CELLS OF HARISH-CHANDRA MODULES FOR REAL CLASSICAL GROUPS

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$G =$ real reductive group, classical type

$K =$ maximal compact subgroup

$\mathfrak{g} =$ complexified Lie algebra

$W =$ Weyl group of $\mathfrak{g}$

$G_0 := G_0(\mathfrak{g}, K) =$ complexified Grothendieck group of finitely generated $(\mathfrak{g}, K)$ modules of trivial infinitesimal character

Study as W-module, via coherent continuation representation; look at (Harish-Chandra) *cells*, spanned by classes of simple modules in G_0

Motivations:

- Vogan's program for computing irreducible characters as combinations of standard characters
- Understand (graded) Casselman-Jacquet and Bernstein functors, which interchange HC cells and cells in category $\mathcal{O}$ in a W-equivariant way; exploit well-known module structure of the latter kind of cell
- Parametrize HC modules with a fixed annihilator; e.g. ladder representations
- Illustrate the behavior of Kazhdan-Lusztig theory in a different module category

Cells in category $\mathcal{O}$ (KL cells)

Type A: KL cells are irreducible as modules; all irreducible modules occur.

Types B,C,D: To a KL *double* cell D there corresponds a pair L, R of n-dimensional vector spaces over F_2 with canonical bases $\{\ell_i\}, \{r_i\}$ and a perfect pairing $(\cdot, \cdot)$. Thus $L \oplus R$ becomes a symplectic space over F_2. Then one-sided KL cells inside D correspond to Lagrangean subspaces of $L \oplus R$ of the form $S \oplus S^\perp$, where both S and $S^\perp$ are spanned by sums of *consecutive* basis vectors. Each vector in $S \oplus S^\perp$ corresponds to a constituent of the cell, which occurs with multiplicity one.

Two easy examples: $S = L, S = 0$ (Springer, Lusztig cells)

HC cells

In general: HC cells lie inside KL double cells.

Type A: HC cells are again irreducible.

Types B,C,D: HC cells again multiplicity-free as W-modules; each contained in some KL double cell D as above, so parametrized by some subset H of the sum $L \oplus R$.

Conjecture: H again takes the form $T \oplus T^{\perp}$ for some $T \subset L$ (NOT necessarily spanned by sums of consecutive basis vectors); or at least H is a Lagrangean subspace of $L \oplus R$.

Theorem: For any KL left cell C lying in D, parametrized by some $C' := S \oplus S^{\perp}$, the intersection $H \cap C'$ is a subspace of C', which parametrizes simple HC modules with annihilator I, the primitive ideal corresponding to C. If I is replaced by its image under a wall-crossing operator, the dimension of the intersection $H \cap C'$ changes by one.

More precisely: HC cells for

- $Sp(p,q)$ or $SO^*(2n)$ are all Springer, parametrized by real nilpotent orbits.
- $Sp(2n,R)$ or $SO(p,2n-p)$ are all Lusztig.
- $SO(p,2n+1-p)$ are all Springer.
- $SO_e(p,n-p)$ need NOT be isomorphic to KL cells; especially mysterious if n is even.

Also know the behavior of Vogan IC4 duality for types A, B, C: involves certain disconnected groups, many of which have HC cells not isomorphic to KL cells. Do NOT know, for example, the decomposition of the full coherent continuation representation for even Spin groups.

Ingredients of proof:

- Asymptotic Hecke algebra of Lusztig: group algebra of W has many elementary abelian 2-groups sitting inside it which send simple HC modules to simple modules (to prove this, must use deep positivity results of Springer on Hecke algebra and adapt them to the HC setting via Lusztig-Vogan)
- Casian: W-equivariant injection $G_0 \to (\mathbb{C}W)^m$, some m, sending HC cells to double cells, and behaving well with respect to annihilators
- Barbasch-Vogan: formula for full coherent continuation representation as a direct sum of induced representations
- Littlewood-Richardson Rule for inducing representations of $S_p \times S_q$ to S_{p+q}
- Formula of Adams-Barbasch-Vogan for counting blocks in the Grothendieck group