EXCEPTIONAL THETA CORRESPONDENCE OVER FINITE FIELDS

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1. Lusztig Series

Let k be the finite field of characteristic p with q elements, and $\overline{k}$ a fixed algebraic closure of k. Let G be a connected reductive group over $\overline{k}$, and assume for simplicity that G has connected center, and that p is a good prime for G. Suppose that $F : G \longrightarrow G$ is a Frobenius map such that $G^F := G(k)$ is a finite group of Lie type. In this section, we briefly recall the theory of Deligne and Lusztig on the complex representation theory of $G(k)$.

Recall that associated to the pair (G, F), there is a dual pair (G^*, F^*) [Ca]. In our case, G^* has simply-connected derived group. Let $Irr(G(k))$ denote the set of equivalences classes of irreducible representations of $G(k)$ (over $\mathbb{C}$ or $\overline{\mathbb{Q}}_l$, with $l \neq p$). Then it is a result of Deligne and Lusztig that:

$$Irr(G(k)) = \bigcup_{(s^*)\subset G^*} Irr(G(k), s^*) \tag{1.1}$$

a disjoint union over the F^*-stable semi-simple classes (s^*) in the dual group G^*. Each set $Irr(G(k), s^*)$ is called a geometric conjugacy class or a Lusztig series.

Example: The set $Irr(G, 1)$ is the set of unipotent characters, which include the trivial representation and the Steinberg representation.

It was shown further by Lusztig that there is a bijection (almost canonical)

$$Irr(G(k), s^*) \longrightarrow Irr(C_{G^*}(s^*), 1) \tag{1.2}$$

Hence, in each Lusztig series, there is a character corresponding to the Steinberg character of $C_{G^*}(s^*)$ under the above bijection (though the bijection above is not always canonical, the preimage of the Steinberg representation is). This is the unique generic character in the series. Recall that an irreducible character of $G(k)$ is said to be generic if it is a component of the Gelfand-Graev representation, which is multiplicity-free, and the generic representations are characterized by the fact that their dimensions have the form:

$$q^N + (\text{lower powers of } q)$$

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where N is the number of positive roots of G. Hence we see that generic characters can be parametrized by F^*-stable semi-simple classes in the dual group G^*. We shall denote by $\chi^G_{s^*}$ the generic character corresponding to the class (s^*).

2. Reflection Representation

Now assume that G is simple and split over k with rank l. In this section, we focus attention on a very special representation Π of $G(k)$, which is called the reflection representation. Π can be characterized by one of the following two statements:

(i) Π is the unipotent principal series representation whose space of Borel-fixed vectors is isomorphic, as a representation of the Weyl group W of G, to the reflection representation ξ of W.

(ii) Let P be any parabolic subgroup, and suppose the Levi factor of P has semi-simple rank r. Then,

$$\langle \Pi, Ind_P^G 1\rangle = l - r$$

If, further, G is simply-laced, it was shown by Lusztig that, at least for p a good prime for G, we have:

$$\Pi = \frac{1}{|W|}\sum_{w\in W}\xi(w)R_w \tag{2.1}$$

Here, $R_w := R_{T_w,1}$ is a Deligne-Lusztig character. From this, one deduce that, for a semi-simple element $s \in G(k)$,

$$Tr(s|\Pi) = Tr(F|M(s)) \tag{2.2}$$

where $M(s)$ is the motive of $C_G(s)^0$ [Gr].

Example: $dim(\Pi) = \sum_i q^{e_i}$, where the e_i's are the exponents of W. Note that if e_1 is the largest exponent, then $2e_1$ is the dimension of the minimal nilpotent orbit in the Lie algebra of G. Hence we are justified to call Π the **minimal representation** of $G(k)$.

3. Dual Pair Correspondence

In view of the discussion in the previous section, we can use Π to obtain correspondences of representatons for dual pairs. This is the finite field analogue of the results of Magaard and Savin [M-S] over p-adic fields. We shall concentrate only on the pair

$$G_2 \times PGL_3 \subset E_6$$

Here all three groups are of adjoint type. We refer the reader to [G] for more details.

Let π be a generic representation of PGL_3. Then we let $\Theta(\pi)$ be the set of irreducible representations π' of G_2, counted with multiplicity, such that

$\pi \otimes \pi'$ is a component of $\Pi|_{G_2 \times PGL_3}$. Let $\Theta_g(\pi) \subset \Theta(\pi)$ be the subset of generic representations. Then we have:

Proposition 3.1. *Assume that $p \geq 5$. Then*

$$\Theta_g(\chi_{s^*}^{PGL_3}) = \{\chi_{i(s^*)}^{G_2}\}$$

where

$$i : SL_3 \longrightarrow G_2$$

is the natural inclusion of dual groups.

Remark: Note that $PGL_3(k)$ has q^2 generic characters, each of which has degree $(q^3 + ...)$. Also, each generic character of $G_2(k)$ has degree $(q^6 + ...)$. Thus the proposition accounts for a subspace of Π of dimension $(q^{11} + ...)$. Since the dimension of Π is $q^{11} + q^8 + q^7 + q^5 + q^4 + q$, the proposition accounts for a large piece of Π.

Let us give a sketch of the proof of this proposition, which essentially consists of the following three steps:

(i) **Whittaker Vectors.** Let U_{G_2} be the unipotent radical of a Borel subgroup of G_2, and let φ be a character of U_{G_2} in general position. Then we have the following proposition.

Proposition 3.2.

$$\Pi_{U,\varphi} = \textit{ Gelfand-Graev representation of } PGL_3(k)\ .$$

As a corollary, we have:

Corollary 3.3. *If π is a generic representation of $PGL_3(k)$, then $\Theta_g(\pi)$ is a singleton set.*

(ii) **Jacquet Modules.** In view of (i), it remains to chack that the single element in $\Theta_g(\chi_{s^*}^{PGL_3})$ is indeed $\chi_{i(s^*)}^{G_2}$. We first do this for non-cuspidal representations. Then this reduces to a computation of Jacquet functors. In this case, the computation in [M-S] works over a finite field as well. As an example, let us describe one of the results in [M-S].

Let $P = MN$ be a maximal parabolic subgroup of E_6, with M of type D_5 and N abelian. Then one sees easily, for reasons of dimension, that:

$$\Pi|_P = \Pi_N \oplus \mathbb{C}[\Omega]$$

where $\Pi_N = 1 \oplus \Pi(M)$ ($\Pi(M)$ is the minimal representation of M), and Ω is the minimal non-trivial M-orbit on $\overline{N}$. Moreover, if ψ is a fixed non-trivial character of k, and $\langle \cdot, \cdot \rangle$ is a non-degenerate G-invariant form on the Lie algebra of G, for example the Killing form, then we can identify $\overline{N}$ and the dual $N^\vee$ of N as follows:

$$\psi_x(n) = \psi(\langle n, x \rangle), \quad n \in N, x \in \overline{N}$$

Now for $f \in \mathbb{C}[\Omega]$, then

$$\Pi(n)f(x) = \psi_x(n)f(x), \quad n \in N$$

$$\Pi(m)f(x) = f(m^{-1}(x)), \quad m \in M$$

Note that $(G_2 \times PGL_3) \cap P = G_2 \times LU$, where LU is a maximal parabolic of PGL_3, so that $L = GL_2$. Using the above description of $\Pi|_P$, it was shown by Magaard and Savin that:

Proposition 3.4. *As a $G_2 \times L$-module,*

$$\mathbb{C}[\Omega]_U \cong \mathrm{Ind}_{L\times P_2}^{L\times G_2}\mathbb{C}[GL_2] \oplus \mathrm{Ind}_{B\times P_1}^{L\times G_2}\mathbb{C}[GL_1]$$

Here B is the Borel subgroup of L, P_2 is the Heisenberg parabolic of G_2, and P_1 is the other maximal parabolic of G_2.

In the above proposition, only the first summand on the right hand side can contain generic representation of $G_2(k)$. From this, it is easy to check that the correspondence for non-cuspidal representations is indeed functorial with respect to the natural inclusion of dual groups.

(iii) **Base Change/Shintani Descent.** It remains to verify the proposition for cuspidal characters. For these, we shall use base change to reduce to the case of non-cuspidal representations. Let us recall briefly the notion of base change.

Let k_m be the extension of k of degree m. Then there is a norm map $N_m : G(k_m) \to G(k)$ which maps the F-conjugacy classes of $G(k_m)$ bijectively onto the conjugacy classes of $G(k)$. Hence, if χ is an F-class function on $G(k_m)$, then the function $Sh_m(\chi)(N_m(g)) := \chi(g)$ is a class function on $G(k)$, called the Shintani descent of χ. Note that Sh_m defines a vector space isomorphism between the space of F-class function on $G(k_m)$ and the space of class function on $G(k)$. More importantly, it is an isometry for the natural inner products on the two spaces:

$$\langle Sh_m(\chi_1), Sh_m(\chi_2)\rangle_{G(k)} = \langle \chi_1, \chi_2\rangle_{G(k_m)}$$

In particular, if χ is an F-stable representation of $G(k_m)$, it extends to a representation of $G(k_m) \rtimes \langle F\rangle$ (non-uniquely of course), and the function

$$g \mapsto \chi(Fg)$$

is an F-class function on $G(k_m)$, and so we have $Sh_m(\chi(F\cdot))$, which of course may not be the character of a representation of $G(k)$. However we have the following results of Digne-Michel and Digne [D]:

Proposition 3.5. *(i) Let Π_m be the reflection representation of $G(k_m)$. Then there is an extension of Π_m to $G(k_m) \rtimes \langle F\rangle$ such that:*

$$Sh_m(\Pi_m(F\cdot)) = \Pi$$

(ii) If p is a good prime for G and $(m,p)=1$, then there is an extension of the Deligne-Lusztig charcater R_{T_m,θ_m}, where $T_m = T(k_m)$ and $\theta_m = \theta \circ N_m$, such that:

$$Sh_m(R_{T_m,\theta_m}(F\cdot)) = R_{T(k),\theta}$$

Here, θ is a character of $T(k)$.

In our case, we need to assume $p \geq 5$ to apply this proposition. Using these results and the fact the Sh_m is an isometry, one reduces to the case of non-cuspidal representations of the groups over k_m, and the results in (ii) apply. This completes the proof of the proposition.

Remarks: (i) We note that the computations in (i) and (ii) above works over a p-adic field as well. However, we do not know how to adapt (iii) to the p-adic case, or whether it is possible to do so.

(ii) Essentially the same computations give us similar results for the dual pairs $G_2 \times PU_3 \subset {}^2E_6$ and $G_2 \times PGSp_6 \subset E_7$. Since every semi-simple element of G_2 is contained in either a SL_3 or a SU_3, we see that every generic character of $G_2(k)$ is lifted from either $PGL_3(k)$ or $PU_3(k)$ by the theta correspondence.

4. Classical Case

We would like to conclude with a short survey of the results for the classical dual pairs over finite fields. Let W be the Weil representation of Sp_{2n}, and let $H_1 \times H_2 \subset Sp_{2n}(k)$ be a classical dual pair. Hence, $(H_1, H_2) = (GL_m, GL_{m'})$, $(U_m, U_{m'})$ or $(Sp_{2m}, O_{m'})$. In [Sr], Srinivasan obtained the decomposition of the uniform part of $W|_{H_1\times H_2}$ for most of the pairs above. It follows from her results that the Howe duality correspondence respects Lusztig series [A-M-R]. For example, unipotent representations correspond.

Adams and Moy [A-M] then showed that cuspidal unipotent representations correspond in the case of first occurrence in a tower of dual pairs. More recently, Aubert, Michel and Rouquier [A-M-R], starting from the main result in [Sr], were able to describe the correspondence completely, in a combinatorial manner, except in the case of symplectic-orthogonal pairs, for which they stated a conjecture.

The results for the exceptional dual pairs is an analogue for these results, though it is certainly less complete than the classical case. One should perhaps try to obtain an analogue of the result in [Sr].

References

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