The Plancherel formula for real reductive symmetric spaces

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1 Introduction

Let G be a real reductive group in the Harish-Chandra class, σ an involution of G, θ a Cartan involution of G commuting with σ, K the subgroup of fixed points of θ, H an open subgroup of the subgroup of fixed points of σ. The goal is :
(a) to decompose the left regular representation of G in $L^2(G/H)$ into an Hilbert integral of irreducible unitary representations.
(b) to decompose the Dirac measure at eH, where e is the neutral element of G, into an integral of H-fixed distribution vectors.
(c) to make the simultaneous spectral decomposition of the elements of the algebra $\mathbb{D}(G/H)$ of left invariant differential operators under G on G/H. In other words, one wants to write the elements of $L^2(G/H)$ with the help of joint eigenfunctions under $\mathbb{D}(G/H)$.
These problems were solved for the "group case" (i.e. the group viewed as a symmetric space) by Harish-Chandra [H-C1],[H-C2], [H-C3].
For the case where G is a complex group and H is a real form, this was solved by P. Harinck(cf. [H], see also [D4]).
Before the solution of the general case, they were the only known cases of arbitrary real rank.
I found the last step of the general case during the Fall 1995.
The work of J. Bernstein, [Be], on the support of the Plancherel measure and the study of truncated inner products, [D2], appeared to be crucial. The work of Casselman on truncation [Cass] was influential, even if, eventually, the point of view of J. Arthur, [A], par. 8, was used.
At the same period E. van den Ban and H. Schlichtkrull found the Paley-Wiener theorem for smooth, compactly supported and K-finite functions on real reductive symmetric spaces. Using the regularity theorem for normalized Eisenstein integrals (joint work with J. Carmona, [CD2], see Theorem 6), they were able to deduce from their proof the Plancherel formula. They informed me that they have got during the Spring 1997 a proof of the regularity theorem independent from ours.
Also, in response to my announcement, T. Oshima informed me that he got a proof of the Plancherel formula since 1984.

2 Temperedness of the spectrum

Let $\mathfrak{g}$ be the Lie algebra of G, etc...Let $\mathfrak{s}$ (resp. $\mathfrak{q}$) be the space of elements in $\mathfrak{g}$ which are antiinvariant under the differential of θ (resp. σ). Let $\mathfrak{a}_\emptyset$ be a maximal abelian subspace of $\mathfrak{s} \cap \mathfrak{q}$. If P is a $\sigma\theta$-stable parabolic subgroup of G, containing $A_\emptyset := exp\, \mathfrak{a}_\emptyset$, we denote by $P = M_P A_P N_P$ its Langlands σ-decomposition. More precisely A_P is the subgroup of the elements a of the split component of the Levi factor $L_P = P \cap \theta(P)$ such that $\sigma(a) = a^{-1}$. Here M_P is larger than for the usual Langlands decomposition. In order to simplify the exposition we will make the following simplifying assumption:

Hypothesis: For any P as above, HP is the unique open (H, P)-double coset.

When $\sigma = \theta$ (the case of a riemannian symmetric space) or $G = G_1 \times G_1$ and σ exchanges the factors (the "group case"), this hypothesis is satisfied.
To get the Plancherel formula, it is useful to use K-finite functions. They are often replaced by τ-spherical functions. Here (τ, V_τ) is a finite dimensional unitary representation of K and a τ-spherical function on G/H is a function $f : G/H \to V_\tau$ such that $f(kx) = \tau(k)f(x)$, $k \in K$, $x \in G/H$.
Some spaces of τ-spherical functions on G/H play a crucial role in the theory, namely:
(a) $\mathcal{C}(G/H,\, \tau)$: the Schwartz space of τ-spherical functions on G/H which are rapidly decreasing as well as their derivatives by elements of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.
(b) $\mathcal{A}(G/H,\, \tau)$: the space of smooth τ-spherical functions on G/H which are $\mathbb{D}(G/H)$ finite. Here $\mathcal{A}$ is used to evoke automorphic forms.
(c) $\mathcal{A}_{temp}(G/H,\, \tau)$: the space of elements of $\mathcal{A}(G/H,\, \tau)$ which have tempered growth as well as their derivatives by elements of $U(\mathfrak{g})$. Integration of functions on G/H defines a pairing between $\mathcal{A}_{temp}(G/H,\, \tau)$ and $\mathcal{C}(G/H,\, \tau)$.
(d) $\mathcal{A}_2(G/H,\, \tau)$: the space of square integrable elements of $\mathcal{A}(G/H,\, \tau)$. This is a subspace of the three preceeding spaces.
One has:

Theorem 1 (cf. [D2], Proposition 1):
The space $\mathcal{A}_2(G/H,\, \tau)$ is finite dimensional.

This is deduced from the theory of discrete series for G/H due to M. Flensted-Jensen [F-J] and T. Oshima, T. Matsuki [OM]. One has also to use the behaviour of the discrete series under certain translation functors, studied by D. Vogan [V] and a result of H. Schlichtkrull [S] on the minimal K-types of certain discrete series.
The next result follows from the work of J. Bernstein [Be] on the support of the Plancherel measure.

Theorem 2 ([CD1], Appendice C):
Every function in $\mathcal{C}(G/H,\, \tau)$ can be canonically desintegrated as an integral of elements of $\mathcal{A}_{temp}(G/H,\, \tau)$.

This information appeared to be crucial at the end of our proof.

3 The continuous spectrum: Eisenstein integrals

Let $P = MAN$ the Langlands σ-decomposition of a $\sigma\theta$-stable parabolic subgroup P of G. Let ρ_P be the half sum of the roots of $\mathfrak{a}$ in $\mathfrak{n}$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ be such that the real part of $\lambda - \rho_P$ is strictly dominant with respect to the roots of $\mathfrak{a}$ in $\mathfrak{n}$. Let τ_M be the restriction of τ to $M \cap K$. Then, if $x \in G/H$ and $\psi \in \mathcal{A}_2(M/M \cap H, \tau_M)$, the following integral is convergent:

$$E(P, \psi, \lambda)(x) := \int_K \tau(k^{-1})\Psi_\lambda(kx)dk,$$

where $\Psi_\lambda(x) = 0$ if $x \notin PH$, and $\Psi_\lambda(x) = a^{-\lambda+\rho_P}$ if $x = namH$ with $n \in N$, $a \in A$, $m \in M$. Moreover $E(P, \psi, \lambda)$ is an element of $\mathcal{A}(G/H, \tau)$. Eisenstein integrals are the τ-spherical version of K-finite functions of the form: $gH \mapsto < \pi'(g)\xi, v >$, where π' is the contragredient of a generalized principal series π, ξ is a certain H-fixed distribution vector of π, v is a K-finite vector of π.

Theorem 3 (J.L. Brylinski,D., [BrD]):
The function $\lambda \mapsto E(P, \psi, \lambda)$ admits a meromorphic continuation in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. This meromorphic continuation , denoted in the same way, multiplied by a suitable product, p_ψ, of functions of type $\lambda \mapsto (\alpha, \lambda) + c$, where α is a root of $\mathfrak{a}$ and $c \in \mathbb{C}$, is holomorphic around $i\mathfrak{a}*$.

This meromorphic continuation is an interesting feature of the theory. For the case $G_1 \times G_1/G_1$ (the "group case") it comes down to the meromorphic continuation of intertwining integrals (cf. [BrD]).
Our proof uses D-modules arguments.
The case where P is minimal had been treated separately by E. van den Ban [B1] and G. Olafsson [Ol]. One has also to mention the earlier work of T. Oshima and J. Sekiguchi [OSe] on the spaces of type G/K_ε.
Also, in a joint work with J.Carmona [CD1], we were able to find an alternative proof which gives a functional equations for the H-fixed distribution vectors of generalized principal series (cf. [W], chapter 10, for similar results relative to intertwining integrals, and [B2] for the case where P is minimal). This proof uses Bruhat's thesis and tensoring by finite dimensional modules. This implies rough estimates for Eisenstein integrals, which generalize those obtained by E. van den Ban when P is minimal [B2].

Theorem 4 ([D1]) :
If $\lambda \in i\mathfrak{a}^*$ is such that $E(P, \psi, \lambda)$ is well defined, then it is tempered, i.e. is an element of $\mathcal{A}_{temp}(G/H, \tau)$

This a natural result but the proof is quite long. It uses the behaviour under translation functors of H-fixed distribution vectors of discrete series and of generalized principal series, and also of the Poisson transform. Moreover the duality of M. Flensted-Jensen, [F-J], and a criteria of temperedness due to Oshima [O] play a crucial role.
With the help of this theorem and using techniques due to E. van den Ban [B2], the rough estimates for Eisenstein integrals can be improved to get uniform sharp estimates

for $p_\psi(\lambda)E(P,\psi,\lambda),$, $\lambda \in i\mathfrak{a}^*$ (cf. [D1]).

4 C-functions

Let P be as above and let L be equal to MA. The theory of the constant term, due to J. Carmona, gives a linear map from $\mathcal{A}_{temp}(G/H,\,\tau)$ into $\mathcal{A}_{temp}(L/L\cap H,\,\tau_L)$, $\varphi \mapsto \varphi_P$, characterized by :

$$lim_{t\to+\infty}\,\delta_P^{1/2}((exp\,tX)l)\varphi((exp\,tX)l) - \varphi_P((exp\,tX)l) = 0,$$

where $l \in L/L\cap H$, $X \in \mathfrak{a}_P$ is P-dominant and δ_P is the modular function of P.
Let Q be a $\sigma\theta$-stable parabolic subgroup of G with the same θ-stable Levi subgroup L than P. Let $W(\mathfrak{a})$ be the group of automorphisms of $\mathfrak{a}$ induced by an element of $Ad(G)$. One defines meromorphic functions on $\mathfrak{a}^*_{\mathbb{C}}$, $\lambda \mapsto C_{Q|P}(s,\lambda)$ with values in $End(\mathcal{A}_2(M/M\cap H,\tau_M))$ such that :

$$E(P,\psi,\lambda)_Q(ma) = \sum_{s\in W(\mathfrak{a})}(C_{Q|P}(s,\lambda)\psi)(m)a^{-s\lambda},\ m\in M,\ a\in A,\ \lambda\in i\mathfrak{a}^*,$$

or rather for λ in an open dense subset of $i\mathfrak{a}^*$.
The C-functions allow to normalize Eisenstein integrals as folllows:

$$E^0(P,\psi,\lambda) := E(P,C_{P|P}(1,\lambda)^{-1}\psi,\lambda)$$

5 Truncation, Maass-Selberg relations and the regularity of normalized Eisenstein integrals

Let P be as above and let $P' = M'A'N'$ be the Langlands σ-decomposition of an other $\sigma\theta$-stable parabolic subgroup of G. Let ψ (resp. ψ') be an element of $\mathcal{A}_2(M/M\cap H,\tau_M)$ (resp. $\mathcal{A}_2(M'/M'\cap H,\tau_{M'})$). One chooses p_ψ as in Theorem 3, such that the product of p_ψ with the C-functions are holomorphic in a neighbourhood of $i\mathfrak{a}^*$ which is a product of $i\mathfrak{a}^*$ with a neighbourhood of 0 in $\mathfrak{a}^*$. We do the same for ψ'. One defines $F(\lambda) := p_\psi(\lambda)E(P,\psi,\lambda)$. One defines similarly F'.
One assumes, for the rest of the article, that G is semisimple. This is just to simplify the exposition. One chooses $T \in \mathfrak{a}_\emptyset$, regular with respect to the roots of $\mathfrak{a}_\emptyset$ in $\mathfrak{g}$. Let C_T^1 be the convex hull of $W(\mathfrak{a}_\emptyset)T$ and let C_T be equal to the subset $K(expC_T^1)H$ of G/H.

Theorem 5 ([D2]):
(i) One gets an explicit expression $\omega^T(\lambda,\lambda')$, involving the C-functions (see an example below) and vanishing when A and A' are not conjugate under K, which is asymptotic to

$$\Omega^T(\lambda,\lambda') := \int_{C_T}(F(\lambda)(x),F'(\lambda')(x))dx$$

when T goes to infinity and $\lambda\in i\mathfrak{a}^*$, $\lambda'\in i\mathfrak{a}'^*$. More precisely for $\delta>0$ there exists $C>0$, $k\in\mathbb{N}$ and $\varepsilon>0$, such that for all T satisfying $\mid\alpha(T)\mid\geq\delta\mid\mid T\mid\mid$ for every root α of $\mathfrak{a}_\emptyset$ in $\mathfrak{g}$, one has:

$$\mid\Omega^T(\lambda,\lambda')-\omega^T(\lambda,\lambda')\mid\leq C(1+\mid\mid\lambda\mid\mid)^k(1+\mid\mid\lambda'\mid\mid)^k e^{-\varepsilon\mid\mid T\mid\mid}$$

(ii) Moreover ω^T is analytic in $(\lambda, \lambda') \in i\mathfrak{a}^* \times i\mathfrak{a}'^*$.

This generalizes a result of J. Arthur for the group case [A], Theorem 8.1. My proof is quite similar, but I was able to avoid his use of the Plancherel formula.
(ii) is an easy consequence of (i). In fact, the explicit form of ω^T implies that its it meromorphic around $i\mathfrak{a}^* \times i\mathfrak{a}'^*$. Moreover Ω^T is holomorphic, hence locally bounded, around $i\mathfrak{a}^* \times i\mathfrak{a}'^*$. From the inequality in (i), one deduces that ω^T is locally bounded, hence holomorphic, around $i\mathfrak{a}^* \times i\mathfrak{a}'^*$.
We will now show, on an example, how the explicit form of ω^T and its analyticity in $(\lambda, \lambda') \in i\mathfrak{a}^* \times i\mathfrak{a}'^*$ imply the Maass-Selberg relations.
Let σ be equal to θ, $H = K$, and τ be the trivial representation. Let P, P' be minimal parabolic subgroup of G. Then $dim\mathcal{A}_2(M/M \cap H, \tau) = 1$ and the C-functions are scalar valued. One assumes $\mathfrak{g}$ to be semisimple and that the dimension of A is one. Then $W(\mathfrak{a})$ has 2 elements, ± 1, and one has the following explicit expression of ω^T:

$$\omega^T(\lambda, \lambda') = p_\psi(\lambda) p_{\psi'}(\lambda') \sum_{s=\pm 1, s'=\pm 1} e^{s\lambda T - s'\lambda' T} C_{P|P}(s, \lambda) \overline{C_{P|P}(s', \lambda')} (s\lambda - s'\lambda')^{-1}$$

Thus $\omega^T(\lambda, \lambda')$ is the sum of a product of $(\lambda - \lambda')^{-1}$ by an analytic function with a product of $(\lambda + \lambda')^{-1}$ by an analytic function. The analyticity at (λ, λ) implies easily that the factor behind $(\lambda - \lambda')^{-1}$ vanishes for $\lambda = \lambda'$. Hence we get $\mid C_{P|P}(1, \lambda) \mid^2 = \mid C_{P|P}(-1, \lambda) \mid^2$, $\lambda \in i\mathfrak{a}^*$. This is one of the Maass-Selberg relations(cf. [D2], Theorem 2, and the work with J. Carmona [CD2], Theorem 2 for the general case, see [B1], [B2] for the case where P is minimal). These relations imply that the C-functions attached to normalized Eisenstein integrals are unitary, when defined, for λ purely imaginary. Hence they are locally bounded . This implies that they are holomorphic around the imaginary axis. This implies in particular some holomorphy property of the constant term of normalized Eisentein integrals. From this, with the help of [BCD], one deduces:

Theorem 6 (Regularity theorem for normalized Eisenstein integrals, J. Carmona, D.[CD2]): The normalized Eisenstein integrals are holomorphic in a neighbourhood of the imaginary axis.

6 Fourier transform and wave packets

Theorem 7 (J. Carmona, D., [CD2]):
For $f \in \mathcal{C}(G/H, \tau)$, one has $\mathcal{F}^0_P f \in \mathcal{S}(i\mathfrak{a}^*) \otimes \mathcal{A}_2(M/M \cap H, \tau_M)$, where $\mathcal{F}^0_P f$ is characterized by:

$$((\mathcal{F}^0_P f)(\lambda), \psi) = \int_{G/H} (f(x), E^0(P, \psi, \lambda)(x)) dx, \quad \lambda \in i\mathfrak{a}^*, \quad \psi \in \mathcal{A}_2(M/M \cap H, \tau_M)$$

Here $\mathcal{S}(i\mathfrak{a}^*)$ is the usual Schwartz space.

This theorem follows from the sharp estimates of Eisenstein integrals.

Theorem 8 (E. van den Ban,J.Carmona, D., [BCD]):
If Ψ is an element of $\mathcal{S}(i\mathfrak{a}^*) \otimes \mathcal{A}_2(M/M \cap H, \tau_M)$, one has $\mathcal{I}_P^0 \in \mathcal{C}(G/H, \tau)$, where :

$$\mathcal{I}_P^0(x) := \int_{i\mathfrak{a}^*} E^0(P, \Psi(\lambda), \lambda) d\lambda, \quad x \in G/H$$

This theorem follows from the regularity theorem and of the joint work with E. van den Ban and J. Carmona, [BCD].
Now we want to compute $\mathcal{F}_{P'}^0 \mathcal{I}_P^0$. For this purpose one has to study the integral:

$$I := \int_{G/H} (\int_{i\mathfrak{a}^*} \alpha(\lambda) E^0(P, \psi, \lambda)(x) d\lambda, E^0(P', \psi', \lambda')(x)) dx$$

One truncates the integral on G/H to C_T and let T goes to infinity (far from the walls). Let us denote the truncated inner product of the normalized Eisenstein integrals by ${}^0\Omega^T(\lambda, \lambda')$. Using Fubini's theorem one has:

$$I = lim_{T \to \infty} \int_{i\mathfrak{a}^*} \alpha(\lambda) {}^0\Omega^T(\lambda, \lambda') d\lambda$$

As for unnormalized Eisentein interals, one has an asymptotic evaluation of ${}^0\Omega^T(\lambda, \lambda')$ by an explicit expression ${}^0\omega^T(\lambda, \lambda')$. One can replace ${}^0\Omega^T$ by ${}^0\omega^T$ in the previous formula. By using an expression of ${}^0\omega^T(\lambda, \lambda')$, viewed as a distribution in λ, for fixed λ' , involving Fourier transforms of cones (cf. [D2], Theorem 3) one gets:

Theorem 9 (J. Carmona, D., [CD2]) :
Let $\mathbb{F}$ be a set of representative of σ-association classes of $\sigma\theta$-stable parabolic subgroups. Here σ-association means that the $\mathfrak{a}$ are conjugated under K. Define:

$$\mathbb{P}_\tau = \sum_{P \in \mathbb{F}} ((W(\mathfrak{a}_P))^{-1} \mathcal{I}_P^0 \mathcal{F}_P^0.$$

Then $\mathbb{P}_\tau$ is an orthogonal projection operator in $\mathcal{C}(G/H, \tau)$ endowed with the L^2 scalar product.

7 The Plancherel formula

Essentially, the Plancherel formula is contained in the following:

Theorem 10 (D., [D3]):
The projection $\mathbb{P}_\tau$ is the identity operator on $\mathcal{C}(G/H, \tau)$.

The proof goes as follows. If $\mathbb{P}_\tau$ was not the identity, using Theorem 1, one could find a non zero element of $\mathcal{A}_{temp}(G/H, \tau)$ which is orthogonal to the image of $\mathbb{P}_\tau$. Then, generalizing Theorem 5 to the truncated inner product of an Eisenstein integral with a general element of $\mathcal{A}_{temp}(G/H, \tau)$, this orthogonality can be explicitely described (cf.

the evaluation of I before Theorem 9) . As a result, this function has to be zero, a contradiction which proves the theorem.
The theorem translates to K-finite functions, involving representations and H-fixed distribution vectors.
The theorem can also be expressed with the unnormalized Eisenstein integrals. Then there are certain Plancherel factors involved. They are linked, as in the group case , to the intertwining integrals. Following the apprach of A. Knapp and G. Zuckerman, [KZ], their computation is reduced to find an embedding of discrete series into principal series attached to minimal parabolic subgroups.

8 Bibliographie

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