Square Integrable Harmonic Forms and Unitary Representations

L. Barchini

1. Introduction

Perhaps the most important unsolved problem about the representation theory of semisimple Lie groups concerns the unitary dual. That is, one wishes to classify and construct the irreducible unitary representations for such a group. This note concerns the latter part of the problem and is intended as an introduction to the techniques of [BZ].

An interesting class of irreducible unitary representations consists of the representations associated to an elliptic orbit. Associated to these orbits is the representation of G occuring on the Dolbeault cohomology space $H^S(G/L, \mathbf{C}_\lambda)$. The Dolbeault space is a maximal globalization in the sense of Schmid. It follows that the unitary globalization lies inside the cohomology space. Thus, it is reasonable to look for "L_2" representatives for cohomology classes and define an invariant inner product.

Often the only invariant hermitian metric available is indefinite. The indefinite metric can be used to define a global invariant hermitian form on the space of differential forms. Of course the integral defining this global form must converge. In [BZ] the authors choose representatives for each K-finite cohomology class for which the global form converges.

In case G/L is an indefinte Kähler symmetric space they obtain a stronger result. They build a Hilbert space of square integrable differential forms containing representatives of the K-finite cohomology classes. On this L_2-space the integral defining the global invariant form converges and this form is semidefinite. (The authors did not yet prove in general that the invariant form is nonzero).

The technique is to use an explicit intertwining operator, constructed in [BKZ] and [B1], giving an integral formula for harmonic forms on G/L. We review the construction in section 7. See [B2] for an elementary introduction. Square integrability is shown by using Harish-Chandra type estimates.

The results on these notes are joint work of the author and R. Zierau. The notes were writen by R. Donley.

2. Dolbeault Cohomology

Dolbeault cohomolgy provides a framework for constructing unitary representation; although the full space cannot carry a unitary structure in general, one can hope to extract a smaller subspace which does. We indicate the general setting.

The first step in the construction consists of a homogeneous complex manifold G/L and a choice of complex structure. These manifolds will be open G-orbits in a generalized flag manifold.

Next one constructs a homogeneous holomorphic vector bundle

$$G \times_L V \to G/L,$$

where V is a finite-dimensional representation space for L. Associated to this bundle is a $\bar{\partial}$-operator. The choice of complex structure gives a splitting

$$T(G/L)^{\mathbb{C}}_{eL} = T^{1,0}_{eL}(G/L) + T^{0,1}_{eL}(G/L).$$

Let $C^{0,i}(G/L, V)$ denote the space of smooth V-valued $(0,i)$-forms. The $\bar{\partial}$-operator carries

$$\bar{\partial}_i : C^{0,i}(G/L, V) \to C^{0,i+1}(G/L, V)$$

and satisfies $\bar{\partial}_i \circ \bar{\partial}_{i-1} = 0$.

The $(0,i)$-th Dolbeault cohomology is defined as

$$H^{0,i}(G/L, V) = Ker\ \bar{\partial}_i / Im\ \bar{\partial}_{i-1}.$$

At this stage, technical problems need to be considered.

Problem 1 Show $H^{0,i}(G/L, V)$ is a "good" topological vector space. That is, show $Im\ \bar{\partial}$ is closed.
Problem 2 "Unitarize the cohomology". The cohomology spaces $H^{0,i}(G/L, V)$ are maximal globalizations. The unitary globalization lies inside the cohomology space.

For problem 1, solved by H.-W. Wong, the Dolbeault spaces indeed have Frechét topologies. To address problem 2, we formulate a strategy as follows (after [RSW]):

(1) show that the K-finite cohomology classes admit "harmonic representatives,"
(2) control difficulties encountered since the natural Hermitian L-invariant form may not be positive definite, and
(3) show that the "harmonic representatives" are square-integrable.

3. Examples for G/L

Example 1 Let $Z = Gr_r(\mathbb{C}^{p+q})$ be the Grassmanian manifold of r-dimensional planes in $\mathbb{C}^{p+q}$. As a homogeneous space,

$$Z = G^{\mathbb{C}}/Q,$$

where $G^{\mathbb{C}} = GL(p+q, \mathbb{C})$ and Q is the subset of $G^{\mathbb{C}}$ consisting of block upper triangular matrices with block diagonal subgroup

$$L^{\mathbb{C}} = GL(r, \mathbb{C}) \times GL(p+q-r, \mathbb{C}).$$

Here $GL(k, \mathbb{C})$ denotes the space of invertible $k \times k$-matrices with complex entries.

On $\mathbb{C}^{p+q}$ we introduce the Hermitian form

$$\langle z, w\rangle = \sum_{i=1}^{p} z_i \bar{w}_i - \sum_{j=p+1}^{p+q} z_j \bar{w}_j .$$

The subgroup $U(p,q)$ of $G^{\mathbb{C}}$ preserves $\langle \cdot, \cdot \rangle$. When $r < p$, define D_r to be the (open) $U(p,q)$-orbit of eQ in Z. Geometrically, D_r is the set of r-dimensional planes in $\mathbb{C}^{p+q}$ which are positive definite with respect to $\langle \cdot, \cdot \rangle$. As a smooth homogeneous space,

$$D_r \cong U(p,q)/(U(r) \times U(p-r,q)).$$

This space is a holomorphic orbit; it allows a holomorphic embedding into some $\mathbb{C}^N$.

Example 2 Consider the generalized flag manifold $Gr_2(\mathbb{C}^4)$ and define a Hermitian form on $\mathbb{C}^4$ by

$$\Phi(z,w) = z_1\bar{w}_1 + z_2\bar{w}_2 - z_3\bar{w}_3 - z_4\bar{w}_4 .$$

Let

$$SU(2,2) = \{x \in GL(4,\mathbb{C}) \ : \ x \text{ preserves } \Phi, \ \det x = 1\},$$

and

$$D_{+-} = \{z \in Gr_2(\mathbb{C}^4) \ : \ \Phi|_z \text{ has signature } (1,1)\}.$$

Thus D_{+-} can be characterized as the space of planes in $\mathbb{C}^4$ with each basis element chosen from exactly one of P^+ or P^-, the subspaces of positive and negative lines in $(\mathbb{C}^4, \Phi)$. Geometrically we have

$$D_{+-} \cong P^+ \times P^- / \sim,$$

where $(X,Y) \sim (W,Z)$ if $\operatorname{span}_{\mathbb{C}}\{X,Y\} = \operatorname{span}_{\mathbb{C}}\{W,Z\}$; as a homogenous space,

$$D_{+-} = SU(2,2) \cdot \langle e_1, e_4\rangle \cong SU(2,2)/S(U(1,1) \times U(1,1)),$$

where e_i denotes the i-th standard basis vector for $\mathbb{C}^4$. Although D_{+-} is a complex manifold, it does not allow a holomorphic embedding into some $\mathbb{C}^N$. Hence D_{+-} is called a nonholomorphic orbit.

Example 3 For $n > 2$, consider $\mathbb{C}^{n+1}$ with the nondegenerate symmetric form

$$\varepsilon(z,w) = (\sum_{i=1}^{n} z_i w_i) - z_{n+1} w_{n+1}.$$

Define a Hermitian form

$$\langle z, w \rangle = \varepsilon(z, \bar{w})$$

and define $SO(n,1)$ to be the set of real matrices which preserve this form. Define the smooth quadric

$$Z = \{[z] \in \mathbb{C}P^n \ : \ \varepsilon(z,z) = 0\};$$

as a homogenous space,

$$Z = SO(n+1,\mathbb{C})/Q$$

where

$$Q = Stab_{SO(n+1,\mathbb{C})}(1, i, 0, \ldots, 0).$$

Let D be the (open) $SO(n,1)$-orbit of eQ in Z. As a homogeneous space,

$$D \cong SO(n,1)/(SO(2) \times SO(n-2,1)).$$

This is the subset of Z consisting of z such that

$$\langle z, z \rangle > 0.$$

For the general construction of G/L, one needs:

(1) a "linear" connected semisimple Lie group G with complexification $G^{\mathbb{C}}$,
(2) a maximal compact subgroup $K \subset G$, and
(3) the associated Lie algebras $\mathfrak{g}_0, \mathfrak{k}_0$ and their complexifications $\mathfrak{g}, \mathfrak{k}$.

As a convention, real Lie algebras will have a zero subscript; deleting this subscript denotes the complexification. Item (2) yields a Cartan involution for G and $\mathfrak{g}_0$; we denote the latter by θ.

Let T be a torus in K and $L = Z_G(T)$. Then L is connected with reductive Lie algebra $\mathfrak{l}_0$. The homogeneous space G/L carries several homogeneous complex structures and we fix one. Such a choice is equivalent to choosing a θ-stable parabolic subalgebra $\mathfrak{q}$ in $\mathfrak{g}$ such that

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}.$$

Form the analytic subgroup Q in $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{q}$.

Letting z_0 denote eQ, we have

$$\begin{aligned} G/L \text{ (open orbit) } &\subset G \cdot z_0 \\ &\subset G^{\mathbb{C}}/Q \text{ (generalized flag manifold)} \\ &= G^{\mathbb{C}} \cdot z_0. \end{aligned}$$

The choice of $\mathfrak{q}$ identifies $\mathfrak{u}$ with the antiholomorphic tangent space of $G^{\mathbb{C}}/Q$ at eQ. Since G/L can be identified with an open subset of $G^{\mathbb{C}}/Q$, the complex structure on $G^{\mathbb{C}}/Q$ can be pulled back to G/L. Hence

$$T^{\mathbb{C}}_{eL}(G/L) \cong \mathfrak{u} \oplus \bar{\mathfrak{u}}$$

and

$$T^{0,1}_{eL}(G/L) \cong \mathfrak{u}.$$

4. Homogeneous Vector Bundles on G/L

Given a representation (π, V) on L, there exists a natural homogeneous vector bundle $G \times_L V \to G/L$. Here $G \times_L V$ is the quotient space $G \times V/\sim$, where $\sim$ is the equivalence relation

$$(gl, v) \sim (g, \pi(l)v) \text{ for } v \in V, g \in G, l \in L.$$

Sections of this bundle can be considered as a space of functions on G that satifsy a certain translation property with respect to L. Notions of continuity or smoothness of a section translate into the corresponding property on the function. Thus we define the smooth sections

$$\begin{aligned}\mathcal{E}(G/L, V) &= ind_L^G(V) \\ &= \{f : G \to V \ : \ f(gl) = \pi(l)^{-1} f(g) \text{ for } g \in G, l \in L\}.\end{aligned}$$

The correspondence from sections to functions

$$\gamma \mapsto (\phi_\gamma : G \to V)$$

is given by

$$\gamma(gL) = (g, \phi_\gamma(g)).$$

In either case, G acts compatibly by left translations.

With a fixed complex structure $\mathfrak{u}$ on G/L, holomorphic sections on $G \times_L V$ translate into smooth functions on G as above with the property that $X\phi = 0$ for all $X \in \mathfrak{u}$. (We refer the reader to [K3] for more details.)

5. Dolbeault Cohomology and Harmonic Forms

Since the $\bar{\partial}$-operator commutes with the action of G upon smooth forms, we can consider cohomology as a method for constructing representations.

Define

$$\mathcal{E}^{0,i}(G/L, V) = \{\text{smooth } f : G \to V \otimes \wedge^i \mathfrak{u}^* \mid f(gl) = (\pi(l) \otimes Ad(l))^{-1} f(g)\}.$$

There exists an operator

$$\bar{\partial} : \mathcal{E}^{0,i}(G \times_L V) \to \mathcal{E}^{0,i+1}(G \times_L V)$$

with the property that $\bar{\partial}^2 = 0$. A precise definition of this operator in terms of root vectors can be found in [GS].

Suppose $G = K$ is compact and connected. Fix a maximal torus T, form the set of roots $\Delta(\mathfrak{k}, \mathfrak{t})$, and choose a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t})$. Define the Weyl group of K by $W_K = N_K(T)/T$, and let $l(w)$ denote the length of the element w.

Borel-Weil-Bott Theorem [Bo]. *Let (π, V) be the T–character $(\chi_\lambda, \mathbb{C}_\lambda)$, where λ is the differential of χ_λ. Set $\delta = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$.*

(1) *If $\langle \lambda + \delta, \alpha \rangle = 0$ for some $\alpha \in \Delta(\mathfrak{k}, \mathfrak{t})$ then $H^{0,k}(K/T, \mathbb{C}_\lambda) = 0$ for all k.*
(2) *If not, then there exists a unique $w \in W_K$ such that $\mu = w(\lambda + \delta) - \delta$ is dominant with respect to $\Delta^+(\mathfrak{k}, \mathfrak{t})$. Then*

$$H^{0,k}(K/T, \mathbb{C}_\lambda) = \begin{cases} V^\mu & \text{if } k = l(w), \\ 0 & \text{otherwise.} \end{cases}$$

Here V^μ denotes the representation of K with highest weight μ.

Because K/L is a compact complex manifold, these results can be reformulated using Kodaira-Hodge theory. In a natural manner, K/L carries a positive definite measure. Using this, one can define an adjoint $\bar\partial^*$ to $\bar\partial$ which also commutes with the K-action. The space of strongly harmonic $(0, i)$-forms is defined by

$$\mathcal{H}^{0,i}(K/L, \mathbb{C}_\lambda) = \{\omega \in \mathcal{E}^{0,i}(G \times_L \mathbb{C}_\lambda) \mid \omega \in \text{ Ker } \bar\partial \cap \text{ Ker } \bar\partial^*\}.$$

Because K/L is compact and ω is smooth, these forms are automatically square-integrable; thus, a potential unitary structure is evident. The Kodaira–Hodge theorem asserts that the natural map

$$\mathcal{H}^{0,i}(K/L, \mathbb{C}_\lambda) \to H^{0,i}(K/L, \mathbb{C}_\lambda)$$

is an isomorphism. Since this map is K-equivariant and we have identified the latter space, the former space gives a unitary realization of the representations in question. Explicit formulas for these harmonic forms can be found in [GS].

To attack the case of G noncompact, one needs more sophisticated techniques. In any case, the situation for Problem 1 is optimal; it turns out that $\bar\partial$ always has closed range (via [Wo]) for the constructions in question.

Early work of Schmid ([S1]) addresses this question in the case where $\operatorname{rank} G = \operatorname{rank} K$, $L = T$ and $\pi = \chi$ is a character of T. Wong was ([Wo]) was able to find similar results for general G without the rank condition. (See also [AR].)

6. Unitary Realizations on Cohomology

With respect to Problem 2, the nonvanishing Dolbeault cohomology representations of interest are not unitarizable unless we are in the compact case. In fact, these representations are always "maximal globalizations" ([S3]) of their underlying $(\mathfrak{g}, K)$-modules; this important notion unfortunately moves one away from the unitarity problem. We recall techniques for achieving unitarity.

When L is compact, G/L has a G-invariant positve definite metric $\langle \cdot, \cdot \rangle_{inv}$. Using this metric and integration, one can study the smooth square-integrable forms on G/L. In this space, one can define an adjoint $\bar\partial^*$ to $\bar\partial$. Define

$$\mathcal{H}_2^{0,i}(G/L, \mathbb{C}_\lambda) = \{\omega \in C^{0,i}(G/L, \mathbb{C}_\lambda) \mid \omega \in \text{Ker } \bar\partial \cap \text{Ker } \bar\partial^*, \ \omega \text{ square-integrable}\}.$$

Many remarkable properties of these spaces were proven by Schmid ([S2]); these spaces give a realization of the discrete series representations of G.

When L is noncompact, the situation becomes more complicated but not entirely hopeless. Two forms are defined on $T_{eL}^{\mathbb{C}}(G/L)$ as follows:

$$\langle X, Y\rangle_{inv} = B(X, \bar{Y}),$$
$$\langle X, Y\rangle_{pos} = -B(X, \theta\bar{Y}).$$

We use the same notation for the associated Hermitian forms on $C^{0,i}(G/L, \mathbb{C}_\lambda)$. Several problems need to be circumnavigated. The G-invariant form used in the compact case is now indefinite, which complicates finding a Hilbert space. A candidate for a positive definite form $\langle\cdot,\cdot\rangle_{pos}$ exists which is not G-invariant; thus the unitary structure remains unseen. [RSW] gives an outline to make this situation useful:

(1) use the indefinite form to define $\bar{\partial}^*$ and the space of "strongly harmonic forms" (forms which lie in Ker $\bar{\partial}\cap$ Ker $\bar{\partial}^*$); consider distribution solutions to the vanishing of $\bar{\partial}$ and $\bar{\partial}^*$,

(2) use $\langle\cdot,\cdot\rangle_{pos}$ to define a space of square-integrable forms; define

$$H_2^{0,i}(G/L, \mathbb{C}_\lambda) = \{\text{strongly harmonic, } L^2_{\langle\cdot,\cdot\rangle_{pos}}\ forms\},$$

and

(3) find conditions under which one obtains a G-invariant nontrivial Hilbert space.

7. Intertwinors and Cohomology

The previous section suggests putting a unitary structure on some space of harmonic forms; we give a method for accessing and using such harmonic spaces.

(1) Construct "strongly harmonic representatives" for each K-finite cohomology class; we want a surjection

$$(\text{Ker } \bar{\partial}\cap \text{Ker } \bar{\partial}^*)_{K-fin} \to H^S(G/L, \mathbb{C}_\lambda)_{K-fin} \to 0.$$

Integral intertwining operators can be used to construct such cohomology classes; [BKZ] addresses this problem when real rank $G =$ real rank L, and [Ba] handles the case without the real rank condition. We note that $S = \dim_{\mathbb{C}}(\mathfrak{u}\cap\mathfrak{k})$, the degree of nonvanishing cohomology.

(2) Define an appropriate "L^2-cohomology space" with respect to $\langle\cdot,\cdot\rangle_{inv}$ and show that the representatives are square-integrable with respect to $\langle\cdot,\cdot\rangle_{pos}$.

(3) Inside this space, we need to show that G acts continuously, that some subspace is semidefinite with respect to the norm associated to the indefinite form $\langle\cdot,\cdot\rangle_{inv}$, and that G acts unitarily.

Theorem 7.1. *Assuming a positivity condition on λ and that ρ_L is regular, each K-finite cohomology class admits a strongly harmonic representative ω for which $\langle\omega,\omega\rangle_{inv} < \infty$.*

[BKZ] and [Ba] handle the "strongly harmonic" part of the theorem; [BZ] gives "$\langle\omega,\omega\rangle_{inv} < \infty$."

(Idea of proof) For purposes of exposition, we suppose that rank G = rank L. The space we are trying to unitarize has as its underlying $(\mathfrak{g}, K)$-module the cohomologically induced module $A_{\mathfrak{q}}^S(\lambda)$. Using the Langlands classification (or some version of the subquotient theorem), $A_{\mathfrak{q}}^S(\lambda)$ is infinitesimally equivalent to a quotient of some principal series representation $I(W) = ind_P^G(W)$.

One can construct an intertwining operator

$$\mathcal{S} : I(W) \to \mathcal{E}^{0,S}(G\times_L, \mathbb{C}_\lambda) = ind_L^G(\mathbb{C}_\lambda^\# \otimes \wedge^S \mathfrak{u}^*)$$

by

$$(\mathcal{S}f)(x) = \int\limits_{L\cap K} l \cdot T[f(xl)]\ dl$$

where $T : W \to \mathbb{C}_\lambda^\# \otimes \wedge^S \mathfrak{u}^*$ is $L \cap P$-equivariant. The space $\mathbb{C}_\lambda^\#$ denotes a uniform parameter shift of L-action on $\mathbb{C}_\lambda$. With some work, one can show

(1) Im $\mathcal{S} \subset$ Ker $\bar{\partial} \cap$ Ker $\bar{\partial}^*$,
(2) $\mathcal{S}$ is onto at the K-finite level,
(3) $\langle \mathcal{S}f, \mathcal{S}f\rangle_{inv} = c\langle f, f\rangle_{pos}$ for some constant c.

The construction of $\mathcal{S}$ is based on the "heuristic principle" of [K2] for building integral intertwining operators.

Theorem 7.2. *Suppose that real rank G = real rank L, and that ρ_L is dominant. If G/L is a semisimple symmetric space then*

(1) $H_2^{0,S}(G/L, \mathbb{C}_\lambda^\#) \neq 0$,
(2) *$q : H_2^{0,s}(G/L, \mathbb{C}_\lambda^\#)_{K-fin} \to H^{0,s}(G/L, \mathbb{C}_\lambda^\#)_{K-fin}$ is surjective,*
(3) *$\overline{(Im\ \mathcal{S})}$ is a subspace of $H_2^{0,s}(G/L, \mathbb{C}_\lambda^\#)$ where $\langle\cdot,\cdot\rangle_{inv}$ is semidefinite,*
(4) *the quotient*

$$H_2^{0,s}(G/L, \mathbb{C}_\lambda^\#) = \overline{(Im\ \mathcal{S})}/Ker\{q : \overline{(Im\ \mathcal{S})} \to H^{0,s}(G/L, \mathbb{C}_\lambda^\#)\}$$

is a nontrivial Hilbert space with a continuous G-action and is positive definite with respect to the induced $\langle\cdot,\cdot\rangle_{inv}$.

8. Square-integrability Computations

We consider the case where G/L is a semisimple symmetric space with involution σ; σ is chosen to commute with θ. We assume real rank $\mathfrak{g}_0$ = real rank $\mathfrak{l}_0$.

First we give an explicit formula for $\langle\cdot,\cdot\rangle_{pos}$. It uses the decomposition ([Mo])

$$G = K \cdot \exp(\mathfrak{l}_0^\perp \cap \mathfrak{p}_0) \cdot \exp(\mathfrak{l}_0 \cap \mathfrak{p}_0),$$

where $\mathfrak{l}_0^\perp$ denotes the orthocomplement of $\mathfrak{l}_0$ with respect to the Killing form. We have

$$|\mathcal{S}f|_{pos}^2 = \int\limits_{G/L} |\mathcal{S}f(k'(x) \cdot \exp\zeta(x))|_{pos}^2\ dx$$

with $k'(x) \in K$ and $\zeta(x) \in \mathfrak{l}_0^\perp \cap \mathfrak{p}_0$.

Several ideas are key to what follows. These are

(1) avoid the integral formulas for symmetric spaces,
(2) use the "growth" of $f \in I(W)$ to conclude square-integrability, and
(3) relate $\langle \cdot, \cdot \rangle_{pos}$ to integrals on $\bar{N}$ and use classical results of Gindikin and Karpelevic, and Harish-Chandra for estimates.

We recall the change of variables formula. Let

$$x = k(x)\mu(x)n(x)e^{H(x)}$$

be the decomposition of an element $x \in G$ according to the decomposition

$$G = K\exp(\mathfrak{m}_0 \cap \mathfrak{p}_0)NA.$$

When Haar measures are properly normalized, one has

$$\int_K f(k)\ dk = \int_{\bar{N}} f(k(\bar{n}))e^{-2\rho_G(H(\bar{n}))}\ d\bar{n},$$

where f is a smooth function on K right invariant under $K \cap M$. With this normalization in place,

$$\int_{G/A} f(x)\ dx = \int_{K\times M\times\bar{N}} f(km\bar{n})\ dk\, dm\, d\bar{n}.$$

Analogous statements hold for the decomposition

$$g = \bar{k}(g)\bar{\mu}(g)\bar{n}(x)e^{\bar{H}(g)}$$

with respect to $K\exp(\mathfrak{m}_0 \cap \mathfrak{p}_0)\bar{N}A$.

(When M is noncompact, one further needs

Theorem 8.1. *If F is a $\mathcal{Z}(\mathfrak{g})$-finite K-finite function in $L^2(G)$ then for all $r > 0$, there exists a constant C_r such that*

$$|F(x)| \leq C_r\ \Xi_0(x)\ (1 + ||x||)^{-r}.$$

Here Ξ_0 is defined in [HC]. When F is in the Schwartz space, the conclusion is true for all derivatives of F.)

Finally we have a decomposition formula relating σ and θ. Suppose $\theta\sigma(x) = x$. Then

(1) $x = k(x)n(x)e^{H(x)} \in KNA$,
(2) $\sigma(x) = \sigma(k(x))\sigma(n(x))e^{H(x)}$ since $A \subset L$,
(3) $x = \theta\sigma x = \theta\sigma(k(x)) \cdot \theta\sigma(n(x)) \cdot e^{-H(x)} \in K\bar{N}A$.

It follows that $\bar{H}(x) = -H(x)$.

We evaluate $|\cdot|^2_{pos}$ on the image of $\mathcal{S}$. First we note (for fixed $x \in K \cdot \exp(\mathfrak{l}_0^\perp \cap \mathfrak{p}_0)$ and with respect to the Hermitian form on $T_{xL}(G/L)$)

$$\begin{aligned}
\langle \mathcal{S}f(x), \mathcal{S}f(x)\rangle_{pos} &= \left| \int\limits_{L\cap K} \langle f(xl), \phi\rangle \langle \omega_s, \mathcal{S}f(xl)\rangle_{pos}\ dl \right| \\
&= \left| \int\limits_{L\cap K} \langle f(xl), \phi\rangle \int\limits_{L\cap \bar{N}} \overline{\langle f(xl\bar{n}_L), \phi\rangle}\ d\bar{n}_L\, dl \right| \\
&= \left| \int\limits_{L\cap K} e^{-(\rho_G+\rho_L)H(xl)} \int\limits_{L\cap \bar{N}} e^{-(\rho_G+\rho_L)H(xl\bar{n}_L)} \right. \\
&\qquad\qquad \left. \times \langle f(k(xl)), \phi\rangle \overline{\langle f(k(xl\bar{n}_L)), \phi\rangle}\ dl\ d\bar{n}_L \right| \\
&\leq C \int\limits_{L\cap K} \int\limits_{L\cap \bar{N}} e^{-(\rho_G+\rho_L)H(xl)} e^{-(\rho_G+\rho_L)H(xl\bar{n}_L)}\ dl\, d\bar{n}_L \\
&= C \int\limits_{L\cap K} \int\limits_{L\cap \bar{N}} \{e^{-(\rho_G+\rho_L)H(xl)} e^{-(\rho_G+\rho_L)\bar{H}(xl)}\} \\
&\qquad\qquad \times e^{-(\rho_G+\rho_L)H(\bar{n}(xl\bar{n}_L))}\ dl\ d\bar{n}_L \\
&= C \int\limits_{L\cap K} \int\limits_{L\cap \bar{N}} e^{-(\rho_G+\rho_L)H(\bar{n}(xl\bar{n}_L))}\ d\bar{n}_L\, dl
\end{aligned}$$

Thus we can compute

$$\begin{aligned}
&\int\limits_{G/L} |\mathcal{S}f(k(x)exp\zeta(x))|^2_{pos}\ dx \\
&\qquad \leq C \int\limits_{G/L} \int\limits_{(L\cap K)\times(L\cap\bar{N})} e^{-(\rho_G+\rho_L)H(\bar{n}(k(x)exp\zeta(x)k\bar{n}))}\ dk\, d\bar{n}_L\, dx \\
&\qquad \leq C \int\limits_{G/L} \int\limits_{L/A} e^{-(\rho_G+\rho_L)H(\bar{n}(k(x)exp\zeta(x)l))}\ dl\, dx \\
&\qquad = C \int\limits_{G/L} \int\limits_{L/A} e^{-(\rho_G+\rho_L)H(\bar{n}(xl))}\ dl\, dx \\
&\qquad = C \int\limits_{G/A} e^{-(\rho_G+\rho_L)H(\bar{n}(x))}\ dx \\
&\qquad = C' \int\limits_{\bar{N}} e^{-(\rho_G+\rho_L)H(\bar{n})}\ d\bar{n} \\
&\qquad < \infty.
\end{aligned}$$

The last inequality follows from the dominance condition on ρ_L. See [K1], Ch. VII.

References

[AR] R. Aguilar-Rodriguez, Connections between representations of Lie groups and sheaf cohomology, Ph. D. dissertation, Harvard University (1987).

[B1] L. Barchini, Szegö Mappings, harmonic Forms, and Dolbeault cohomology, *J. Funct. Anal.* **118** (1993), 351–406.

[B2] L. Barchini, Strongly harmonic differential forms on elliptic orbits *in* "The Penrose Transform and Analytic Cohomology in Representation Theory", *Contemp. Math.* **154** (1993), AMS, Providence, 77–88.

[BKZ] L. Barchini, A. W. Knapp, and R. Zierau, Intertwining operators into Dolbeault cohomology representations, *J. Funct. Anal.* **107** (1992), 302–341.

[BZ] L. Barchini and R. Zierau, Square integrable harmonic forms and representation theory, preprint (1996).

[Bo] R. Bott, Homogeneous Vector Bundles, *Ann. of Math.* **66** (1957), 203–248.

[GS] P. Griffiths and W. Schmid, Locally homogeneous complex manifolds, *Acta Math.* **123** (1969), 253–302.

[HC] Harish-Chandra, Spherical functions on a semi-simple Lie group I, *Amer. J. of Math.* **80** (1958), 241–310.

[K1] A. W. Knapp, "Representation Theory of Lie Groups: An Overview Based on Examples", Princeton Univ. Press, Princeton, 1986.

[K2] A. W. Knapp, Imbedding Discrete Series in $L^2(G/H)$, *in* "Harmonic Analysis on Lie Groups (Sandjerg Estate, August 26-30, 1991)", Copenhagen University Mathematics Institute, Report Series 1991, No. 3, Copenhagen, 1991, pp. 27–29.

[K3] A. W. Knapp, Introduction to represntations in analytic cohomology *in* "The Penrose Transform and Analytic Cohomology in Representation Theory", *Contemp. Math.* **154** (1993), AMS, Providence, 1–19.

[Mo] G. D. Mostow, Some new decomposition theorems for semi-simple Lie groups, *Memoirs of the AMS* **14** (1955), 31–54.

[RSW] J. Rawnsley, W. Schmid, and J. A. Wolf, Singular unitary representations and indefinite harmonic theory, *J. Func. Anal.* **51** (1983), 1–114.

[S1] W. Schmid, Homogeneous complex manifolds and representation theory *in* "Representation Theory and Harmonic Analysis on Semisimple Lie groups," Math. Surveys and Monographs, American Mathematical Society (1989), Providence, 223–286.

[S2] W. Schmid, L^2-cohomology and the discre series, *Annals of Math.* **103**, 375–394.

[S3] W. Schmid, Boundary value problems for group invariant differential equations, *Elie Cartan et les mathématiques d'ajourd'hui (Lyon, 25–29 juin 1984), Astérisque, Numéro hors série* (1985), 311–321.

[Wo] H.-W. Wong, Dolbeault cohomologies and Zuckerman modules associated with finite rank representations, *J. Func. Anal.* **90** (1995), 428–454.

[Zi] R. Zierau Unitarity of certain Dolbeault cohomology representations *in* "The Penrose Transform and Analytic Cohomology in Representation Theory", *Contemp. Math.* **154** (1993), AMS, Providence, 239–259.