# THE MOTIVE OF A REDUCTIVE GROUP

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## 1. Motives and $L$-functions

Let $k$ be any field, $k^s$ a fixed separable closure of $k$, and $Gal(k^s/k)$ the Galois group of $k^s$ over $k$. Let $V$ be a finite dimensional vector space over $\mathbb{Q}$, and let

$$\rho : Gal(k^s/k) \longrightarrow GL(V) \tag{1.1}$$

be a continuous Galois representation on $V$. By definition, $\rho$ factors through a finite Galois extension of $k$.

The category of such continuous Galois representations is equivalent to the category of Artin motives, which are those motives corresponding to zero dimensional varieties over $k$. Hence we shall call such a $(\rho, V)$ an Artin motive over $k$. As we do not need the theory of motives in general, it is best to take (1.1) as the definition of an Artin motive.

Now if $k$ is a finite field, local field or global field, we have an $L$-function attached to any Artin motive $(\rho, V)$ in the following way:

- If $k$ is finite of order $q$, then let

$$\sigma : x \mapsto x^q, \quad x \in k$$

be a topological generator of $Gal(k^s/k)$. Let $F = \sigma^{-1}$ be the Frobenius element. Note that the eigenvalues of $F$ on $V$ are roots of unity. Now we define:

$$L(V, s) = det(1 - Fq^{-s}|V)^{-1} \tag{1.2}$$

for $s \in \mathbb{C}$.

- If $k$ is a non-archimedean local field, then we have the exact sequence:

$$1 \longrightarrow I \longrightarrow Gal(k^s/k) \longrightarrow \widehat{\mathbb{Z}} \cdot F \longrightarrow 1$$

where $I$ is the inertia subgroup, and $\widehat{\mathbb{Z}} \cdot F$ is the absolute Galois group of the residue field of $k$, with $F$ chosen as above. Now we define:

$$L(V, s) = det(1 - Fq^{-s}|V^I)^{-1} \tag{1.3}$$

where $q$ is the order of the residue field of $k$, and $s \in \mathbb{C}$. Note that this $L$-function has no zeros, but may have poles; for example, at $s = 0$ when $V^{Gal(k^s/k)} \neq 0$.

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• If $k = \mathbb{R}$, we let $V^+$ be the $(+1)$-eigenspace of the action of complex conjugation on $V$, and $V^-$ the $(-1)$-eigenspace. Let

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2) \tag{1.4}$$

where $\Gamma$ is the usual gamma function. Then we define

$$L(V,s) = \Gamma_{\mathbb{R}}(s)^{dim(V^+)} \cdot \Gamma_{\mathbb{R}}(s+1)^{dim(V^-)} \tag{1.5}$$

If $k = \mathbb{C}$, we define

$$L(V,s) = (\Gamma_{\mathbb{R}}(s) \cdot \Gamma_{\mathbb{R}}(s+1))^{dim(V)} \tag{1.6}$$

• If $k$ is a global field and $v$ is a place of $k$, then

$$Gal(k_v^s/k_v) \hookrightarrow Gal(k^s/k)$$

as the decomposition group at $v$. This embedding is well-defined up to conjugation, and so we can regard $(\rho, V)$ as an Artin motive over $k_v$, by restriction. Let $L_v(V,s)$ be the $L$-function of this Artin motive, as defined in (1.3), (1.5) and (1.6). Now we define

$$L(V,s) = \prod_v L_v(V,s) \tag{1.7}$$

where the product is taken over all places of $k$. This product converges absolutely for $Re(s) > 1$.

It was proved by Artin and Brauer that the above product has a meromorphic continuation, which we shall also denote by $L(V,s)$, to all $s \in \mathbb{C}$, and satisfies a functional equation with respect to the substitution $s \mapsto 1-s$. It is conjectured that $L(V,s)$ is entire, except possibly at $s = 0$ and $s = 1$, and that all the zeroes of $L(V,s)$ satisfy $Re(s) = \frac{1}{2}$. When $k$ is a function field, this was proved by Weil.

If $S$ is a finite set of places of $k$, containing the set $S_\infty$ of all archimedean places, we set

$$L_S(V,s) = \prod_{v \notin S} L_v(V,s) \tag{1.8}$$

which is called the partial $L$-function. By results of Euler and Siegel, $L_S(V,n)$ is a rational number if $n \leq 0$ is a non-positive integer.

**Example:** If $V = \mathbb{Q}$ is the trivial representation, and $S = S_\infty$, then $L_S(V,s) = \zeta_k(s)$, the zeta function of the global field $k$.

This completes our definition of the $L$-function attached to an Artin motive. For our purposes, we need to consider motives which are slightly more general than Artin motives, namely the Artin-Tate motives. If $V$ is an Artin motive over $k$, and $n \in \mathbb{Z}$, there is an Artin-Tate motive $V(n)$, obtained from

$V$ by $n$ Tate twists. Without giving the precise definition of $V(n)$, let us simply define its $L$-function to be

$$L(V(n), s) = L(V, s+n) \tag{1.9}$$

If $F$ is a Frobenius element in $Gal(k^s/k)$ (with $k$ one of the fields above), then the eigenvalues of $F$ on $V(n)$ are of the form $\zeta q^{-n}$, with $\zeta$ a root of unity. Hence $V(n)$ is not a rational representation of $Gal(k^s/k)$ (unless $n = 0$). However there is an $l$-adic representation $V_l(n)$ with these eigenvalues for all $l \neq char(k)$.

## 2. The Motive of $G$

Let $G$ be a connected reductive group over $k$. In this section, we show how to attach an Artin-Tate motive $M$ to $G$. For more details, see [Gr].

First, assume that $G$ is quasi-split, and fix $T \subset B$, a maximal torus contained in a Borel subgroup, both defined over $k$. Then

$$E = Hom_{k^s}(T, \mathbb{G}_m) \otimes \mathbb{Q} \tag{2.1}$$

is a representation of $W \rtimes Gal(k^s/k)$, where $W = N_G(T)/T$ is the Weyl group of $T$ in $G$. Let

$$Sym^\bullet(E)^W := R = \mathbb{Q} \oplus \mathbb{R}_+ \tag{2.2}$$

where $R_+$ denotes the set of invariants of $W$ of positive degree. Also, let

$$R_+/R_+^2 := V = \oplus_{d \geq 1} V_d \tag{2.3}$$

Then $V$ is a graded representation of $Gal(k^s/k)$. It is a theorem of Steinberg that $V$, without its grading, is isomorphic to $E$ as a $Gal(k^s/k)$-module. Now we make the following definition:

**Definition:** The motive of $G$ is

$$M = \oplus_d V_d(1-d) \tag{2.4}$$

If $G$ is not quasi-split, we define the motive of $G$ to be the motive of its quasi-split inner form. Let $M^\vee$ denote the dual motive to $M$, then we have

$$M^\vee(1) = \oplus_d V_d(d) \tag{2.5}$$

If $k$ is finite, local or global, we have the $L$-function values:

$$L(M) := L(M, 0) = \prod_{d \geq 1} L(V_d, 1-d) \tag{2.6}$$

and

$$L(M^\vee(1)) := L(M^\vee(1), 0) = \prod_{d \geq 1} L(V_d, d) \tag{2.7}$$

Similarly, we may define $L_S(M)$ in the global case; when $S$ contains $S_\infty$, this is either a rational number or infinity.

**Examples:** (i) If $G = T$ is a torus, then $V = E = V_1$, and $M$ is an Artin motive. In this case, $L(M) = L(V, 0)$.

(ii) If $G = GL_n$, then $E = \mathbb{Q}^n$ as a Galois representation, and $W \cong S_n$ acts by permuting the $n$ factors. Thus $V_1 = V_2 = ... = V_n = \mathbb{Q}$, and

$$M = \oplus_{i=1}^{n} \mathbb{Q}(1 - i).$$

(iii) If $G = U_n$, a quasi-split unitary group associated to the separable quadratic extension $K/k$, and $\varepsilon : Gal(K/k) \longrightarrow \pm 1$ is the non-trivial character of $Gal(K/k)$, then

$$M = \mathbb{Q}(\varepsilon) \oplus \mathbb{Q}(-1) \oplus \mathbb{Q}(\varepsilon)(-2) \oplus ... \oplus \mathbb{Q}(\varepsilon^n)(1 - n)$$

Now we give an alternative definition of $M$, using the $L$-group of $G$ [Gr2]. Let $\widehat{G}$ be the dual group of $G$, which is defined and split over $\mathbb{Q}$. Then, corresponding to our fixed choice of $T \subset B$, we have a pinning of $\widehat{G}$:

$$\widehat{T} \subset \widehat{B} \subset \widehat{G}$$

$$e_{\hat{\alpha}} : \mathbb{G}_a \cong \widehat{\mathbb{U}}_{\hat{\alpha}} \tag{2.8}$$

for all simple roots $\hat{\alpha}$ (with respect to $\widehat{T} \subset \widehat{B}$). There is an action of $Gal(k^s/k)$ on $\widehat{G}$ preserving the pinning. Under this setting, the $W \rtimes Gal(k^s/k)$-module $E$ discussed above is simply $Lie(\widehat{T})$, which is the centralizer in $\widehat{\mathfrak{g}} = Lie(\widehat{G})$ of a regular semi-simple element in $Lie(\widehat{T})$. Now let:

$$N = \oplus_{\hat{\alpha} > 0} Lie(e_{\hat{\alpha}})(1) \in \widehat{\mathfrak{g}} \tag{2.9}$$

Then $N$ is a regular nilpotent element fixed by $Gal(k^s/k)$. Hence, $Gal(k^s/k)$ acts on $V'$, the centralizer of $N$ in $\widehat{\mathfrak{g}}$. Moreover this Galois representation is graded. Indeed, the line $\mathbb{Q}N$ is normalized by a one-dimensional torus: if

$$2\rho = \sum_{\alpha > 0} \alpha : \mathbb{G}_m \longrightarrow \widehat{\mathbb{T}}$$

then $2\rho(\mathbb{G}_m)$ normalizes $\mathbb{Q}N$. Let $V'_d$ be the subspace of $V'$ on which $\mathbb{G}_m$ acts via:

$$\lambda \mapsto \lambda^{2(d-1)}, \quad \lambda \in \mathbb{Q}^{\times}$$

Then we have:

$$V' = \oplus_d V'_d \cong V = \oplus_d V_d$$

as graded Galois modules. Hence we can use $V'$ to define the motive $M$ of $G$.

## 3. Haar Measures

In this section, we give some applications of the motive of $G$ and show how it arises in the harmonic analysis on $G(k)$. In particular, we relate various natural Haar measures on $G(k)$ by using the $L$-function of $M$.

First let $k$ be a finite field with $q$ elements. Then Steinberg showed that:

$$\frac{\#G(k)}{q^{dim(G)}} = \prod_d det(1 - Fq^{-d}|V_d) = L(M^\vee(1))^{-1} \tag{3.1}$$

We can interpret this formula as follows. The finite group $G(k)$ can be regarded as a discrete group or as a compact group. As a discrete group, the most natural invariant measure on $G(k)$ is $\mu_{G,disc}$, which gives each element of $G(k)$ measure 1, but as a compact group, the natural measure is $\mu_{G,cpct}$, which gives the whole group measure 1. Then (3.1) amounts to giving the ratio of the two measures:

$$\frac{\mu_{G,disc}}{\mu_{G,cpct}} = \#G(k) = \frac{q^{dim(G)}}{L(M^\vee(1))} \tag{3.2}$$

We shall discuss similar results when $k$ is local or global. The local results are reinterpretations of results of Serre, Steinberg, Kottwitz and Tate, whereas the global results are inspired by the work of Harder and Prasad.

Now assume that $k$ is a local field of characteristic zero. Then $G(k)$ is locally compact and hence the Haar measure exists. There is a natural invariant measure $\mu_G$ (not necessarily positive) on $G(k)$, called the Euler-Poincaré measure, which was introduced and studied by Serre [Se]. The measure $\mu_G$ can be characterized as follows. Let $\Gamma \subset G(k)$ be any discrete, torsion-free, co-compact subgroup. Such groups exist by results of Borel and Harder. Then $\mu_G$ is characterized by:

$$\int_{\Gamma\backslash G(k)} \mu_G = \chi(\Gamma) := \sum_i (-1)^i dim H^i(\Gamma, \mathbb{Q}) \tag{3.3}$$

for all $\Gamma$. Note that $\mu_G$ can be zero. Indeed we have the following theorem:

**Theorem 3.4.** *The following are equivalent:*

*(i)* $\mu_G \neq 0$.

*(ii)* $G$ *has a maximal anisotropic torus over* $k$.

*(iii)* $G$ *has representations in the discrete series.*

*(iv)* $L(M) \neq \infty$.

*If the above conditions hold, and $k$ is non-archimedean, then* $\mu_G = (-1)^{rank(G)}\mu_{St}$, *where* $\mu_{St}$ *is the Plancherel measure of the Steinberg representation of* $G(k)$.

We note that the equivalence of (i) and (ii) is due to Serre [Se]. The theorem suggests that we consider the measure $L(M) \cdot \mu_G$, when they are both defined. This measure is actually positive when $G$ is quasi-split.

Another natural Haar measure on $G(k)$ is $|\omega_G|$, which, for $k$ non-archimedean, is defined by an invariant differential of top degree using Bruhat-Tits theory [Gr]. For our purposes, it suffices to say that if $G$ is unramified, and $G(A)$ is a hyperspecial maximal compact subgroup of $G(k)$, then $\omega_G$ is an invariant differential of top degree on $G(A)$, which has good reduction modulo the maximal ideal. We refer the reader to [Gr] for the definition in the archimedean case. Now the following theorem relates the measures $\mu_G$ and $|\omega_G|$, and can be regarded as a functional equation for the $L$-function of $M$.

**Theorem 3.5.**

$$L(M) \cdot \mu_G \cdot c(G) \cdot e(G) = L(M^\vee(1)) \cdot |\omega_G|$$

*where* $e(G) = \pm 1 \in Br_2(k)$ *is the Kottwitz invariant [K], and*

$$c(G) = \begin{cases} \#H^1(k,G), & \textit{if } k \textit{ is non-archimedean;} \\ \frac{\#H^1(k,T)}{\#ker(H^1(k,T)\to H^1(k,G))}, & \textit{if } k = \mathbb{R}. \end{cases}$$

Now we come to the case when $k$ is a number field, with adele ring $\mathbb{A}$. Let $S \supset S_\infty$ be a finite set of places of $k$ such that for all $v \notin S$, $G_v := G \otimes k_v$ is unramified. Also assume that $\#S \geq 2$. For $v \notin S$, let $\mu_v$ be the Haar measure on $G(k_v)$ giving $G(A_v)$, a hyperspecial maximal compact subgroup, volume 1. For $v \in S$, let $\mu_v$ be the Euler-Poincaré measure. Then define:

$$\mu_S = \prod_v \mu_v \tag{3.6}$$

which is an invariant measure on $G(\mathbb{A})$. Also let $|\omega|$ be the Tamagawa measure of $G(\mathbb{A})$. Then the global analogue of Theorem (3.5) is:

**Theorem 3.7.**

$$\mu_S = \frac{L_S(M)}{\prod_{v\in S} c(G_v)} \cdot |\omega|$$

Note that $\mu_S/|\omega|$ is a rational number.

**Example:** If $G = SL_2$, and $S = S_\infty$, then $M = \mathbb{Q}(-1)$, and $\mu_S = \zeta_k(-1)|\omega|$.

## References

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[Se] J.P. Serre, *Cohomologie des Groupes Discrets*, Serre's Collected Works Vol 2.