AN APPROACCH TO A CONJECTURE OF JACQUET

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1. Introduction

In this talk, we shall provide an approach to a conjecture of Jacquet, by means of periods of residual representations from certain Eisenstein series. First we recall Jacquet's conjecture and some relevant work by D. Prasad and by M. Harris and S. Kudla.

1.1. **Jacquet's Conjecture.** Let F be a number field and $\mathbb{A} = \mathbb{A}_F$ the ring of adeles of F. Let (π_i, V_i) $(i = 1, 2, 3)$ be irreducible cuspidal automorphic representations of $GL_2(\mathbb{A})$ with $\prod_{i=1}^{3} \omega_{\pi_i} = 1$, where ω_{π_i} is the central character of π_i. Then one has the restricted tensor product decomposition for π_i $(i = 1, 2, 3)$

$$\pi_i = \otimes'_{v\in\Omega_F} \pi_{i,v}.$$

We need following notations:

$$S := \{v \in \Omega_F \mid \pi_{i,v}, i = 1, 2, 3, \text{belong to discrete series}\}$$

and

$$\mathcal{D}_S := \{D \mid \text{quaternion algebras over } F, \text{which is ramified only at } v \in S\}.$$

By the global Jacquet-Langlands correspondence, for each quaternion algebra $D \in \mathcal{D}_S$, there is an irreducible automorphic representation π_i^D of $D^\times(\mathbb{A})$ for $i = 1, 2, 3$, which corresponds to π_i, respectively.

Now, for given three irreducible cuspidal automorphic representations (π_i, V_i) $(i = 1, 2, 3)$ of $GL_2(\mathbb{A})$ with $\prod_{i=1}^{3} \omega_{\pi_i} = 1$, one can have the triple product L-function $L(s, \pi_1 \otimes \pi_2 \otimes \pi_3)$. The integral representation for the triple product L-functions was first given by P. Garrett in [G] and more generally by I. Piatetski-Shapiro and Rallis in [PSR].

Conjecture 1 (Jacquet). *The triple product L-function $L(s, \pi_1 \otimes \pi_2 \otimes \pi_3)$ does not vanish at $s = \frac{1}{2}$ if and only if there is at least one quaternion algebra $D \in \mathcal{D}_S$ such that for some $f_i^D \in \pi_i^D$, $i = 1, 2, 3$, the following period integral*

$$\int_{\mathbb{A}_F^\times D(F)\backslash D(\mathbb{A})} f_1^D(g) f_2^D(g) f_3^D(g) dg$$

does not vanish.

This is the summary for my talk at Seattle meeting, July, 1997.

Remark 1. *Since the pair* $(PGL_2 \times PGL_2, PGL_2)$ *can be viewed as the pair* $(SO(4), SO(3))$*, the Jacquet conjecture can be related to the more general conjecture of Gross and Prasad for the pair* $(SO(n), SO(n-1))$ [GP].

1.2. **The Work of D. Prasad.** Let (π_i, V_i) $(i = 1,2,3)$ be irreducible admissible representations of $GL_2(F_v)$ with $\prod_{i=1}^3 \omega_{\pi_i} = 1$. D. Prasad proved in [Pr] the following theorem.

Theorem 1 (Prasad). *If at least one of the representations* (π_i, V_i) $(i = 1,2,3)$ *does not belong to the discrete series of representations of* $GL_2(F_v)$*, then*

$$\dim_{\mathbb{C}} Hom_{GL_2(F_v)}(V_1 \otimes V_2 \otimes V_3, \mathbb{C}) = 1.$$

If all of the representations (π_i, V_i) $(i = 1,2,3)$ *belong to the discrete series of representations of* $GL_2(F_v)$*, then, by the local Jacquet-Langlands correspondence, for each* i*, there is an irreducible repressentation* (π_i^D, V_i^D) *of* $D^\times(F_v)$*, where* D *is the nonsplit quaternion algebra over* F_v*, such that*

$$\dim_{\mathbb{C}} Hom_{D^\times(F_v)}(V_1^D \otimes V_2^D \otimes V_3^D, \mathbb{C}) = 1.$$

Remark 2. (1) *When* $F_v = \mathbb{R}$*, it is assumed in* [Pr] *that at least one of the representations* (π_i, V_i) $(i = 1,2,3)$ *belongs to the discrete series of representations of* $GL_2(F_v)$.

(2) *When all of the representations* (π_i, V_i) $(i = 1,2,3)$ *are supercuspidal, then the existence of the* $GL_2(F_v)$ *(or* $D^\times(F_v)$*) invariant trilinear functionals can be determined by the corresponding local* ϵ*-factors. This is the local evidence for the Jacquet conjecture.*

1.3. **The Work of M. Harris and S. Kudla.** Based on the work of P. Garrett and the work of I. Piatetski-Shapiro and S. Rallis on the integral representation for the triple product L-function, Harris and Kudla established in [HK] the 'Main Identity', which relates the central critical value of the triple product L-function and the trilinear period integrals, by means of Seesaw duality, Siegel-Weil formula, and a theorem of Shimizu. The Jacquet conjecture follows from the 'Main Identity' if one knows explicitly the analytic properties of the relevant local zeta integrals, which was determined in [HK] for the cases when the Ramanujan conjecture is valid.

2. Approach by means of Periods of Residual Representations

We shall provide an approach by means of constant terms of Eisentein series and the periods of residual representations.

2.1. G_2**-periods of Residual Representations.** The study of periods of residual representations associated to Eisenstein series was initiated by Jacquet and Rallis in [JR] for the case of $(GL(2n), Sp(2n))$, which was motivated by the devolopment of various trace formulas for symmetric spaces. In [Jng], some cases of non-symmetric space, spherical type was computed explicitly, essentially following the idea in [JR]. The following is one of the cases from [Jng].

Let $G = PGO(4,4)$ and $P = M_2N_2$ the standard maximal parabolic subgroup of G with $M_2 = GL_2 \times PGO(2,2)$. Let π_i, $i = 1,2,3$, be irreducible cuspidal automorphic representations of $GL_2(\mathbb{A})$ with $\omega_{\pi_i} = 1$. Then the tensor product $\sigma = \pi_1 \otimes \pi_2 \otimes \pi_3$ can be viewed as an irreducible cuspidal automorphic representation of $M_2(\mathbb{A})$. From the cuspidal data (M_2, σ), one can form an Eisenstein series as follows:

$$E(g;\Phi_{\sigma,s},s) := \sum_{\gamma\in P\backslash G} \Phi_{\sigma,s}(\gamma g).$$

It is known from Langlands theory of Eisenstein series that the Eisenstein series $E(g;\Phi_{\sigma,s},s)$ attached to the cuspidall data (M_2,σ) defines a meromorphic function in s over $\mathbb{C}$ and has finitely many possible simple poles when the real part of s is non-negative.

Let $H = G_2$ be the split group of type G_2, which can be embedded into G via triality. We are interested in the following period integral

$$\mathcal{P}_{G_2}(res_{s=s_0}E(\cdot,\Phi_{\sigma,s},s)) := \int_{G_2(F)\backslash G_2(\mathbb{A})} res_{s=s_0}E(h;\Phi_{\sigma,s},s)dh.$$

Theorem 2 ([Jng]). *With notations as above, we have*

(1) *If there is a $\Phi_{\sigma,s}$ such that $\mathcal{P}_{G_2}(res_{s=s_0}E(\cdot,\Phi_{\sigma,s},s)) \neq 0$, then $s_0 = \frac{1}{2}$.*
(2) *There is a $\Phi_{\sigma,s}$ such that $\mathcal{P}_{G_2}(res_{s=\frac{1}{2}}E(\cdot,\Phi_{\sigma,s},s)) \neq 0$ if and only if there exists $\phi_{\pi_i} \in V_i$ for $i = 1,2,3$, such that the following period integral*

$$\int_{PGL_2(F)\backslash PGL_2(\mathbb{A})} \phi_{\pi_1}(g)\phi_{\pi_2}(g)\phi_{\pi_3}(g)dg \neq 0.$$

Corollary 1. *Let π_i, $i = 1,2,3$, be irreducible cuspidal automorphic representations of $GL_2(\mathbb{A})$ with trivial central character, such that the residue at $s = \frac{1}{2}$ of the Eisenstein series $E(g;\Phi_{\sigma,s},s)$ has non-zero G_2-period $\mathcal{P}_{G_2}(res_{s=\frac{1}{2}}E(\cdot,\Phi_{\sigma,s},s))$. Then, for each local place v of the number field F, at least one of the three local components at v, $\pi_{1,v}$, $\pi_{2,v}$ and $\pi_{3,v}$, of π_1, π_2 and π_3, respectively, does not belong to the discrete series of representations of $GL_2(F_v)$.*

Proof. By the second part of the Theorem, the period gives rise to a nontrivial global $GL_2(\mathbb{A})$-invariant trilinear functional over the space $V_1\otimes V_2\otimes V_3$. Then, after restricting to each local place v, we have a nontrivial $GL_2(F_v)$-invariant trilinear functional over the space $V_{1,v}\otimes V_{2,v}\otimes V_{3,v}$. Finally, the claimed result follows from the theorem of Prasad. □

2.2. **Constant Term of Eisenstein Series.** By Langlands theory of constant terms of Eisenstein series [L], we can relate our nonvanishing result of the G_2-period of the residual representations to the nonvanishing of the central critical value of the triple product L-function $L(s,\pi_1\otimes\pi_2\otimes\pi_3)$.

We consider the constant term along the maximal parabolic subgroup P_2 of the Eisentein series $E(g;\Phi_{\sigma,s},s)$, which is given by

$$E_{P_2}(g;\Phi_{\sigma,s},s)=\Phi_{\sigma,s}(g)+\mathcal{M}_{\sigma,s}(\Phi_{\sigma,s})(g),$$

where $\mathcal{M}_{\sigma,s}(\Phi_{\sigma,s})$ is the intertwining operator associated the longest Weyl group element in this situation. Then the nonvanishing of the residue at $s=\frac{1}{2}$ of the Eisenstein series $E_{P_2}(g;\Phi_{\sigma,s},s)$ is equivalent to the nonvanishing of the residue at $s=\frac{1}{2}$ of $\mathcal{M}_{\sigma,s}(\Phi_{\sigma,s})(g)$.

Following the theory of Langlands-Shahidi [Sh], the local c-function at each unramified local place v of the local component of the intertwining operator $\mathcal{M}_{\sigma,s}(\Phi_{\sigma,s})$ is

$$\frac{L(s,\pi_{1,v}\otimes\pi_{2,v}\otimes\pi_{3,v})\zeta_v(2s)}{L(s+1,\pi_{1,v}\otimes\pi_{2,v}\otimes\pi_{3,v})\zeta_v(2s+1)}.$$

Then, by proving a theorem of Casselman-Shahidi type [CS] on the normalization of the local intertwining operators, we obtain that the nonvanishing of the residue at $s=\frac{1}{2}$ of $\mathcal{M}_{\sigma,s}(\Phi_{\sigma,s})(g)$ is equivalent to the nonvanishing of the value at $s=\frac{1}{2}$ of the triple product L-function $L(s,\pi_1\otimes\pi_2\otimes\pi_3)$.

Combining with the theorem in [Jng], we will have the following theorem.

Theorem 3 ([Jng1]). *The nonvanishing of the period integral*

$$\int_{PGL_2(F)\backslash PGL_2(\mathbb{A})}\phi_{\pi_1}(g)\phi_{\pi_2}(g)\phi_{\pi_3}(g)dg$$

for some $\phi_{\pi_i}\in V_i$ for $i=1,2,3$ implies that

(1) *for each local place v of the number field F, at least one of the three local components at v, $\pi_{1,v}$, $\pi_{2,v}$ and $\pi_{3,v}$, of π_1, π_2 and π_3, respectively, does not belong to the discrete series of representations of $GL_2(F_v)$;*

(2) *the nonvanishing of the value at $s=\frac{1}{2}$ of the triple product L-function $L(s,\pi_1\otimes\pi_2\otimes\pi_3)$.*

Remark 3. *This theorem can be viewed as one direction of the conjecture of Jacquet when S is an empty set. The other direction is more involved and the work is in progress. It will be very interesting to formulate a similar approach which is able to deal with the general case when S is not empty.*

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