# Hecke Algebras of Classical Groups over $p$-adic field and Supercuspidal Representations

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## Abstract

Let $G$ be a classical group defined over $p$-adic field $k$. In order to understand the admissible representations of $G$, we construct a family of Hecke algebras on $G$. As a corollary, we also get new supercuspidal representations. When $G$ is a symplectic group in $2n$ variables over $k$, we describe some of these algebras explicitly in terms of generators and relati ons. Our constructions are valid when the residual characteristic of $k$ is large.

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## INTRODUCTION

$k$ : $p$-adic field with $\mathrm{char}(\overline{k}) = p$, $\overline{k} = \mathbb{F}_q$

$G$ : (connected) reductive group over $k$,

Jacquet, H-C $\Rightarrow$ all irreducible admissible representation of $G$ can

be realized as a subrepresentation of the $\mathrm{Ind}_P^G \sigma$ with $P = MN$ and $\sigma$ supercuspidal representation of $M$.

(e.g) Spherical principal series can be understood by Iwahori Hecke algebras (Casselman, Borel)

$P = B$, $\sigma = \nu$: unramified char of $B/N$

$\mathrm{Ind}_B^G \nu$ can be understood via $\mathcal{H}(G//I)$

$$\begin{array}{c} \text{Irreducible subquotients of} \\ \{\ \mathrm{Ind}_B^G \nu \mid \nu\text{:unramified}\ \} \end{array} \longleftrightarrow \begin{array}{c} \text{finite dimensional simple} \\ \text{representations of } \mathcal{H}(G//I) \end{array}$$

$$W \longleftrightarrow W^I$$

[KL] classifies the simple modules of $\mathcal{H}(G//I)$ when $G$ is split wit h connected center.

**Definition.** Let $(J, \rho)$ be a pair of open compact subgroup $J$ and its representation $\rho$. Define a *spherical function algebra* associated to $(J, \rho)$ :

$$\mathcal{H}(G//J,\rho) = \left\{ \begin{array}{l} f \in C_c^\infty(G, \mathrm{End}(\rho)) \parallel \\ f(jgj') = \rho(j)f(g)\rho(j'), \forall j, j' \in J, g \in G \end{array} \right\}$$

This is a Hilbert algebra with convolution and *-operation;

$$f^*(g) = f(g^{-1})^*, \qquad \langle f_1, f_2 \rangle = \mathrm{Tr}(f_1 * f_2^*(1)).$$

$$\begin{array}{ccc} \text{Irr admissible repns } \tau \text{ of } G & \longleftrightarrow & \text{finite dim simple repns} \\ \check{\rho} \subset \tau | J & & \text{of } \mathcal{H}(G//J,\rho) \\ V & \longleftrightarrow & (V_\rho \otimes V)^J \end{array}$$

**Goal.** Construct as many as spherical Hecke algebras to describe as many as admissible representations. Those are usually isomorphic to (generalized) affine Hecke algebras. Moreover, for some $(M, \sigma)$ ($M$ : a Levi subgroup, $\sigma$ : supercuspidal representation of $M$), they d escribe

$$\{ \ \mathrm{Ind}_P^G \sigma \otimes \psi \mid \psi : \text{ unramified } \}.$$

Let $\widetilde{W} = NT/T_0$ be an affine Weyl group and let $v : S/\sim \longrightarrow \mathbb{Z}^+$ be the weight function. Then a *(generalized) affine Hecke algebra* $\mathcal{H}$ associated to $\widetilde{W}$ is linearly spanned by elements parametrized by $w \in \widetilde{W}$

| | |
|---|---|
| generators | $f_{s_i}$ |
| length preserving | $f_w * f_{w'} = f_{ww'}$, $l(ww') = l(w) + l(w')$ |
| quadratic relation | $f_{s_i} * f_{s_i} = q^{v(s_i)} f_1 + (q^{v(s_i)} - 1) f_{s_i}$ |
| braid relation | $f_{s_i} * f_{s_j} * \cdots = f_{s_j} * f_{s_i} * \cdots$ |

Introduce Indexed Affine Dynkin diagram reflecting the quadratic relation

$\Longrightarrow$ Affine Dynkin diagram + weights :

**Optimal result :** Construct the family of $(J, \rho)$ such that

(i) $\mathcal{H}(G//J, \rho)$ is an Affine Hecke algebra.

(ii) For any $\tau \in \widehat{G}^{\mathrm{adm}}$, there exists unique $(J, \rho)$ from the family such that $\rho \subset \tau|J$.

(iii) If $\rho \subset \mathrm{Ind}_P^G \sigma \otimes \psi$, all subquotients of $\mathrm{Ind}_P^G \sigma \otimes \psi'$ contain $\rho$.

This result is established for $GL_n$ by [HM] and [BK].

## Main Result

$k$ : $p$-adic field with involution $\sigma$ with $k^\sigma = k_0$

$V$ : finite dimensional vector space over $k$ with nondegenerate sesquilinear form $f_V$

$G$ : $G(V, f_V)$, group of isometry (or similitude) on $(V, f_V)$

**Assumption on $k$.**

(1) $\frac{1}{\dim_k(V)} > \frac{2\mathrm{ord}(p)}{p-1}\frac{p}{p-1} + \frac{1}{p-1}$

$\rightarrow \begin{cases} \text{convergence of exp on some "big" lattice, } \mathcal{R}_p \\ \text{and its image}(\exp \mathcal{R}_p) \text{ to be a group.} \end{cases}$

(2) $k$ contains enough roots of 1 $\Rightarrow$ Maximal tori are split over tamely ramified abelian extensions.

**Basic Data.** $\Sigma = (\Gamma, P_0, \varrho)$,

$$\begin{cases} \Gamma \in \mathfrak{G} : \text{s.s. of certain "truncated form"} \\ P_0 \subset C_G(\Gamma) : \text{parahoric subgroup} \\ \varrho : \text{cuspidal representation of reductive quotient of } P_0 \end{cases}$$

Here, $C_G(\Gamma)$ is a (central) product of $G(W, f_V|W)$ with $W \subset V$ and general linear groups, unitary groups defined over some finite extension of $k$.

**Theorem 1.** *For a given triple* $\Sigma = (\Gamma, P_0, \varrho)$ *as above, we construct a pair* $(J_\Sigma, \rho_\Sigma)$ *of an open compact subgroup* $J_\Sigma$ *and its representation* $\rho_\Sigma$ *such that the Hecke algebra* $\mathcal{H} = \mathcal{H}(G//J_\Sigma, \rho_\Sigma)$ *associated to* $(J_\Sigma, \rho_\Sigma)$ *is spanned by functions* $f_w$ *(uni que up to constant) whose support is a single double coset of* $J_\Sigma w J_\Sigma$ *represented by an element of the affine Weyl group of* $C_G(\Gamma)$*.*

**Corollary.** *If* $G' = C_G(\Gamma)$ *is compact* mod $Z_G$, $\mathrm{Ind}_{Z'_G \cdot J_\Sigma}^G \tilde{\rho}_\Sigma$ *is a supercuspidal representation, where* $\tilde{\rho}_\Sigma$ *is any representat ion on* $Z'_G \cdot J_\Sigma$ *extending* $\rho_\Sigma$ *on* $J_\Sigma$*.*

*More generally, if* $G'$ *does not have any* $GL$ *factors and if* $P'_0$ *in* $\Sigma$ *is maximal parahoric in* $G'$, $c$-$\mathrm{Ind}_{Z_{G'} \cdot J_\Sigma}^G \tilde{\rho}_\Sigma$ *is irreducible and supercuspid al. where* $Z_{G'}$ *is the center of* $G'$ *and where* $\tilde{\rho}_\Sigma$ *is the representation of on* $Z \cdot J_\Sigma$ *extending* $(\rho_\Sigma, J_\Sigma)$*.*

**Remark.**

$(J_\Sigma, \rho_\Sigma)$ in Theorem 1 is a Bushnell-Kutzko type ;

BK-types for principal series for split reductive groups are constructed b y A. Roche.

**Remark and Conjecture.**

(1) We get some new supercuspidal representations, which are not necessarily as sociated to a character of a tamely ramified torus [Morris], nor can be detected by elliptic unrefined $K$-types [Adler].

We have constructed all known supercuspidal representations under the above assumptions on $k$. Moreover, we can conjecture they give all supercuspidal representations.

(2) The above Hecke algebras parametrize all admissible representations of $G$ in the sense of (ii) in the introduction.

Fix $\Omega$ an additive character of $k_0$ with conductor $\mathcal{O}_{k_0}$. Then

$$\theta : \mathfrak{G} \xrightarrow{\sim} \widehat{\mathfrak{G}}$$

$$\theta(x)(y) = \Omega(\langle x, y \rangle)$$

Especially, $$\theta(\Gamma) \in \widehat{\mathfrak{G}}.$$

We construct $\mathcal{R}_p \subset \mathfrak{G}$ ; a "big" lattice where exp is defin ed. Then $\theta(\Gamma)|\mathcal{R}_p$ defines a character.

*Sketch.* Let $\mathfrak{T}$ be a maximal and maximally split torus in $\mathfrak{G}$ containing *Gamma*. Decompose $\mathfrak{G}$ as $k_0$-rational root sub-spaces of $\mathfrak{T}$ ;

$$\mathfrak{G} = \mathfrak{T} \oplus \sum \mathfrak{G}_\varepsilon.$$

We find lattice structures on each $\mathfrak{G}_\varepsilon$ and

$$\mathcal{R}_p = \mathcal{P}_1 + \sum \mathcal{L}_\varepsilon.$$

(e.g) If $\mathfrak{T}$ is a "principal" torus, $\mathcal{R}_p$ is the $\mathcal{O}_{k_0}$-L ie algebra of the 1-st congruence subgroup of some parahoric subgroup.

$$
\begin{array}{ccc}
\begin{array}{c} R = P_0 \cdot R_p, \quad \mathcal{R}_p \\ P_0 \cdot J_p \end{array} & & \begin{array}{c} \theta(\Gamma)|\mathcal{R}_p \\ \mathcal{J}_p = \mathrm{rad}\mathcal{J}'_p \end{array} \\
\Big\downarrow & \begin{array}{c} \mathrm{Ad}\ g(\Gamma + \mathcal{R}_p^\sharp) \\ = (\Gamma + \mathcal{R}_p^\sharp) \end{array} & \\
\begin{array}{c} J_{\Gamma} = P_0 \cdot (J_{\Gamma})_p \\ (J_{\Gamma})_p = \exp(\mathcal{J}_{\Gamma})_p \end{array} & \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \theta(\Gamma)|(\mathcal{J}'_{\Gamma})_p \\
 & \begin{array}{c} \rho_{\Gamma} \\ \mathcal{J}_p = \mathcal{J}'_p \\ \to \text{character} \\ \mathcal{J}_p \supsetneq \mathcal{J}'_p \\ \to \text{Heisenberg} \end{array} & \Big\uparrow \\
\begin{array}{c} \exp(\mathcal{J}'_p) \\ \cup \\ \mathrm{Stab}_R(\theta(\Gamma)|\mathcal{R}_p) \\ = H_{\Gamma} = P_0 \cdot (H_{\Gamma})_p \\ (\mathfrak{H}_{\Gamma})_p \end{array} & \begin{array}{c} X \to \theta(\Gamma)(\log X) \\ \\ \\ \rho_{\Sigma} = \rho_{\Gamma} \otimes \varrho \\ \\ \end{array} & \begin{array}{c} \mathcal{J}'_p \\ \cup \\ \mathrm{Ad}\ R\text{-invariant} \\ [\mathcal{J}'_p, \mathcal{J}'_p] \subset \ker(\theta(\Gamma)) \\ \\ \theta(\Gamma)|(\mathfrak{H}_{\Gamma})_p \neq 1 \end{array}
\end{array}
$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

**Crucial Lemma.**

$$\mathrm{Supp}(\mathcal{H}(G//J_{\Sigma}, \rho_{\Sigma})) \subset J_{\Sigma} C_G(\Gamma) J_{\Sigma} = J_{\Sigma} \widetilde{W}' J_{\Sigma}.$$

**Theorem 2.** *Let $G$ be a symplectic group in $2n$ variables over $k$. Let $P_0$ is minimal parahoric and $\varrho = 1$ in $\Sigma = (\Gamma, P_0, 1)$. Then we have an $L^2$-isomorphism*

$$\eta : \mathcal{H}(G//J_{\Sigma}, \rho_{\Sigma}) \xrightarrow{\sim} \mathcal{H}(C_G(\Gamma)//P_0, \chi)$$

*where $\chi$ is a suitable tamely ramified character of $P_0$.*

**Remark.** $\mathcal{H}(C_G(\Gamma)//I_0, \chi)$ is a generalized affine Hecke algebra (not necessarily Iwahori Hecke Algebra).