theta(z_1, \dots, z_n) = \text{diag}(\frac{1}{z_n}, \dots, \frac{1}{z_1})$ , and  $Z^r = \{I\}$  if  $n$  is odd, or  $\{\pm I\}$  if  $n$  is even.

Here is an example of each case.

**Example 7.9** There is only one real form in this inner class of  $SL(5, \mathbb{R})$ :

```
main: type
Lie type: A4 sc s
main: strongreal
there is a unique real form: sl(5,R)
cartan class (one of 0,1,2): 0
Name an output file (hit return for stdout):
```

```
real form #0: [0] (1)
```

Here  $Z^r = \{I\}$  and

$$(7.10) \quad \mathcal{X}_\delta = \{\text{diag}(x, y, z) \mid \text{diag}(x/z, 1, z/x) = I\} / \{\text{diag}(xz, y^2, xz)\} = I$$

There is only one strong real form in this case.

**Example 7.11** Now consider the inner class of  $SL(4, \mathbb{R})$ , which also contains the real form  $SL(2, \mathbb{H})$ . In this case  $Z = \pm I$ .

```
main: type
Lie type: A3 sc s
main: strongreal
(weak) real forms are:
0: sl(2,H)
1: sl(4,R)
enter your choice: 0
there is a unique conjugacy class of Cartan subgroups
Name an output file (hit return for stdout):
```

```
there are 2 real form classes:
```

```
class #0:
real form #1: [0,1] (2)
```

```
class #1:
real form #0: [0] (1)
real form #0: [1] (1)
```

Taking  $z = I$  (this is `class #1`) we have

$$\begin{aligned}\mathcal{X}_\delta(I) &= \{\text{diag}(z, w, w, z) \mid z^2 w^2 = 1\} / \{\text{diag}(ad, bc, bc, ad) \mid abcd = 1\} \\ &= \{I, \text{diag}(1, -1, -1, 1)\}.\end{aligned}$$

These elements are not conjugate by  $W$ , so there are two strong real forms mapping to the real form  $SL(2, \mathbb{H})$  (`real form #0`).

On the other hand if  $z = -I$  (`class #0`) then

$$\begin{aligned}\mathcal{X}_\delta(I) &= \{\text{diag}(z, w, -w, -z) \mid z^2 w^2 = 1\} / \{\text{diag}(ad, bc, -bc, -ad) \mid abcd = 1\} \\ &= \{\text{diag}(1, 1, -1, -1), \text{diag}(-1, -1, 1, 1)\}.\end{aligned}$$

These two elements are conjugate by  $W$ , so there is only one strong real form mapping to the real form  $SL(4, \mathbb{R})$  (`real form #1`).

## 8 Weyl Groups

Fix  $(G, \gamma)$ ,  $\tau \in \mathcal{I}_W$ ,  $\xi \in \tilde{\mathcal{X}}_\tau$ , and set  $K = G^{\theta_\xi}$ . Recall (cf. Section 2)  $W(K, H)$  is the “real” Weyl group; it is isomorphic to  $W(G(\mathbb{R}), H(\mathbb{R})) = \text{Norm}_{G(\mathbb{R})}(H(\mathbb{R})) / H(\mathbb{R})$  where  $G(\mathbb{R})$  is a real form of  $G$  corresponding to  $K$ .

We briefly recall some constructions from [1, Section 8], also see [3, Proposition 4.16]. We have

$$(8.1) \quad W(K, H) \simeq (W_C)^\tau \rtimes (W(M \cap K, H) \times W_r).$$

Here

- $W_r$  is the Weyl group of the system of real roots;
- $(W_C)^\tau$  is the Weyl group of a certain root system constructed using complex roots ([3, Proposition 3.12]);
- $W(K \cap M, H) \simeq W_{i,c} \rtimes \mathcal{A}(H) \subset W_i$ , where  $W_{i,c}$  is the Weyl group of the root system of compact imaginary roots, and  $\mathcal{A}(H)$  is a certain two-group.

To describe  $W(K, H)$  it is therefore sufficient to describe  $W_r, W_{i,c}, (W_C)^\tau$  and  $\mathcal{A}(H)$ . The first three are Weyl groups of root systems, and the last is an abelian two-group.

The `realweyl` command describes these three root systems, denoted `W_r`, `W_ic`, `WC` respectively; and  $\mathcal{A}(H)$ , denoted `A`.

**Example 8.2** As usual we start with  $SL(2, \mathbb{R})$ :

```
main: type
Lie type: A1 sc s
main: realweyl
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
cartan class (one of 0,1): 0
Name an output file (return for stdout, ? to abandon):
```

```
real weyl group is  $W^C \cdot ((A \cdot W \setminus_{ic}) \times W^R)$ , where:
 $W^C$  is trivial
A is trivial
 $W \setminus_{ic}$  is trivial
 $W^R$  is trivial
```

This is the compact Cartan subgroup, for which  $W(K, H)$  is trivial.

```
real: realweyl
cartan class (one of 0,1): 1
Name an output file (return for stdout, ? to abandon):
```

```
real weyl group is  $W^C \cdot ((A \cdot W \setminus_{ic}) \times W^R)$ , where:
 $W^C$  is trivial
A is trivial
 $W \setminus_{ic}$  is trivial
 $W^R$  is a Weyl group of type A1
```

For the split Cartan subgroup the only non-trivial factor is  $W_r$ , and  $W(K, H) = W_r \simeq \mathbb{Z}/2\mathbb{Z}$ .

**Example 8.3** There is a small change when we compute the Weyl group for the compact Cartan subgroup of  $PSL(2, \mathbb{R})$  instead:

```
main: type
Lie type: A1 ad s
main: realweyl
```

```

(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
cartan class (one of 0,1): 0
Name an output file (return for stdout, ? to abandon):

```

```

real weyl group is W^C.((A.W_ic) x W^R), where:
W^C is trivial
A is an elementary abelian 2-group of rank 1
W_ic is trivial
W^R is trivial

```

In this case  $A = \mathbb{Z}/2\mathbb{Z}$ , so  $W(K, H) = \mathbb{Z}/2\mathbb{Z}$ . Recall  $PSL(2, \mathbb{R}) \simeq SO(2, 1)$  is disconnected; the non-trivial Weyl group element is given by an element in the non-identity component.

**Example 8.4** Here is an example, for the split group of type  $D_{2m}$ , in which the group  $\mathcal{A}(H)$  is quite large. We take the Cartan subgroup  $(S^1)^2 \times (\mathbb{C}^\times)^{m-1}$  (cf. Example 5.7). In the notation at the beginning of [1, Section 12] we have

$$\begin{aligned}
(8.5) \quad & \Delta_i = A_1^{m-1} \times D_2 \simeq A_1^{m+1} \\
& \Delta_r = A_1^{m-1} \\
& \Delta_C = A_{m-2} \times A_{m-2} \\
& W_i \simeq (\mathbb{Z}/2\mathbb{Z})^{m+1}, W_r \simeq (\mathbb{Z}/2\mathbb{Z})^{m-1} \\
& W_{i,c} = 1 \\
& W_C \simeq S_{m-1} \times S_{m-1}, (W_C)^\tau \simeq S_{m-1} \\
& W^\tau \simeq S_{m-1} \times [(\mathbb{Z}/2\mathbb{Z})^{m+1} \times (\mathbb{Z}/2\mathbb{Z})^{m-1}].
\end{aligned}$$

Here  $S_{m-1}$  acts trivially on the final two factors of  $(\mathbb{Z}/2\mathbb{Z})^{m+1}$ .

So far this discussion is independent of isogeny.

The real Weyl group is the same as  $W^\tau$ , with the factor  $W_i = (\mathbb{Z}/2\mathbb{Z})^{m+1}$  replaced by  $\mathcal{A}(H)$ , which is a subgroup satisfying

$$(8.6) \quad 1 = W_{i,c} \subset \mathcal{A}(H) \subset W_i = (\mathbb{Z}/2\mathbb{Z})^{m+1}.$$

The group  $\mathcal{A}(H)$  depends on the isogeny. If  $G$  is simply connected it is as small as possible, and for  $G$  adjoint it is as large as possible, in fact equal to

$W_i$ . So if  $G(\mathbb{R})$  is the adjoint group  $PSO(n, n)$  we have  $\mathcal{A}(H) \simeq (\mathbb{Z}/2\mathbb{Z})^{m+1}$ , and  $W(K, H) = W^\tau$ . On the other hand if  $G = Spin(n, n)$  then  $\mathcal{A}(H) \simeq (\mathbb{Z}/2\mathbb{Z})^{m-1}$ .

We can compute this information using the atlas software:

```
empty: type
Lie type: D8 ad s
main: realform
(weak) real forms are:
0: so(16)
1: so(14,2)
2: so(12,4)
3: so*(16)[0,1]
4: so*(16)[1,0]
5: so(10,6)
6: so(8,8)
enter your choice: 6
real: realweyl
cartan class (one of 0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15): 5
Name an output file (hit return for stdout):

real weyl group is W^C.((A.W_ic) x W^R), where:
W^C is isomorphic to a Weyl group of type A2
A is an elementary abelian 2-group of rank 5
W_ic is trivial
W^R is a Weyl group of type A1.A1.A1
```

Here is the relevant output from the `cartan` command:

```
Cartan #5:
canonical twisted involution: 6,7,8,6,5,6,7,8,6,4,...[truncated]
split: 0; compact: 2; complex: 3
twisted involution orbit size: 3360
imaginary root system: A1.A1.A1.A1.A1
real root system: A1.A1.A1
complex factor: A2
```

If  $H$  is a fundamental Cartan subgroup then  $H_K = H \cap K$  is a Cartan subgroup of  $K$ , and it is not hard to see that  $W(G, H) \simeq W(K, H_K)$ . In

particular if  $K$  is connected this is the Weyl group of the root system of  $H_K$  in  $K$ . It is interesting to consider this in the case of real forms of  $E_6$  of unequal rank.

**Example 8.7** First suppose  $K$  is of type  $F_4$ :

```
main: type
Lie type: E6 sc u
main: realweyl
(weak) real forms are:
0: e6(f4)
1: e6(R)
enter your choice: 0
there is a unique conjugacy class of Cartan subgroups
Name an output file (return for stdout, ? to abandon):

real weyl group is W^C.((A.W_ic) x W^R), where:
W^C is isomorphic to a Weyl group of type A2
A is trivial
W_ic is a Weyl group of type D4
W^R is trivial
```

Thus  $W(K, H) \simeq W(A_2) \times W(D_4) \simeq S_3 \times W(D_4)$ . The action of  $S_3$  on  $W(D_4)$  is induced by the action of  $S_3$  on the Dynkian diagram of  $D_4$ . By the remarks above we have obtained the classical isomorphism

$$(8.8) \quad W(F_4) \simeq S_3 \times W(D_4).$$

**Example 8.9** Something similar happens for the split real form of  $E_6$ , i.e.  $K$  is of type  $C_4$ :

```
real: type
Lie type: E6 sc s
main: realform
(weak) real forms are:
0: e6(f4)
1: e6(R)
enter your choice: 1
real: realweyl
```

cartan class (one of 0,1,2,3,4): 0  
 Name an output file (return for stdout, ? to abandon):

real weyl group is  $W^{\mathbb{C}}((A.W_{ic}) \times W^{\mathbb{R}})$ , where:  
 $W^{\mathbb{C}}$  is isomorphic to a Weyl group of type A2  
 A is an elementary abelian 2-group of rank 2  
 $W_{ic}$  is a Weyl group of type A1.A1.A1.A1  
 $W^{\mathbb{R}}$  is trivial

Thus

$$(8.10) \quad \begin{aligned} W(C_4) &\simeq S_3 \times (\mathbb{Z}_2^2 \times \mathbb{Z}_2^4) \\ &\simeq S_4 \times \mathbb{Z}_2^4. \end{aligned}$$

**Example 8.11** Here is an example of the Weyl group of a complex group. Fix basic data  $(G_1 \times G_1, \gamma)$  where for some group  $G_1$ , with  $\gamma$  the inner class of the complex group. Thus  $G(\mathbb{R}) = G_1(\mathbb{C})$ . If  $H_1$  is a Cartan subgroup of  $G_1$  then  $H = H_1 \times H_1$  is a Cartan subgroup of  $G$ , and  $W(G, H) \simeq W(G_1, H_1) \times W(G_1, H_1)$ , and  $W(K, H) = W(G_1, H_1)^\Delta$  (the diagonal embedding). In the notation of the output of `realweyl`  $W^{\mathbb{C}} = W(G)$  and  $(W^{\mathbb{C}})^\tau = W(G_1, H_1)$ .

empty: type  
 Lie type: A3.A3 sc C  
 main: realweyl  
 there is a unique real form:  $\mathfrak{sl}(4, \mathbb{C})$   
 there is a unique conjugacy class of Cartan subgroups  
 Name an output file (hit return for stdout):

real weyl group is  $W^{\mathbb{C}}((A.W_{ic}) \times W^{\mathbb{R}})$ , where:  
 $W^{\mathbb{C}}$  is isomorphic to a Weyl group of type A3  
 A is trivial  
 $W_{ic}$  is trivial  
 $W^{\mathbb{R}}$  is trivial

generators for  $W^{\mathbb{C}}$ :  
 2,5  
 3,6  
 1,4

Simple roots 1, 2, 3 give the first copy of  $G_1$ , and 4, 5, 6 give the second. The elements  $s_1s_4, s_2s_5$  and  $s_3s_6$  generate the  $W(G_1, H_1)^\Delta \simeq W(A_3)$ .

## 9 $K$ orbits on the flag variety and the parameter space $\mathcal{X}$

Fix  $(G, \gamma)$ ,  $\xi \in \tilde{\mathcal{X}}$  and let  $K = G^\xi$ . The space  $K \backslash G/B$  of  $K$ -orbits on  $G/B$  is finite. It is described by the `kgb` command.

**Example 9.1** We first consider  $SL(2, \mathbb{R})$ . See [1, Example 12.20].

```
main: type
Lie type: A1 sc s
main: kgb
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
kgbsize: 3
Name an output file (return for stdout, ? to abandon):
0:   1   2 [n]  0
1:   0   2 [n]  0
2:   2   * [r]  1  1
```

Here  $G(\mathbb{R}) = SL(2, \mathbb{R})$ ,  $K(\mathbb{R}) = S^1$  and  $K = K(\mathbb{C}) = \mathbb{C}^\times$ . This acts on  $G/B = P^1(\mathbb{C}) = \mathbb{C} \cup \infty$  by  $\mathbb{C}^\times \ni z : w \rightarrow z^2 w$ . There are three orbits of this action:  $0, \infty$  and  $\mathbb{C}^\times$ . These are orbits #0,1,2 respectively (the first entry in each line) in the output of `kgb`. In this case the penultimate entry in each line gives the dimension of the orbit.

**Example 9.2** Here is  $PGL(2, \mathbb{R})$ ; see [1, Example 12.25] for more detail.

```
main: type
Lie type: A1 ad s
main: kgb
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
kgbsize: 2
Name an output file (return for stdout, ? to abandon):
```

```

0:    0    1  [n]  0
1:    1    *  [r]  1  1

```

If  $G(\mathbb{R}) = PGL(2, \mathbb{R})$  there are only two orbits, one open and one closed. In this case  $K = O(2, \mathbb{C})$  acts on  $G/B = P^1(\mathbb{C})$ . The orbits are the same as in the previous example, except that an element from the non-identity component of  $K$  takes  $z$  to  $\frac{1}{z}$ , and 0 to  $\infty$ .

We now explain the other information given in the output. Let  $n$  be the semisimple rank of  $G$ . The simple roots are labelled  $1, \dots, n$ . Recall (2)  $K \backslash G/B \hookrightarrow \mathcal{X}^r$ . Each row  $i$  of output corresponds to an element  $x = x_i$  of  $\mathcal{X}^r$ , and an orbit  $\mathcal{O}_i$  of  $K$  on  $G/B$ . Fix a row  $i$  with corresponding  $x = x_i$ .

Suppose  $\alpha$  is a (simple) root. Then  $\alpha$  is real, complex, compact imaginary, or non-compact imaginary with respect to  $x$  (see Section 2 and [1, Section 12]). The term in brackets in row  $i$  of the output gives this information for each simple root:  $\mathbf{r}, \mathbf{C}, \mathbf{n}$  or  $\mathbf{c}$ , respectively. There are  $n$  terms in the brackets.

After the first entry  $i:$ , the next  $n$  columns give the cross action of the simple roots. An entry  $j$  in column  $k$  of row  $i$  means that the cross action of the  $k^{\text{th}}$  simple root takes  $x_i$  to  $x_j$ .

The next  $n$  columns give Cayley transforms by non-compact imaginary roots. There is an entry in column  $k$  of row  $i$  only if the  $k^{\text{th}}$  simple root is non-compact imaginary for  $x_i$ ; in this case an entry  $j$  means this Cayley transform takes  $x_i$  to  $x_j$ . These Cayley transforms are single valued (the inverse, multivalued Cayley transforms are not listed).

The penultimate entry in row  $i$  is  $\dim(\mathcal{O}_i) - \dim(\mathcal{O}_0)$ . The closed orbits are listed first; there is at least one such orbit (**#0**), and all such orbits have the same dimension.

Recall there is a map  $p : \mathcal{X}^r \rightarrow \mathcal{I}_W \subset W^\Gamma$ . Then  $p(x_i) = w\delta$  for some  $w \in W$ , given by the last entry in the row as a product of simple reflections.

Here is the  $SL(2, \mathbb{R})$  example again:

```

0:    1    2  [n]  0
1:    0    2  [n]  0
2:    2    *  [r]  1  1

```

Here is the information we can read off from this output. Orbits **#0** and **#1** are closed and 0-dimensional, and orbit **#2** is one dimensional. The first two orbits are interchanged by the cross action of the unique simple root.

For orbits #0,#1 this root is non-compact, and the Cayley transform takes each of this orbits to orbit #2.

See [1, Example 14.19] for a detailed discussion of  $Sp(4, \mathbb{R})$ .

**Example 9.3** Here is  $SL(3, \mathbb{R})$ :

```
main: type
Lie type: A2 sc s
main: kgb
there is a unique real form: sl(3,R)
kgbsize: 4
Name an output file (hit return for stdout):
0:  2  1  *  *  [CC]  0
1:  1  0  3  *  [nC]  1  2,1
2:  0  2  *  3  [Cn]  1  1,2
3:  3  3  *  *  [rr]  2  1,2,1
```

This is not an equal rank case; the dimension of the closed orbit is 1, and the dimension of the unique open orbit is  $1 + 2 = 3$ , the dimension of  $G/B$ .

**Example 9.4** Here are the equal rank real forms of  $SL(4, \mathbb{C})$ . See Example 7.6.

```
empty: type
Lie type: A3 sc c
main: realform
(weak) real forms are:
0: su(4)
1: su(3,1)
2: su(2,2)
enter your choice: 2
real: kgb
kgbsize: 21
Name an output file (return for stdout, ? to abandon): kgb_SU22

real: realform
(weak) real forms are:
0: su(4)
1: su(3,1)
```

```

2: su(2,2)
enter your choice: 1
real: kgb
kgbsize: 10
Name an output file (return for stdout, ? to abandon): kgb_SU31

```

```

real: realform
(weak) real forms are:
0: su(4)
1: su(3,1)
2: su(2,2)
enter your choice: 0
real: kgb
kgbsize: 1
Name an output file (return for stdout, ? to abandon): kgb_SU4

```

The number of  $K$  orbits on  $G/B$  is 21 for  $SU(2, 2)$ , 10 for  $SU(3, 1)$  and 1 for  $SU(4)$ . Recall (cf. Example 7.6) there are two strong real forms mapping to  $SU(4)$  and  $SU(3, 1)$ , and one mapping to  $SU(2, 2)$ . This means that the set  $K \backslash G/B$  for  $SU(3, 1)$ , of order 10, appears twice in  $\mathcal{X}^r$ , and similiary for  $SU(4)$ . Therefore the order of  $\mathcal{X}^r$  is

$$(9.5) \quad 2 \times 1 + 2 \times 10 + 1 \times 21 = 43.$$

This agrees with the counting done a different way in Example 7.6 (cf. (7.7)).

**Example 9.6** We do the same example again for equal rank forms of the adjoint group  $PSL(4, \mathbb{C})$ .

```

main: type
Lie type: A3 ad c
main: realform
(weak) real forms are:
0: su(4)
1: su(3,1)
2: su(2,2)
enter your choice: 2
real: kgb
kgbsize: 12

```

```

Name an output file (return for stdout, ? to abandon): kgb_PSU22
real: realform
(weak) real forms are:
0: su(4)
1: su(3,1)
2: su(2,2)
enter your choice: 1
real: kgb
kgbsize: 10
Name an output file (return for stdout, ? to abandon): kgb_PSU31
real: realform
(weak) real forms are:
0: su(4)
1: su(3,1)
2: su(2,2)
enter your choice: 0
real: kgb
kgbsize: 1
Name an output file (return for stdout, ? to abandon): kgb_PSU4

```

In this case  $\mathcal{X}^r$  has order  $12 + 10 + 1 = 23$ . See Example 6.2

**Example 9.7** Complex Groups.

Suppose  $G(\mathbb{C}) = G_1(\mathbb{C}) \times G_1(\mathbb{C})$  and  $\theta(g, h) = (h, g)$ . Then  $G(\mathbb{R}) = G_1(\mathbb{C})$ ,  $K(\mathbb{C}) = G(\mathbb{C})^\theta = G_1(\mathbb{C})^\Delta \simeq G_1(\mathbb{C})$ , and  $K(\mathbb{R})$  is the compact form of  $G_1(\mathbb{C})$ . The map  $\phi(g, h) = gh^{-1}$  takes  $G_1(\mathbb{C}) \times G_1(\mathbb{C})$  to  $G_1(\mathbb{C})$ . It is well known, and not hard to see, that this induces an isomorphism

$$(9.8) \quad G_1(\mathbb{C})^\Delta \backslash G_1(\mathbb{C}) \times G_1(\mathbb{C}) / B_1(\mathbb{C}) \simeq B_1(\mathbb{C}) \backslash G_1(\mathbb{C}) / B_1(\mathbb{C}).$$

Thus the orbits of  $K$  on  $G/B$  are in bijection with orbits of  $B_1(\mathbb{C})$  on  $G_1(\mathbb{C})/B_1(\mathbb{C})$ , which is described by the well known theory of the Schubert calculus. In particular the orbits are parametrized by  $W$ .

For example suppose  $G(\mathbb{R}) = SL(n, \mathbb{C})$ , i.e. take  $G_1(\mathbb{C}) = SL(n, \mathbb{C})$  and  $G(\mathbb{C}) = SL(n, \mathbb{C}) \times SL(n, \mathbb{C})$ . Then  $K(\mathbb{C}) = SL(n, \mathbb{C})$  and  $K(\mathbb{R}) = SU(n)$ .

**Example 9.9** Here is the real group  $SL(3, \mathbb{C})$ :

```
main: type
```

```

Lie type: A2.A2 sc C
main: kgb
there is a unique real form: sl(3,C)
kgbsize: 6
Name an output file (hit return for stdout):
0:  2  1  2  1  *  *  *  *  [CCCC]  0
1:  4  0  3  0  *  *  *  *  [CCCC]  1  2,4
2:  0  3  0  4  *  *  *  *  [CCCC]  1  1,3
3:  5  2  1  5  *  *  *  *  [CCCC]  2  2,1,3,4
4:  1  5  5  2  *  *  *  *  [CCCC]  2  1,2,4,3
5:  3  4  4  3  *  *  *  *  [CCCC]  3  1,2,1,3,4,3

```

Since all roots are complex there are no Cayley transforms. The cross action of the Weyl group  $W(A_2) \simeq S^3$  is simply transitive. The last column gives the Weyl group element  $(w, w)$  as a product of simple reflections; just taking the first half of each entry we see  $W = \{id, s_2, s_1, s_2s_1, s_1s_2, s_1s_2s_1\}$ . See Example 8.11.

## 10 Representation Theory: Blocks and the two-sided parameter space

The `block` command gives information about blocks of representations. Recall (Section 2) a block of representations of  $G(\mathbb{R})$  is a set of representations with fixed infinitesimal character. An irreducible representation is determined by an element  $(x, y)$  of  $\mathcal{Z}^r$ , and the block containing this representation corresponds to  $\mathcal{X}[x] \times \mathcal{X}^\vee[y] \subset \mathcal{Z}^r$ . Thus a block is determined by a pair: (strong real form of  $G$ , strong real form of  $G^\vee$ ).

There is an important subtle point here. Fix  $x \in \mathcal{X}^r$  with corresponding real group  $G(\mathbb{R})$ . Suppose  $y, y' \in \mathcal{X}^{\vee r}$  define different strong real forms ( $y$  is not  $G^\vee$ -conjugate to  $y'$ ) but the same (weak) real form. Then the blocks  $\mathcal{B}, \mathcal{B}'$  for  $G(\mathbb{R})$  defined by  $(x, y)$  and  $(x, y')$  are isomorphic, but not equal. The output of `block` does not distinguish between these two blocks, and is therefore determined by the choice of a *weak* real form only. The result is that the output of `block` only shows one block here; there are  $k$  isomorphic blocks.

The cases of real forms of  $SL(2, \mathbb{C})$  and  $PSL(2, \mathbb{C})$  are explained in detail in [1, Example 12.20]. Here is a brief summary.

**Example 10.1**  $SL(2, \mathbb{R})$  and  $SU(2)$ :

```
main: type
Lie type: A1 sc s
main: realform
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
real: block
possible (weak) dual real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
Name an output file (return for stdout, ? to abandon):
0(0,1):  1    (2,*)    [i1]  0
1(1,1):  0    (2,*)    [i1]  0
2(2,0):  2    (*,*)    [r1]  1  1
real: block
possible (weak) dual real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 0
Name an output file (return for stdout, ? to abandon):
0(2,0):  0    (*,*)    [rn]  1  1
```

Thus  $SL(2, \mathbb{R})$  has two blocks, of size 3 and 1 respectively. We can take the infinitesimal character to be  $\rho$ , and the big block to be the two discrete series representations and the trivial representation. The singleton block is the irreducible principal series representation (with odd  $K$ -types) with infinitesimal character  $\rho$ .

Of course  $SU(2)$  has a single block, consisting of a finite dimensional representation (the trivial representation if we take infinitesimal character  $\rho$ ):

```
real: realform
(weak) real forms are:
0: su(2)
1: sl(2,R)
```

```

enter your choice: 0
real: block
there is a unique dual real form choice: sl(2,R)
Name an output file (return for stdout, ? to abandon):
0(0,1): 0    (*,*)    [ic]  0

```

**Example 10.2** The groups  $PSL(2, \mathbb{R}) \simeq SO(2, 1)$  and  $SO(3)$  give the dual picture to the preceding one:

```

Lie type: A1 ad s
main: realform
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
real: block
possible (weak) dual real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
Name an output file (return for stdout, ? to abandon):
0(0,2): 0    (1,2)    [i2]  0
1(1,0): 2    (*,*)    [r2]  1  1
2(1,1): 1    (*,*)    [r2]  1  1

```

The group  $SO(2, 1)$  has a block of three elements (dual to the big block of  $SL(2, \mathbb{R})$ ). We can take the infinitesimal character to be  $\rho$ , and this block consists of the unique discrete series representation, the trivial representation, and the  $\text{sgn}$  representation. These are the only irreducible representation of  $SO(2, 1)$  with infinitesimal character  $\rho$ .

Since  $PSL(2, \mathbb{C})$  is not simply connected, the set  $\Lambda$  of Theorem 2.5 has two elements, which we can take to be  $\rho$  and  $2\rho$ . Now  $SO(2, 1)$  has two irreducible representation  $PS_{\pm}$  at infinitesimal character  $2\rho$ . Both are irreducible principal series;  $PS_{+}$  contains the trivial (lowest)  $K$ -type, and  $PS_{-}$  contains the  $\text{sgn}$   $K$ -type. These two representations differ by tensoring with  $\text{sgn}$ . Both  $PS_{+}$  and  $PS_{-}$  are blocks containing a single irreducible representation.

Here is the relevant `atlas` command:

```

real: block
possible (weak) dual real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 0
Name an output file (return for stdout, ? to abandon):
0(1,0):  0    (*,*)    [rn]  1  1

```

This is an example of the subtlety discussed above. There are two strong real forms of (the dual group)  $SL(2, \mathbb{C})$  mapping to the compact form  $SU(2)$  (see Example 10.1); call them  $SU(2, 0)$  and  $SU(0, 2)$ . Then one block (say  $\{PS_+\}$ ) is dual to the trivial representation of  $SU(2, 0)$ , and the other ( $\{PS_-\}$ ) is dual to the trivial representation of  $SU(0, 2)$ . See the table at the end of [1, Section 12], and Example 10.4.

We illustrate the other information in the output of `block` by looking at  $Sp(4, \mathbb{R})$ . See [1, Example 14.19].

**Example 10.3** main: type

```

Lie type: C2 sc s
main: block
(weak) real forms are:
0: sp(2)
1: sp(1,1)
2: sp(4,R)
enter your choice: 2
possible (weak) dual real forms are:
0: so(5)
1: so(4,1)
2: so(2,3)
enter your choice: 2
Name an output file (return for stdout, ? to abandon):
0( 0,6):  1  2    ( 6, *) ( 4, *)    [i1,i1]  0
1( 1,6):  0  3    ( 6, *) ( 5, *)    [i1,i1]  0
2( 2,6):  2  0    ( *, *) ( 4, *)    [ic,i1]  0
3( 3,6):  3  1    ( *, *) ( 5, *)    [ic,i1]  0
4( 4,4):  8  4    ( *, *) ( *, *)    [C+,r1]  1  2
5( 5,4):  9  5    ( *, *) ( *, *)    [C+,r1]  1  2
6( 6,5):  6  7    ( *, *) ( *, *)    [r1,C+]  1  1

```

```

7( 7,2):  7  6  (10,11) ( *, *)  [i2,C-]  2  2,1,2
8( 8,3):  4  9  ( *, *) (10, *)  [C-,i1]  2  1,2,1
9( 9,3):  5  8  ( *, *) (10, *)  [C-,i1]  2  1,2,1
10(10,0): 11 10  ( *, *) ( *, *)  [r2,r1]  3  1,2,1,2
11(10,1): 10 11  ( *, *) ( *, *)  [r2,rn]  3  1,2,1,2

```

This block has 12 representations, labelled  $0, \dots, 11$ . These are parametrized by pairs  $(x, y)$ , the second entry on each line, from the corresponding `kgb` commands for  $G$  and  $G^V$ :

```

main: kgb
(weak) real forms are:
0: sp(2)
1: sp(1,1)
2: sp(4,R)
enter your choice: 2
kgbsize: 11
Name an output file (return for stdout, ? to abandon):
0:  1  2  6  4  [nn]  0
1:  0  3  6  5  [nn]  0
2:  2  0  *  4  [cn]  0
3:  3  1  *  5  [cn]  0
4:  8  4  *  *  [Cr]  1  2
5:  9  5  *  *  [Cr]  1  2
6:  6  7  *  *  [rC]  1  1
7:  7  6  10 *  [nC]  2  2,1,2
8:  4  9  *  10 [Cn]  2  1,2,1
9:  5  8  *  10 [Cn]  2  1,2,1
10: 10 10  *  *  [rr]  3  1,2,1,2
main: type
Lie type: C2 ad s
main: kgb
(weak) real forms are:
0: sp(2)
1: sp(1,1)
2: sp(4,R)
enter your choice: 2
kgbsize: 7
Name an output file (return for stdout, ? to abandon):

```

0:	0	1	3	2	[nn]	0
1:	1	0	*	2	[cn]	0
2:	5	2	*	*	[Cr]	1 2
3:	3	4	*	*	[rC]	1 1
4:	4	3	6	*	[nC]	2 2,1,2
5:	2	5	*	6	[Cn]	2 1,2,1
6:	6	6	*	*	[rr]	3 1,2,1,2

Thus  $|K \backslash G / B| = 11$  and  $K^\vee \backslash G^\vee / B^\vee = 7$ .

The next two columns give cross actions of the simple roots, similar to the `kgb` command, followed by two columns for Cayley transforms. These follow from the cross action/Cayley transforms for `kgb` on both the  $G$  and  $G^\vee$  side.

For example  $s_1 \times \pi_0 = \pi_1$ , or in terms of  $(x, y)$  parameters  $s_1 \times (0, 6) = (1, 6)$ . For  $G$  we have  $s_1 \times 0 = 1$  (from the output of `kgb` for  $G$ ) and for  $G^\vee$  we have  $s_1 \times 6 = 6$  (from the output of `kgb` for  $G^\vee$ ).

In this setting (unlike `kgb`) these can be double valued, even for a non-compact imaginary root, since this corresponds to a real root on the dual side.

The term in brackets list the roots as

- compact imaginary: `ic`
- noncompact imaginary, type I: `i1`
- noncompact imaginary, type II: `i2`
- complex: `C+, C-`
- real, not satisfying the parity condition: `rn`
- real, satisfying the parity condition type I: `r1`
- real, satisfying the parity condition type II: `r2`

The penultimate column gives the *length* of this parameter, and the final column is the corresponding twisted involution, exactly as in the `kgb` command for  $x$ .

The `blocksizes` command gives the sizes of all blocks. (As in the example of  $SO(2,1)$  above, if  $G$  is not simply connected there may be more than one block of each size.)

The output of `blocksizes` has one row for each real form of  $G$ , and one column for each real form of  $G^\vee$ .

**Example 10.4** Here are real forms of  $SL(2, \mathbb{C})$ :

```
main: type
Lie type: A1 sc s
main: blocksizes
  0  1
  1  3
```

or adding some labelling by hand:

	SO(3)	SO(2,1)
SU(2)	0	1
SL(2,R)	1	3

Thus  $SU(2)$  has a single block, dual to a block of  $SO(2,1)$ , and  $SL(2, \mathbb{R})$  has two blocks, dual to  $SO(3)$ , and  $SO(2,1)$ , respectively. Since the dual group is adjoint, there is only one block of each size.

**Example 10.5** The corresponding output for  $PSL(2, \mathbb{C})$  is the same, but there is a subtle point here.

```
main: type
Lie type: A1 sc s
main: blocksizes
      SU(2) SL(2,R)
SO(3)  0     1
SO(2,1) 1     3
```

(labelling added). Now the dual group  $SL(2, \mathbb{C})$  is not adjoint, and the map from strong real forms to real forms is not injective (there are two strong real forms mapping to  $SU(2)$ ). Consequently there are two blocks of  $SO(2,1)$  of size 1. See Example 10.2. We could display this information by constructing a table by hand showing *strong* real forms on the dual side:

	SU(2,0)	SU(0,2)	SL(2,R)
SO(3)	0	0	1
SO(2,1)	1	1	3

The output of `block` tends to be concentrated on the quasisplit forms of  $G$  or  $G^V$ :

**Example 10.6** Here is the output of `blocksizes` for of  $Sp(12, \mathbb{C})$ , with the real forms added:

```
Lie type: C6 sc s
main: blocksizes
```

	SO(13)	SO(12,1)	SO(11,2)	SO(10,3)	SO(9,4)	SO(8,5)	SO(7,6)
Sp(6)	0	0	0	0	0	0	1
Sp(5,1)	0	0	0	0	0	0	36
Sp(4,2)	0	0	0	0	0	0	315
Sp(3,3)	0	0	0	0	0	0	680
Sp(12,R)	1	13	108	556	1975	4707	7416

**Example 10.7** In the case of equal rank real forms of  $SO(12, \mathbb{C})$  we get some non-diagonal elements:

```
main: type
Lie type: D6
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,1/2

enter inner class(es): e
main: blocksizes
```

	SO(12)	SO(10,2)	SO*(12)	SO*(12)	SO(8,4)	SO(6,6)
SO(12)	0	0	0	0	0	1
SO(10,2)	0	0	0	0	15	66
SO*(12)	0	0	0	60	0	692
SO*(12)	0	0	60	0	0	692
SO(8,4)	0	15	0	0	300	885
SO(6,6)	1	66	692	692	885	2320

Note that this diagram is symmetric:  $SO(12, \mathbb{C})$  is self-dual (and the equal rank and split inner classes coincide). For a discussion of the two version of  $\mathfrak{so}^*(12)$  see Example 4.4.

**Example 10.8** We break the symmetry between  $G$  and  $G^\vee$  of the previous example by taking  $Spin(12, \mathbb{C})$ , which is dual to  $PSO(12, \mathbb{C})$ .

```
main: type
Lie type: D6 sc s
main: blocksizes
      PSO(12) PSO(10,2) PSO*(12) PSO*(12) PSO(8,4) PSO(6,6)
Spin(12)  0      0      0      0      0      1
Spin(10,2) 0      0      0      0      15     87
Spin*(12)  0      0      0      60     0     692
Spin*(12)  0      0      60     0      0     692
Spin(8,4)  0      15     0      0      300    915
Spin(6,6)  1      66     436    436    885    2180
```

There is an interesting phenomenon here. Note that  $Spin(6, 6)$  has a block  $\mathcal{B}_{spin}$  of order 66, dual to a block of  $PSO(10, 2)$ . From the previous example  $SO(6, 6)$  also has a block of order 66, denoted  $\mathcal{B}_{SO}$ , dual to a block for  $SO(10, 2)$ . From the output of the `block` command one can see these blocks are isomorphic.

All representations in a block have the same central character, and the representations  $\mathcal{B}_{Spin}$  factor to the image of  $Spin(6, 6)$  in  $SO(6, 6)$ . Now  $Spin(6, 6)$  is connected, and  $SO(6, 6)$  has two components, so the image of this map has index 2 in  $SO(6, 6)$ . Therefore it is not obvious that  $\mathcal{B}_{Spin}$  and  $\mathcal{B}_{SO}$  should be isomorphic.

The explanation is seen by looking at the dual side. The map  $SO(10, 2) \rightarrow PSO(10, 2)$  is surjective. It follows easily that the dual blocks are isomorphic, hence the blocks themselves are isomorphic.

We leave it to the reader to see that the symmetry between the two versions of  $SO^*(12)$  is broken by taking the “non-diagonal” quotient of  $SO(12, \mathbb{C})$ . See Examples 3.7 and 4.4.

**Example 10.9** We conclude with a large example, for the (simply connected) real form of  $E_6$  with  $K$  of type  $A_5 \times A_1$ .

```

main: typex
Lie type: E6 sc e
main: block
(weak) real forms are:
0: e6
1: e6(so(10).u(1))
2: e6(su(6).su(2))
enter your choice: 2
possible (weak) dual real forms are:
0: e6(f4)
1: e6(R)
enter your choice: 0
Name an output file (return for stdout, ? to abandon):

```

**Block for  $E_6(A_5 \times A_1)$  (simply connected), dual to  $E_6(F_4)$  (adjoint)**

```

0( 851,44): 2 0 0 0 0 1 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C+,rn,rn,rn,rn,C+]
1(1013,43): 4 1 1 1 3 0 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C+,rn,rn,rn,C+,C-]
2(1014,42): 0 2 5 2 2 4 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C-,rn,C+,rn,rn,C+]
3(1165,41): 8 3 3 6 1 3 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C+,rn,rn,C+,C-,rn]
4(1166,40): 1 4 9 4 8 2 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C-,rn,C+,rn,C+,C-]
5(1167,39): 5 5 2 7 5 9 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [rn,rn,C-,C+,rn,C+]
6(1304,38): 11 10 10 3 6 6 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C+,C+,C+,C-,rn,rn]
7(1305,37): 7 14 7 5 14 12 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [rn,C+,rn,C-,C+,C+]
8(1306,36): 3 8 13 11 4 8 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C-,rn,C+,C+,C-,rn]
9(1307,35): 9 9 4 12 13 5 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [rn,rn,C-,C+,C+,C-]
10(1430,34): 18 6 6 10 10 10 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C+,C-,C-,rn,rn,rn]
11(1431,33): 6 18 15 8 11 11 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C-,C+,C+,C-,rn,rn]
12(1432,32): 12 19 12 9 17 7 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [rn,C+,rn,C-,C+,C-]
13(1433,31): 13 13 8 16 9 13 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [rn,rn,C-,C+,C-,rn]
14(1434,30): 14 7 14 14 7 19 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [rn,C-,rn,rn,C-,C+]
15(1536,29): 24 24 11 20 15 15 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C+,C+,C-,C+,rn,rn]
16(1537,28): 16 22 20 13 21 16 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [rn,C+,C+,C-,C+,rn]
17(1538,27): 17 23 17 21 12 23 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [rn,C+,rn,C+,C-,C+]
18(1539,26): 10 11 24 18 18 18 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C-,C-,C+,rn,rn,rn]
19(1540,25): 19 12 19 19 23 14 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [rn,C-,rn,rn,C+,C-]
20(1620,24): 28 26 16 15 25 20 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C+,C+,C-,C-,C+,rn]
21(1621,23): 21 29 25 17 16 27 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [rn,C+,C+,C-,C-,C+]
22(1622,22): 22 16 26 22 29 22 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [rn,C-,C+,rn,C+,rn]
23(1623,21): 23 17 23 27 19 17 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [rn,C-,rn,C+,C-,C-]
24(1624,20): 15 15 18 28 24 24 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C-,C-,C-,C+,rn,rn]
25(1684,19): 32 31 21 25 20 30 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C+,C+,C-,rn,C-,C+]
26(1685,18): 34 20 22 34 31 26 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C+,C-,C-,C+,C+,rn]
27(1686,17): 27 33 30 23 27 21 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [rn,C+,C+,C-,rn,C-]
28(1687,16): 20 34 28 24 32 28 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C-,C+,rn,C-,C+,rn]
29(1688,15): 29 21 31 33 22 33 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [rn,C-,C+,C+,C-,C+]
30(1730,14): 37 36 27 30 30 25 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C+,C+,C-,rn,rn,C-]
31(1731,13): 38 25 29 35 26 36 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C+,C-,C-,C+,C-,C+]
32(1732,12): 25 38 32 32 28 37 (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) (*,*) [C-,C+,rn,rn,C-,C+]

```

```

33(1733,11): 33 27 36 29 33 29 ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) [rn,C-,C+,C-,rn,C-]
34(1734,10): 26 28 34 26 38 34 ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) [C-,C-,rn,C-,C+,rn]
35(1760, 9): 40 35 39 31 40 39 ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) [C+,rn,C+,C-,C+,C+]
36(1761, 8): 41 30 33 39 36 31 ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) [C+,C-,C-,C+,rn,C-]
37(1762, 7): 30 41 37 37 37 32 ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) [C-,C+,rn,rn,rn,C-]
38(1763, 6): 31 32 38 40 34 41 ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) [C-,C-,rn,C+,C-,C+]
39(1777, 5): 43 39 35 36 42 35 ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) [C+,rn,C-,C-,C+,C-]
40(1778, 4): 35 40 42 38 35 43 ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) [C-,rn,C+,C-,C-,C+]
41(1779, 3): 36 37 41 43 41 38 ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) [C-,C-,rn,C+,rn,C-]
42(1786, 2): 44 42 40 42 39 44 ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) [C+,rn,C-,rn,C-,C+]
43(1787, 1): 39 43 44 41 44 40 ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) [C-,rn,C+,C-,C+,C-]
44(1790, 0): 42 44 43 44 43 42 ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) ( *, * ) [C-,rn,C-,rn,C-,C-]

```

(The last two columns of the output of `block`, have been deleted.)

Note that in this case there are no Cayley transforms; all of these representations come from a single Cartan subgroup  $H(\mathbb{R})$ , which is isomorphic to  $S^1 \times S^1 \times \mathbb{C}^\times \times \mathbb{C}^\times$ . Although this group has 5 conjugacy classes of Cartan subgroups (see the `cartan` command), including a compact Cartan subgroup, but not a split one, the dual block is for the adjoint group  $E_6(F_4)$ , which has only one conjugacy class of Cartan subgroups (cf. Example 5.4).

There are 1,791 orbits of  $K$  on  $G/B$  for  $G = E_6(A_5 \times A_1)$ , (as given by the `kgb` command). For the dual group  $E_6(F_4)$  there are 45 such orbits. The fact that there are precisely 45 representations in this block, with distinct  $y$  values, is due to the fact that  $H(\mathbb{R})$  is connected. The fact that 45 distinct  $x$  values  $0 \leq x \leq 1790$  occur is due to the fact that the dual Cartan subgroup is also connected. (The remaining 1746  $x$  values play a role in other blocks of  $E_6(A_5 \times A_1)$ ).

## 11 Appendix: List of commands

- `block`
- `blockd`
- `blocksizes`
- `blockstabilizer`
- `blocku`
- `blockwrite`
- `cartan`

- cmatrix
- components
- corder
- coroots
- coroots\_rootbasis
- dualblock
- dualkgb
- dualmap
- extract-cells
- extract-graph
- gradings
- help
- kgb
- klbasis
- kllist
- klwrite
- poscoroots
- poscoroots\_rootbasis
- posroots
- posroots\_rootbasis
- primkl
- q
- qq

- realform
- realweyl
- rootdatum
- roots
- roots\_rootbasis
- showdualforms
- showrealforms
- simplecoroots
- simpleroots
- special
- strongreal
- test
- type
- wcells
- wgraph

## References

- [1] Jeffrey Adams and F. du Cloux. Algorithms for representation theory of real reductive groups I. preprint.
- [2] David A. Vogan, Jr. *Representations of Real Reductive Lie Groups*, volume 15 of *Progress in mathematics*. Birkhäuser, Boston, 1981.
- [3] David A. Vogan, Jr. Irreducible characters of semisimple Lie groups IV. character-multiplicity duality. *Duke Math. J.*, 49, No. 4:943–1073, 1982.