STRUCTURE OF $\Lambda\mathfrak{g}$ AND DISCRETE SERIES REPRESENTATIONS

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In this talk, $\mathfrak{g}$ is a simple complex Lie algebra, and G is the simply connected complex Lie group with Lie algebra $\mathfrak{g}$. We begin with some history.

In the early 1960's I was trying to classify the abelian subalgebras of $\mathfrak{g}$. For any k-dimensional subspace $\mathfrak{m} \subset \mathfrak{g}$, denote

$$[\mathfrak{m}] = \Lambda^k \mathfrak{m} \subseteq \Lambda^k \mathfrak{g}.$$

Let $M_k \subset \Lambda^k\mathfrak{g}$ be the span of the $[\mathfrak{m}]$'s with $\mathfrak{m}$ abelian, $\dim \mathfrak{m} = k$, and let $M = \oplus_k M_k \subset \Lambda\mathfrak{g}$. Clearly M_k and M are G submodules of $\Lambda\mathfrak{g}$.

In my paper [Ann. Math. 1961] I showed that, as an ungraded G-module, $\Lambda\mathfrak{g}$ is 2^ℓ copies of $V_\rho \otimes V_\rho$, where ℓ is the rank of G and V_ρ is the irreducible module with highest weight ρ=half sum of the positive roots. This is explained on the level of modules as follows. Consider the Clifford algebra $C(\mathfrak{g})$, which is isomorphic to $\Lambda\mathfrak{g}$ as a G-module. Let J be the G invariants in $C(\mathfrak{g})$. It is well known that $\dim J = 2^\ell$. It turns out that $E := End(V_\rho)$ is isomorphic to the Clifford algebra on the space $\{dx : \ x \in \Lambda\mathfrak{g}\}$, where d is the Koszul differential of $\Lambda\mathfrak{g}$, and we have

$$C(\mathfrak{g}) \simeq E \otimes J.$$

Now, the abelian subalgebras are related to the *graded* G-module $\Lambda^k\mathfrak{g}$. Consider the action on $\Lambda\mathfrak{g}$ of the Casimir operator Q, normalized to have eigenvalue 1 on $\Lambda^1\mathfrak{g} = \mathfrak{g}$. That is, we scale the invariant symmetric form used to define Q so that

$$|\rho + \psi|^2 - |\rho|^2 = 1,$$

where ψ is the highest root. The norm of Q acting on $\Lambda^k\mathfrak{g}$ is its highest eigenvalue m_k. In [Topology, 1965] I proved

Theorem. *$m_k \le k$, with equality if and only if $M_k \neq 0$, in which case M_k is the m_k eigenspace of Q on $\Lambda^k\mathfrak{g}$.*

For example, if $k = 2$, we have $m_2 = 2$, and M_2 is the kernel of the bracket $\Lambda^2\mathfrak{g} \longrightarrow \mathfrak{g}$.

Moreover, M generates $\mathfrak{g}$, viewed as a module over the symmetric algebra $S\mathfrak{g}$ via the Chevalley transgression $c : S\mathfrak{g} \longrightarrow \Lambda^{\text{even}}\mathfrak{g}$. This is the degree doubling ring homomorphism given in degree one by $x \mapsto dx$. Let A be the image of c, and let I be the ideal in $\Lambda\mathfrak{g}$ generated by A. Then we have

$$\Lambda\mathfrak{g} = A \wedge M = I \oplus M.$$

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What are the highest weight spaces in M_k? Let $\mathfrak{b}$ be a Borel subalgebra. Let Ξ be an index set for the abelian ideals in $\mathfrak{b}$. For example, if $\mathfrak{g} = \mathfrak{sl}(n)$, the set Ξ may be taken as the set of Young diagrams with fewer than n rows and columns. A diagram indicates the nonzero entries in the corresponding ideal.

In general, for each $\xi \in \Xi$, we have an abelian ideal $\mathfrak{a}_\xi \subset \mathfrak{b}$, and also the set Γ_ξ of positive roots appearing in $\mathfrak{a}_\xi$. Moreover, the highest weight subspaces in M are precisely the $[\mathfrak{a}_\xi]$, for $\xi \in \Xi$, with weight equal to the sum of the roots in Γ_ξ.

It follows that the theorem above actually characterizes abelian ideals: Given any set $\Gamma = \{\phi_1, \ldots, \phi_k\}$ of positive roots, we have

$$|\rho + \phi_1 + \cdots + \phi_k|^2 - |\rho|^2 \leq k,$$

with equality if and only if $\Gamma = \Gamma_\xi$ for some ξ, iff the span of the root vectors from Γ is an abelian ideal.

The number of abelian ideals in $\mathfrak{b}$ was found by Dale Peterson to be $|\Xi| = 2^\ell$. This is also the number of involutions in a maximal torus T, and Peterson also explained this relation, in terms of the affine Weyl group.

Recall that affine roots are the affine functions on the Lie algebra $\mathfrak{h}$ of T, of the form $n\delta + \phi$, where ϕ is a root, n is an integer, and δ is the constant function 1. Let $\Delta^a_\pm$ be the positive/negative affine roots, where the sign of $n\delta + \phi$ is that of ϕ. The affine Weyl group W^a is the group of affine motions of $\mathfrak{h}$ generated by the reflections about the hyperplanes of zeros of the affine roots. W^a permutes the affine roots by composition. It is also a semidirect product $W^a = Y \rtimes W$, where W is the usual Weyl group, and Y is a lattice in the split real form of $\mathfrak{h}$.

We can now describe Peterson's bijection

$$\{\text{abelian ideals in } \mathfrak{b}\} \longrightarrow \{\text{involutions in } T\}.$$

Let $\mathfrak{a}_\xi$ be an abelian ideal. Write $\Gamma_\xi = \{\phi_1, \ldots, \phi_k\}$. Then there is a unique $\sigma \in W^a$ such that the set of positive affine roots made negative by σ is exactly

$$\delta - \Gamma_\xi := \{\delta - \phi_1, \ldots, \delta - \phi_k\}.$$

Decompose $\sigma = z_\sigma w_\sigma$ according to the semidirect product expression of W^a. Then $\exp \pi i z_\sigma$ is the corresponding involution in T. An element $z \in \mathfrak{h}$ is a z_σ if and only if $\exp 2\pi i z = 1$ and all positive root values on z belong to $\{-1, 0, 1, 2\}$.

Now involutions in T give rise to symmetric homogeneous spaces for real forms of G. For simplicity, let us assume the center of G is trivial.

We consider pairs $(\mathfrak{k}, \mathfrak{b})$, where $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$, and $\mathfrak{k} = \mathfrak{g}^\theta$ (fixed points of $Ad\theta$) for an involution $\theta \in T$, such that $\theta(\mathfrak{b}) = \mathfrak{b}$. Let $\mathfrak{p}$ be the -1 eigenspace of θ, so we have

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \qquad \mathfrak{b} = \mathfrak{b}_\mathfrak{k} \oplus \mathfrak{b}_\mathfrak{p},$$

where $\mathfrak{b}_\mathfrak{k} = \mathfrak{b} \cap \mathfrak{k}$, $\mathfrak{b}_\mathfrak{p} = \mathfrak{b} \cap \mathfrak{p}$

Let us say the pair $(\mathfrak{k}, \mathfrak{b})$ is *compatible* , if

$$[[\mathfrak{b}_\mathfrak{p}, \mathfrak{b}_\mathfrak{p}], \mathfrak{b}_\mathfrak{p}] = 0.$$

Then $[\mathfrak{b}_\mathfrak{p}, \mathfrak{b}_\mathfrak{p}]$ is an abelian ideal $\mathfrak{a}_\xi$, for some ξ.

Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{k} \cap \mathfrak{b}$, and a Borel subalgebra $\mathfrak{s}$ of $\mathfrak{k}$, with $\mathfrak{h} \subset \mathfrak{s} \subset \mathfrak{k}$. Let $\mathcal{B}^\mathfrak{s}$ be the set of Borel subalgebras of $\mathfrak{g}$ whose intersection with $\mathfrak{k}$ contains $\mathfrak{s}$.

Even if the center of G is non-trivial, we have the following

Theorem. *The number of compatible pairs* $(\mathfrak{k}, \mathfrak{b})$ *with* $\mathfrak{b} \in \mathcal{B}^{\mathfrak{s}}$ *is equal to the order of the center of* G.

Consider the example $G = E_8$. There are three types of symmetric spaces with $\mathfrak{k}_1 = A_1 \times E_7$, $\mathfrak{k}_2 = D_8$, $\mathfrak{k}_3 = E_8$ (compact form). The sum of the Euler characteristics of the compact forms of G/K_i is $120 + 135 + 1 = 2^8$. Via Peterson's bijection, we have a corresponding partition $\Xi = \Xi_1 \cup \Xi_2 \cup \Xi_3$ whose respective cardinalities are $120, 135, 1$. In each Ξ_i there is a distinguished ξ_i, such that $\mathfrak{a}_{\xi_i} = [\mathfrak{b}^i_{\mathfrak{p}}, \mathfrak{b}^i_{\mathfrak{p}}]$, where $(\mathfrak{k}_i, \mathfrak{b}^i)$ is the unique compatible pair as in the theorem (recall the center of E_8 is trivial). The other abelian ideals are obtained as follows. For simplicity take $i = 2$ and suppress it. There are 135 abelian subspaces $V_j \subset \mathfrak{b}_{\mathfrak{p}}$ which are stable under the adjoint action of $\mathfrak{b}_{\mathfrak{k}}$. The 135 abelian ideals in $\mathfrak{b}$ are exactly the $[\mathfrak{b}_{\mathfrak{p}}, \mathfrak{b}_{\mathfrak{p}}] + V_j$.

The connection with discrete series is as follows. The set $\mathcal{B}^{\mathfrak{s}}$ parametrizes discrete series of various real forms of G, and more precisely, Ξ_i parametrizes discrete series for the i^{th} symmetric space. Let $\theta = \exp(\pi i x)$ be the corresponding involution, and let $\tilde{x}$ be the linear form on $\mathfrak{h}$ determined by x via duality. Then the highest weight of the lowest K-type among all the discrete series corresponding to $\xi \in \Xi_i$ is $\tilde{x} - 2[\xi]$, where $[\xi]$ is the sum of the roots in Γ_ξ.

If the center C of G is non-trivial one does not obtain all the equal rank symmetric spaces by restricting one's self to the involutions in the maximal torus. In order to capture all the symmetric spaces one must extend Petersons theory. In particular drop the condition $\exp 2\pi i z = 1$, and let

$$Z := \{z \in \mathfrak{h} : \ \phi(x) = -1, 0, 1, 2 \text{ for all } \phi \in \Delta^+\}.$$

We have $|Z| = 2^\ell |C|$. Then one aspect of the extension of Petersons theory is an explict surjection $Z \to \Xi$ all of whose fibers have cardinality equal to the order of C. Moreover the fibers of this are exactly the orbits of D on Z, where D is the stabilizer in the extended affine Weyl group of a fundamental simplex.

We close with some open questions. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{b}$. The abelian ideals in $\mathfrak{b}$ all live in $(\Lambda\mathfrak{n})^{\mathfrak{n}}$, a space of mysterious nature. Is it spanned by indecomposable elements? The highest weights of constituents of M seem to be of PRV type, that is, of the form $w_1\rho - w_2\rho$. Are all constituents of $(\Lambda\mathfrak{n})^{\mathfrak{n}}$ of this form?