Star-products and the eigenvalue of the Casimir

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1 Introduction

These are informal notes of a talk on joint work with David Vogan. What it is about is in essence a calculation. A consequence of this calculation is the non-existence of invariant star-products on certain nilpotent orbits.

2 The eigenvalue of the Casimir

Let G be a complex simple Lie group with Lie algebra $\mathfrak{g}$, and let $J \subset U(\mathfrak{g})$ be a primitive ideal, i.e. the annihilator of an irreducible $U(\mathfrak{g})$-module. Let Ω be the Casimir element of $U(\mathfrak{g})$ (with respect to a nondegenerate G-invariant bilinear form B on G); so $\Omega = \sum x_i x_i^*$, where $\{x_i\}$ and $\{x_i^*\}$ are dual bases of $\mathfrak{g}$ with respect to B. Under the natural map $U(\mathfrak{g}) \to U(\mathfrak{g})/J$, the element Ω maps to a scalar c_J. This scalar I will call the eigenvalue of the Casimir.

One would like to construct primitive ideals attached to certain nilpotent orbits in $\mathfrak{g}^*$ – one might call this quantizing the orbit. There is a kind of quantization in symplectic geometry called star-products, which is formally very similar to the construction of primitive ideals. Since nilpotent orbits are symplectic, it is natural to try to construct primitive ideals using star-products.

In fact what I am going to explain is that if we have a star-product satisfying some conditions that are natural from the viewpoint of symplectic geometry, then we do get a primitive ideal J_{star}, and moreover we can actually calculate $c_{J_{star}}$.

This has the satisfying feature that we can calculate a number. This is a little surprising because the theorems about the existence of star-products in symplectic geometry are not very explicit. On the other hand, it has the unsatisfying consequence that in some cases the number $c_{J_{star}}$ is not compatible with known results about primitive ideals, showing that in these cases such star-products do not exist. We are then calculating numbers attached to non-existent star-products. I don't know if they have any significance.

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3 Star-products in symplectic geometry

The notion of star-product was introduced as a way to formulate quantum mechanics (see [BFFLS]). In classical mechanics, the observables are $C^\infty(M)$, where M is a symplectic manifold. In quantum mechanics, the observables are some noncommutative algebra related to $C^\infty(M)$. This algebra is given, in the star-product formulation, as a deformation of the ring $R = C^\infty(M)$. Formally, a star-product is an associative algebra structure $*_t$ on the ring $R[[t]]$ which is t-linear and for $r, s \in R$ takes the form

$$r *_t s = \sum t^i m_i(r, s),$$

where $m_i : R \otimes R \to R$ are bilinear maps satisfying certain conditions. Associativity imposes strong conditions on the m_i; in addition we require:

(1) m_0 is the usual product, and $m_i(\text{constant}, r) = m_i(r, \text{constant}) = 0$ for $i > 0$ (this implies that $1 \in R$ is the multiplicative identity under $*_t$).

(2) $m_1 = \frac{1}{2}\{,\}$, where $\{,\}$ is the Poisson bracket.[1]

(3) m_i is symmetric (resp. anti-symmetric) if i is even (resp. odd).

These conditions are not so serious. More serious is:

(4) The m_i are bidifferential operators, i.e., in local coordinates we have

$$m_i(r, s) = \sum a_{IJ} \frac{\partial^I r}{\partial x^I} \frac{\partial^J s}{\partial x^J}.$$

Finally, there is a result of Lichnerowicz [L] which says that given a star-product (i.e. $*_t$ satisfying these conditions), we can modify it so that:

(5) Each m_i has (bi)-order at most (i, i).

It is difficult to write down star-products explicitly, but for example in $\mathbb{R}^2$, with coordinates x, y and Poisson bracket

$$\{,\} = \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x},$$

we can define a star-product by the rule

$$m_i = \frac{1}{2^i i!} \left(\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} \right)^i. \tag{3.1}$$

This defines a star-product on $\mathbb{R}^2$; a similar formula works for $\mathbb{R}^{2n}$.

In fact, I could use the same formulas to define (in this case) a star-product on $R[[t]]$ where R is the polynomial ring $\mathbb{R}[x, y]$ (rather than $C^\infty(M)$). In this case, I can define a new product $*_1$ on R by

$$r *_1 s = r *_t s|_{t=1}$$

(the sum converges). The algebra $(R, *_1)$ is isomorphic to the noncommutative algebra $R[x, \frac{\partial}{\partial x}]$.

The following theorem was proved by deWilde-Lecomte and independently and later, using a different method, by Fedosov (see [DL], [F], [D]).

Theorem 3.1 *Any symplectic manifold admits a star-product, unique up to (some kind of) equivalence.*

[1] The factor of $\frac{1}{2}$ is usually omitted; it can be eliminated by rescaling the Poisson bracket.

4 Nilpotent orbits

Nilpotent coadjoint orbits in real Lie groups are symplectic manifolds, so it is reasonable to consider star-products on them. We are interested in simple Lie algebras; it is convenient to work with adjoint orbits, which are identified (using a nondegenerate invariant bilinear form) with coadjoint orbits. Let $G_{\mathbb{R}}$ be a real reductive group with Lie algebra $\mathfrak{g}_{\mathbb{R}}$ and let $\mathcal{O}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ be a nilpotent orbit. Then $\mathcal{O}_{\mathbb{R}}$ is a cone (i.e. invariant under dilation), so it makes sense to say a function r on $\mathcal{O}_{\mathbb{R}}$ is homogeneous of degree k: it transforms appropriately under dilation.

If m_0 is the ordinary product on $R = C^\infty(\mathcal{O}_{\mathbb{R}})$, and m_1 the Poisson bracket, and if r_1 and r_2 are homogeneous of degrees k_1 and k_2, then $m_0(r_1, r_2)$ (resp. $m_1(r_1, r_2)$) is homogeneous of degree $k_1 + k_2$ (resp. $k_1 + k_2 - 1$).

Definition 4.1 *A star-product on $\mathcal{O}_{\mathbb{R}}$ is called graded if for r_1, r_2 as above, each $m_i(r_1, r_2)$ is homogeneous of degree $k_1 + k_2 - i$.*

Lemma 4.2 *If $\mathcal{O}_{\mathbb{R}}$ admits a $G_{\mathbb{R}}$-invariant star-product, then it admits a graded $G_{\mathbb{R}}$-invariant star-product.*

5 Complex nilpotent orbits

Most of the rest of this talk will deal with complex nilpotent orbits (i.e. nilpotent G-orbits in $\mathfrak{g}$, with G and $\mathfrak{g}$ as above). These have G-invariant holomorphic symplectic forms. By definition, a holomorphic symplectic form is a nondegenerate 2-form on the holomorphic tangent spaces, which is holomorphic as a section of the bundle $\bigwedge^2 T^*X$. For example, on $\mathbb{C}^2$ there is the holomorphic symplectic form $dz_1 \wedge dz_2$. This is not a real symplectic form on $\mathbb{C}^2$, but its restriction to $\mathbb{R}^2 \subset \mathbb{C}^2$ is $dx_1 \wedge dx_2 - dy_1 \wedge dy_2$, which is a real symplectic form.

On a complex manifold with a holomorphic symplectic form, there is a notion of holomorphic star-product (defined by the obvious analogues of (1)-(5) above).

Now suppose $\mathfrak{g}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$, $G_{\mathbb{R}} \subset G$, $\mathcal{O}_{\mathbb{R}} = G_{\mathbb{R}} \cdot x \subset \mathcal{O} = G \cdot x$.

Lemma 5.1 *$\mathcal{O}_{\mathbb{R}}$ admits a graded $G_{\mathbb{R}}$-invariant star-product if and only if $\mathcal{O}$ admits a graded G-invariant holomorphic star-product.*

6 Deformations and primitive ideals

Most of the results of this section are due to Vogan [V], who observed that the problem of finding certain primitive ideals in $U(\mathfrak{g})$ can be formulated in terms of deformations of algebras.

Suppose $J \subset U(\mathfrak{g})$ is an ideal. Then J inherits a filtration $J_0 \subset J_1 \subset \cdots$ from $U(\mathfrak{g})$, and $\mathrm{gr}\ J = \oplus J_i/J_{i_1} = I$ is a graded ideal in $\mathrm{gr}\ U(\mathfrak{g}) = S(\mathfrak{g})$. The zero set $V(I) \subset \mathfrak{g}^* \simeq \mathfrak{g}$ is a cone. The following result is due to Joseph and Borho-Brylinski.

Theorem 6.1 *If J is a primitive ideal then $V(gr\ J) = \overline{\mathcal{O}} \subset \mathfrak{g}$ is a nilpotent orbit closure.*

What this means is that $\sqrt{\text{gr } J}$ is the ideal $I(\overline{\mathcal{O}})$ of functions vanishing on $\overline{\mathcal{O}}$. It is natural to ask for ideals J with gr $J = I(\overline{\mathcal{O}})$. These would be the largest ideals attached to an orbit, and therefore should be related to the smallest representations attached to an orbit.

Given such an ideal, the ring $A = U(\mathfrak{g})/J$ has a natural filtration such that gr $A = R(\overline{\mathcal{O}}) = S(\mathfrak{g})/I(\overline{\mathcal{O}})$, (here $R(Z)$ denotes the ring of regular functions on the algebraic variety Z). In fact (writing simply $\overline{R}$ for $R(\overline{\mathcal{O}})$) the ring A is isomorphic to $(\overline{R}, *_1)$. Here $*_t = \sum t^i m_i$, where the m_i are G-invariant maps $\overline{R} \otimes \overline{R} \to \overline{R}$, satisfying conditions (1)-(3) above[2] and moreover a condition related to the grading of $\overline{R}$:

(G) m_i has degree $-i$ as an operator, i.e., if r, s are homogeneous elements of $\overline{R}$ of degrees a, b, then $m_i(r, s)$ is homogeneous of degrees $a + b - i$.

I will not prove that from an ideal J one can get $*_t$ on $\overline{R}$. However, it is easy to go the other way, i.e., to construct J from $\overline{R}$ and $*_t$. Indeed, because $\overline{R}$ has no terms of negative degree, $*_1 = *_t|_{t=1}$ defines an algebra structure on $\overline{R}$. There is an algebra map

$$\Phi : U(\mathfrak{g}) \to (\overline{R}, *_1),$$

defined as follows. For $X \in \mathfrak{g}$, $\Phi(X) = X$ viewed as a function on $\mathcal{O} \subset \mathfrak{g}^* \simeq \mathfrak{g}$. The conditions on the m_i imply that

$$\Phi(X) *_1 \Phi(Y) - \Phi(Y) *_1 \Phi(X) = \Phi([X, Y])$$

so Φ extends to a surjective algebra map $U(\mathfrak{g}) \to (\overline{R}, *_1)$. The kernel of this map will be a primitive ideal J with gr $J = I(\overline{\mathcal{O}})$.

6.1 Normality of orbit closures

There is one technical point connected with normality of the orbit closure $\overline{\mathcal{O}}$. This point can be ignored if the orbits we are dealing with have normal closures, as is the case for the examples considered at the end of the paper. Therefore this section is not essential for the rest of the talk.

The technical point is this. If $\overline{\mathcal{O}}$ is normal, then the rings $\overline{R} = R(\overline{\mathcal{O}})$ and $R = R(\mathcal{O})$ coincide; in general R is the normalization of $\overline{R}$. Instead of considering a product $*_t$ on $\overline{R}$ we could consider a product $*_t$ on R, again satisfying conditions (1)-(3) and (G). Since R is in general a proper subset of $\overline{R}$, these notions are different. However, the kernel J of a map $U(\mathfrak{g}) \to (R, *_1)$ will again be a primitive ideal. For this reason, even in the non-normal case it is worth considering $*_t$ on R, as well as on $\overline{R}$.

7 The eigenvalue of the Casimir

The results of the previous section motivate studying G-invariant $*_t$ on $R = R(\mathcal{O})$ satisfying the associativity constraints, plus conditions (1)-(3) and (G). It is natural therefore to

[2] Actually, for condition (3), we need $J^t = J$, where the superscript is the transpose anti-involution of $\mathfrak{g}$. This holds if J is the annihilator of a unitary representation.

ask in addition for (4) and (5) (in the context of algebraic, or holomorphic, bidifferential operators).

Recall that we have identified $\mathfrak{g}$ with $\mathfrak{g}^*$ via a nondegenerate form B. Pick $f \in \mathcal{O} \subset \mathfrak{g}$, and put f into a standard triple $\{h, e, f\}$. Under the action of the $\mathfrak{sl}_2$ they span, decompose $\mathfrak{g}$ into irreducibles as $\mathfrak{g} = \oplus_n V_n^{\oplus a_n}$, where V_n is the irreducible $\mathfrak{sl}_2$-module of highest weight n. In other words, a_n is the multiplicity of V_n in $\mathfrak{g}$.

Theorem 7.1 *Suppose $*_t$ is a G-invariant star-product satisfying conditions (1)-(5) and (G). Then the eigenvalue of the Casimir, denoted $\Phi(\Omega)$, is*

$$\Phi(\Omega) = \frac{dim\ \mathfrak{g}}{B(e,f)} \sum_n a_n C_n. \tag{7.2}$$

Here the numbers C_n depend only on n, and can be explicitly calculated.

I will not give the explicit formula for C_n, but simply note that $C_0 = C_1 = 0$, $C_2 = -1/8$.

8 Outline of proof

To begin, I will make more explicit the identification between $\mathfrak{g}$ and $\mathfrak{g}^*$. Elements of $S(\mathfrak{g})$ are functions on $\mathfrak{g}^*$; hence since $\mathcal{O} \subset \mathfrak{g}^*$, they define functions on $\mathcal{O}$. Since $\mathfrak{g} = S^1(\mathfrak{g})$, elements of $\mathfrak{g}$ define degree 1 functions on $\mathcal{O}$. In particular this is true of h, e, f. Under the identification of $\mathfrak{g}$ with $\mathfrak{g}^*$ by the form B, for $x \in \mathfrak{g}$, the value of x at $f \in \mathcal{O}$, $x(f)$, is $x(f) = B(x, f)$.

8.1 Ingredients of the proof

We can divide the proof into 4 steps:

1. Reduce to understanding m_2.

2. Reduce further to understanding the scalar $m_2(e, f)$.

3. Define a new map $m : R \otimes R \to \mathbb{C}$ by $m(r \otimes s) = m_2(r, s)(f)$. Then we need properties of m following from:
   (a) Associativity of $*_t$;
   (b) G-invariance of m_2;
   (c) The order of m_2 is $(2, 2)$;
   (d) m_2 vanishes on constants.

4. Vanishing property of elements of $\mathfrak{g}^f$ (viewed as elements of R) at $f \in \mathcal{O}$. Here $\mathfrak{g}^f \subset \mathfrak{g}$ is the stabilizer subalgebra of f.

9 Some details

Step 1 consists of the following lemma.

Lemma 9.1 *Choose an orthonormal basis $\{x_i\}$ (with respect to the form B). Then $\Phi(\Omega) = \sum_i m_2(x_i, x_i)$.*

Proof. Since x_i has degree 1 in R, our assumptions on the m_k imply that deg $m_k(x_i, x_i) = 2 - k$. In particular, $m_k(x_i, x_i) = 0$ for $k \geq 3$. Therefore,

$$\Phi(\Omega) = \sum_i x_i^2 + \frac{1}{2}[x_i, x_i] + m_2(x_i, x_i).$$

But $\sum_i x_i^2$ is an invariant positive degree element in $S(\mathfrak{g})$, and therefore vanishes on $\mathcal{O}$. That is, $\sum_i x_i^2$ is 0 as an element of R. Since $[x_i, x_i] = 0$, the result follows. □

For Step 2, we can simplify matters a little by rescaling the form B so that $B(e, f) = 1$; this is harmless since both sides of (7.2) transform in the same way under rescaling. Now, $\mathfrak{g} \subset R$, so

$$m_2|_{\mathfrak{g}\otimes\mathfrak{g}} : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$$

defines a symmetric bilinear form on $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, this form is a multiple λ of B, and the scalar λ is given by $m_2(e, f)$. Thus,

$$\Phi(\Omega) = \sum m_2(x_i, x_i) = \sum \lambda B(x_i, x_i) = \dim\, \mathfrak{g} \cdot m_2(e, f).$$

By our assumptions on m_2, $m_2(e, f)$ is a degree 0 element of R, i.e. a scalar. To evaluate it, it suffices to evaluate it at any point of $\mathcal{O}$, in particular, at the special point $f \in \mathcal{O}$.

We have defined $m : R \otimes R \to \mathbb{C}$ by $m(r, s) = m_2(r, s)(f)$. By the above remarks, Theorem 7.1 will follow from:

Theorem 9.2 *$m(e, f) = \sum a_n C_n$, where a_n and C_n are as above.*

To establish this theorem, we need to carry out Step 3, which gives some properties of the map m. The properties are (the reasons they hold are sketched in Step 3 above):

(a) Suppose $a, b, c \in R$. Then

$$a(f) \cdot m(b, c) - m(ab, c) + m(a, bc) - m(a, b) \cdot c(f) = N(a, b, c),$$

where we define

$$N(a, b, c) = \frac{1}{4}\{\{a, c\}, b\}(f).$$

If $a, b, c \in \mathfrak{g}$ (recall that $\mathfrak{g} \subset R$) then the right hand side is

$$\frac{1}{4}B([a, c], [b, f]). \tag{9.3}$$

(b) For $\alpha \in R \otimes R$, we have $m((\text{ad } f)\alpha) = 0$.

(c) If $a \in R$ vanishes to order ≥ 3 at f, then for all $b \in R$, we have $m(a, b) = 0$.

(d) $m(a + \text{constant}, b) = m(a, b)$.

Finally, Step 4 says: If x is in the stabilizer algebra $\mathfrak{g}^f$, then x, viewed as an element of R, vanishes to order 2 at f.

10 Completion of the proof for $\mathfrak{sl}_2$

We will use the facts from the previous section to complete the proof in the special case where $\mathfrak{g} = \mathfrak{sl}_2$, and $\mathcal{O}$ is the non-trivial nilpotent orbit. We must show that

$$m(e, f) = C_2 = -\frac{1}{8}.$$

Let $I_{f,k} \subset R$ denote the ideal of functions vanishing to order k at $f \in \mathcal{O}$.

Lemma 10.1 $m(e, f) = -\frac{1}{4} m(e, h^2)$.

Proof. Since $h^2 + 4ef$ is invariant, it vanishes on $\mathcal{O}$, so as elements of R we have $ef = -\frac{1}{4}h^2$. As $e(f) = 1$, we have $f = -\frac{1}{4}h^2$ modulo $I_{f,3}$. Since m_2 has order $(2, 2)$, we conclude that $m(e, f) = -\frac{1}{4} m(e, h^2)$. □

Lemma 10.2 $m(e, h^2) = N(e, e, f)$.

Proof. Again using the equation $h^2 + 4ef = 0$ in R, we have

$$e \otimes h^2 = 2e \otimes ef - f \otimes e^2 + (\mathrm{ad}\ f)\alpha$$

for some $\alpha \in R \otimes R$. Therefore,

$$m(e, h^2) = 2m(e, ef) - m(e^2, f) = 2m(e, f) - m(e^2, f),$$

since $ef = f$ modulo $I_{f,3}$ (as f vanishes to order 2 at f). By Step 3 above,

$$\begin{aligned} N(e, e, f) &= e(f)m(e, f) - m(e^2, f) + m(e, ef) - m(e, e)f(f) \\ &= 2m(e, f) - m(e^2, f), \end{aligned}$$

proving the lemma. □

We conclude that $m(e, f) = -\frac{1}{4} N(e, e, f)$. But we can calculate $N(e, e, f)$ from (9.3); this gives $m(e, f) = -\frac{1}{8}$, completing the proof.

11 Nonexistence of star-products

Suppose that for some orbit $\mathcal{O}$, we know the scalars c_J for the primitive ideals J with $\mathrm{gr}\ J = I(\overline{\mathcal{O}})$. We can compare these scalars with the scalar $c_{J_{star}}$ of Theorem 7.1. If $c_{J_{star}}$ does not equal one of the possible c_J, we conclude that $\mathcal{O}$ does not admit a G-invariant star-product with properties (1)-(5) and (G).

In particular, this is the case for the nilpotent orbits $\mathcal{O}_p$ in $\mathfrak{sp}_{2n}$ corresponding to the partitions $(2^p, 1^{2n-2p})$. These orbits have normal closures [H]. For most p, there is a unique completely prime primitive ideal J_p in $U(\mathfrak{g})$ whose associated variety is the orbit closure $\overline{\mathcal{O}_p}$ (the annihilator of a holomorphic discrete series representation studied by Rossi-Vergne and others). This implies that J_p is the only possible ideal with $\mathrm{gr}\ J_p = \overline{\mathcal{O}}_p$. The infinitesimal characters of the ideals J_p are known, so the scalars c_{J_p} can be calculated; and for $p > 1$ they disagree with the scalar produced by Theorem 7.1. We conclude that these orbits have no G-invariant star-products with properties (1)-(5) and (G). For $p = 1$ the orbit admits a double covering by $\mathbb{C}^{2n} - \{0\}$; there is a star-product on $\mathbb{C}^{2n} - \{0\}$ defined by (3.1), and this descends to a star-product on the orbit having all of the above properties.

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