ENDOSCOPY FOR HECKE ALGEBRAS

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1. Introduction. Let G be a p-adic group and let $f \in C_c^\infty(G)$. Let H be an endoscopic group for G. The conjectural theory of endoscopy predicts th e existence of a function $f^H \in C_c^\infty(H)$ whose stable orbital integrals are linear comb inations of orbital integrals of f, the coefficients in the linear combinations being the transfe r factors of Langlands-Shelstad [LS]. For a general function f one does not expect to be able to find f^H explicitly. But if G is unramified and f is bi-invariant under a hype rspecial maximal compact subgroup K of G, then at least there is a conjectural candidate fo r f^H. This conjecture, which is due to Langlands, is known as the fundamental lemma. Recen tly Waldspurger [W] has shown that the fundamental lemma would imply the existence of f^H for general functions f.

It would be desirable to have a greater supply of functions f for which one k nows a conjectural candidate for f^H. For example, Assem, in his work on endoscopic induction of unipotent orbital integrals, was obliged to limit his test functions to those bi-invariant under a hyperspecial maximal compact subgroup. However these are not enough to separate the various unipotent orbits and are thus not completely satisfactory for the purpose he had in mind, which was to test the correctness of his ideas on transfer factors for unipotent elements (the transf er factors of Langlands-Shelstad are defined only for certain semisimple elements, including all regular ones). Motivated by discussions with Assem during his visit to Chicago in the spring of 1996, I looked for more candidate pairs f, f^H and found the ones that are the subject of this lecture.

2. Preliminaries. Let F be a p-adic field and let $\overline{F}$ be an algebraic closure of F. Let $\mathfrak{o}$ be the valuation ring of F. Let G be a split adjoint semisimple group over F. We write $\hat{G}$ for the Langlands dual group of G. Sometimes we write G rather tha n $G(F)$.

3. Stable conjugacy. Recall that a semisimple element $\gamma \in G(F)$ is said to be *strongly regular* if its centralizer G_γ in G is a torus. Let γ, γ' be strongly regular semisimple elements in $G(F)$. Recall that γ, γ' are said to be *stably conjugate* if there exists $g \in G(\overline{F})$ such that $g^{-1}\gamma g = \gamma'$. The inner automorphism $x \mapsto g^{-1}xg$ induces a n isomorphism (over F) from G_γ to $G_{\gamma'}$, and this isomorphism is independent of t he choice of g.

4. Stable orbital integrals. Let γ be a strongly regular sem isimple element of $G(F)$. Choose Haar measures dg and dt on $G(F)$ and $G_\gamma(F)$ respecti vely. For any function $f \in C_c^\infty(G)$ we write $O_\gamma(f)$ for the orbital integral

$$O_\gamma(f) := \int_{G(F)/G_\gamma(F)} f(g\gamma g^{-1})\, dg/dt.$$

Similarly we write $SO_\gamma(f)$ for the stable orbital integral

$$SO_\gamma(f) := \sum_{\gamma'} O_{\gamma'}(f)$$

where the sum is taken over a set of representatives γ' for the conjugac y classes within the stable conjugacy class of γ. We form the orbital integral $O_{\gamma'}$ using dg and the Haar measure dt' on $G_{\gamma'}(F)$ obtained from dt via the isomorphism from G_γ to $G_{\gamma'}$ defined in §3.

5. Endoscopic groups. Let s be a semisimple element in $\hat{G}$ a nd let $\hat{H}$ denote its centralizer, a reductive complex group which is in fact connected by a theorem of Steinberg (since G is adjoint, $\hat{G}$ is simply connected). Let H be the split connected reductive group over F whose dual group is $\hat{H}$; H is called an *endoscopic group* for G. Note that H comes equipped with the element $s \in Z(\hat{H})$, where $Z(\hat{H})$ denote s the center of $\hat{H}$.

6. Transfer factors. Let T_G denote a split maximal torus in G and let W_G denote its Weyl group. The quotient variety T_G/W_G is the space of semisimple conju gacy classes in G, and its set of F-points is the set of semisimple conjugacy classes that are d efined over F. The same remarks apply to H; moreover there is an isomorphism $T_G \simeq T_H$, c anonical up to composition with elements in W_G, and via this isomorphism W_H goes over to a subgroup of W_G. Therefore there is a canonical morphism from T_H/W_H to T_G/W_G and hence a canonical map from the set of semisimple conjugacy classe s in H over F to the set of semisimple conjugacy classes in G over F. One says that a semis imple conjugacy class in H is strongly G-regular if the associated class in G is strongly regul ar.

Let Δ_0 be the transfer factor defined just before Theorem 3.7.A of [LS]. It depends on the choice of an F-splitting for G, but since G is adjoint any two F-sp littings are $G(F)$-conjugate and hence the transfer factor is independent of the choice of F-splitting. Since G and H are split there is a canonical L-homomorphism ${}^LH \to {}^LG$, n amely the obvious inclusion, and of course it is this L-homomorphism we use to form the transfe r factor Δ_0.

7. Matching orbital integrals. We now recall another definition fro m [LS]. Let $f \in C_c^\infty(G)$ and let $f^H \in C_c^\infty(H)$. One says that f and f^H have *matching orbital integrals* if for every strongly G-regular element $\gamma_H \in H(F)$ the following equality holds:

$$SO_{\gamma_H}(f^H) = \sum_{\gamma'} \Delta_0(\gamma_H, \gamma')O_{\gamma'}(f),$$

where γ' runs over a set of representatives for the $G(F)$-conjugacy cla sses of elements in the stable conjugacy class in $G(F)$ associated to the stable conjugacy class o f γ_H in $H(F)$ as in §6. Langlands and Shelstad conjecture that for every $f \in C^{\in}fty_c(G)$ there exists $f^H \in C_c^\infty(H)$ such that f and f^H have matching orbital in tegrals.

8. Statement of fundamental lemma. We now recall from [L] the statement of the (conjectural) fundamental lemma. Fix special maximal compact subgroups

K_G and K_H in $G(F)$ and $H(F)$ respectively. The Satake isomorphism identifies the Hecke algebra $C_c(K_G\backslash G(F)/K_G)$ with the algebra of $\hat{G}$-conjugation invariant regular functions on $\hat{G}$, and the same is true for H. By means of these identifications the obvious restriction map from functions on $\hat{G}$ to functions on $\hat{H}$ determines an algebra homomorphism

$$b : C_c(K_G\backslash G(F)/K_G) \to C_c(K_H\backslash H(F)/K_H).$$

The fundamental lemma is the assertion that for any $f \in C_c(K_G\backslash G(F)/K_G)$ the functions f and $b(f)$ have matching orbital integrals.

9. Deligne-Lusztig representations. We now review a construction from [DL]. Let k be the residue field of F. We now consider the group G over the field k. As before we denote by W_G the Weyl group of G. Let X denote the flag manifold of G, a variety over k. The variety X is a disjoint union of locally closed pieces X_w, one for each element $w \in W_G$. By definition the subset X_w consists of all points $x \in X$ such that x and $F(x)$ are in relative position w (in other words, the pair $(x, F(x))$ lies in the G-orbit in $X \times X$ indexed by w) ; here F denotes the Frobenius morphism $X \to X$. As in Definition 1.5 of [DL] we denote by R_w the virtual character

$$\sum (-1)^i H^i_c(X_w, \overline{\mathbf{Q}}_l)$$

of $G(k)$ (here l is a prime different from the characteristic of k). The virtual character R_w depends only on the conjugacy class of $w \in W_G$ (see Theorem 1.6 in [DL]) and its values are rational integers independent of l (see Proposition 3.3 of [DL]). As in [Lu] we define, for any complex-valued class function ξ on W_G, a complex-valued class function R_ξ on $G(k)$ by

$$R_\xi := |W_G|^{-1} \sum_{w \in W_G} \xi(w) R_w$$

(in this definition we are of course regarding R_w as a $\mathbf{Z}$-valued function on $G(k)$).

10. Some test functions. We now use the class functions R_ξ to define functions f_ξ on $G(F)$. To do this we consider the canonical surjection

$$K_G = G(\mathfrak{o}) \to G(k),$$

which we denote by $x \mapsto \bar{x}$. We then define f_ξ on K_G by putting

$$f_\xi(x) = R_\xi(\bar{x})/\operatorname{vol}(K_G)$$

and extend f_ξ to $G(F)$ by making it 0 outside of K_G.

11. Conjecture. As before we use the natural embedding of W_H as a subgroup of W_G. This embedding is canonical up to inner automorphisms of W_G, and therefore we can restrict a class function ξ_G on W_G to W_H, obtaining a class function ξ_H on W_H. It seems very likely that the functions $f_{\xi_G} \in C^\infty_c(G)$ and $f_{xi_H} \in C^\infty_c(H)$ have matching orbital integrals.

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